



Proving Inequalities with
Difference Substitution
Yu-Dong Wu, Zhi-Hua Zhang
and Yu-Rui Zhang
vol. 8, iss. 3, art. 81, 2007

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PROVING INEQUALITIES IN ACUTE TRIANGLE WITH DIFFERENCE SUBSTITUTION

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Abstract: In this paper, we prove several inequalities in the acute triangle by means of so-called Difference Substitution. As generalization of the method, we also consider an example that the greatest interior angle is less than or equal to 120° in the triangle.

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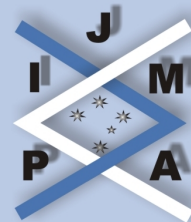
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1. Introduction

In [1, 2], L. Yang suggested the use of Difference Substitution to prove asymmetric polynomial inequalities, as it had been used previously to deal with symmetric ones.

If $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n$ with $n \in \mathbb{N}^*$, then we set

$$(1.1) \quad \begin{cases} x_1 = t_1, \\ x_2 = t_1 + t_2, \\ x_3 = t_1 + t_2 + t_3, \\ \dots\dots\dots \\ x_n = t_1 + t_2 + t_3 + \dots + t_n, \end{cases}$$

where $t_i \geq 0$ for $2 \leq i \leq n$ and $i \in \mathbb{N}^*$.

The expansion (1.1) is so-called a “splitting” transformation, and $\{t_1, t_2, \dots, t_n\}$ is simply the difference sequence of $\{x_1, x_2, \dots, x_n\}$.

In general, for the n -variant polynomials, there are $n!$ different orders of $\{x_1, x_2, \dots, x_n\}$, sorting by size. In the instance of $n = 3$, we let $x \leq y \leq z$, and take

$$(1.2) \quad \begin{cases} x = u, \\ y = u + v, \\ z = u + v + w, \end{cases}$$

where $v \geq 0, w \geq 0$.

Analogously, if $y \leq x \leq z$, then its “splitting” transformation is

$$(1.3) \quad \begin{cases} y = u, \\ x = u + v, \\ z = u + v + w, \end{cases}$$

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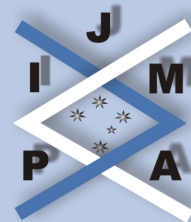
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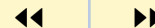
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where $v \geq 0, w \geq 0$.

Sequentially, for $y \leq z \leq x$ or $z \leq x \leq y$ or $z \leq y \leq x$ or $x \leq z \leq y$, we set four similar linear transformations.

For a 3-variant polynomial $F(x, y, z)$, by using the six linear transformations above, we obtain 6 members $P_i(u, v, w)$ with $1 \leq i \leq 6$, and call the set $\{P_1, P_2, \dots, P_6\}$ the Difference Substitution of $F(x, y, z)$ and denote this by $DS(F)$. If all the coefficients of these members $DS(F)$ are nonnegative, then $F \geq 0$ whenever x, y, z all are nonnegative. In other words, F is positive semi-definite on \mathbb{R}_+^3 . Difference substitution is a very valid method for proving inequalities. For more information on Difference Substitution, please refer to [3] and [4].

In this paper, by using Difference Substitution, the authors prove several inequalities in acute triangles.

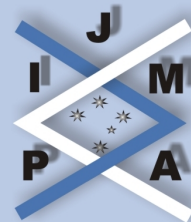
Throughout the paper we denote A, B, C as the interior angles, a, b, c as the side-lengths, S as the area, s as the semi-perimeter, R as the circumradius, r as the in-radius, h_a, h_b, h_c as the altitudes, m_a, m_b, m_c as the medians, and r_a, r_b, r_c as the radii of the described circles of triangle ABC respectively. Moreover, we will customarily use the cyclic sum symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, and $\sum f(a, b) = f(a, b) + f(b, c) + f(c, a)$, etc.

Let us begin with the well-known Walker's inequality [5]. In the acute triangle, show that

$$(1.4) \quad s^2 \geq 2R^2 + 8Rr + 3r^2,$$

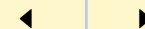
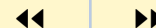
or

$$(1.5) \quad -2a^3b^3 + a^4b^2 - a^4bc + a^5b + ab^5 + b^5c + b^4c^2 \\ - 2b^3c^3 + b^2c^4 - 2c^3a^3 + c^4a^2 + c^5a + c^5b + c^2a^4 \\ + a^5c - ab^4c + a^2b^4 - b^6 - c^6 - a^6 - abc^4 \geq 0.$$



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Let

$$(1.6) \quad \begin{cases} x = \frac{b+c-a}{2} > 0, \\ y = \frac{c+a-b}{2} > 0, \\ z = \frac{a+b-c}{2} > 0. \end{cases}$$

Then inequality (1.4) or (1.5) is equivalent to

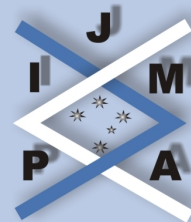
$$(1.7) \quad F(x, y, z) = 6xyz^4 + 2xy^2z^3 + 2xy^3z^2 + 6xy^4z + 2x^2yz^3 + 2x^2y^3z \\ + 2x^3yz^2 + 2x^3y^2z + 6x^4yz - x^4y^2 - x^2z^4 - 2x^3z^3 - x^4z^2 \\ - 2x^3y^3 - y^4z^2 - y^4x^2 - 2y^3z^3 - y^2z^4 - 18x^2y^2z^2 \geq 0.$$

There is no harm in supposing $x \leq y \leq z$ since inequality (1.7) is symmetric for x, y, z . Then, by using (1.2), for the acute triangle, it follows that

$$(1.8) \quad b^2 + c^2 - a^2 = (z+x)^2 + (x+y)^2 - (y+z)^2 = 2[x^2 + (y+z)x - yz] \\ = 2\{u^2 + [(u+v) + (u+v+w)]u - (u+v)(u+v+w)\} \\ = 2(2u^2 - v^2 - vw) > 0,$$

and $F(x, y, z)$ in (1.7) is transformed into

$$(1.9) \quad F(x, y, z) \\ = P(u, v, w) \\ = (2u^2 - v^2 - vw) [(4v^2 + 4w^2 + 4vw)u^2 \\ + (8v^3 + 20vw^2 + 12v^2w + 8w^3)u + 4v^4 + 8v^3w + 2w^4 \\ + 18vw^3 + 22v^2w^2] + (24v^3w^2 + 36v^2w^3 + 12vw^4)u \\ + 34v^3w^3 + 19v^2w^4 + 2vw^5 + 17v^4w^2.$$



Obviously $F(x, y, z) = P(u, v, w) \geq 0$ from (1.8) and $u > 0, v \geq 0, w \geq 0$, i.e., inequality (1.4) or (1.5) is true.

Now, let us consider another semi-symmetric inequality [6] in the acute triangle

$$(1.10) \quad \cos(B - C) \leq \frac{h_a}{m_a}.$$

It is equivalent to

$$(1.11) \quad -a^4 + (3b^2 + 3c^2)a^2 - 2(b - c)^2(b + c)^2 \geq 0,$$

and from (1.6), this equals

$$(1.12) \quad F(x, y, z) \\ = (-y^2 - z^2 + 14yz)x^2 - (y + z)(z^2 - 14yz + y^2)x + yz(y + z)^2 \geq 0.$$

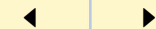
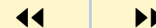
Calculating $DS(F)$, it consists of 3 polynomials with $u > 0, v \geq 0, w \geq 0$ as follows

$$(1.13) \quad P_1(u, v, w) \\ = 40u^4 + 112u^3v + 108u^2v^2 + 56u^3w + 14u^2w^2 + 40uv^3 + 20uvw^2 \\ + 60uv^2w + 108u^2vw + 8v^3w + 5v^2w^2 + vw^3 + 4v^4,$$

$$(1.14) \quad P_2(u, v, w) \\ = (2u^2 - v^2 - vw)(20u^2 + (24w + 52v)u + 53v^2 + 6w^2 + 52vw) \\ + (72v^3 + 36vw^2 + 108v^2w)u + 57v^2w^2 + 51v^4 + 6vw^3 + 102v^3w,$$

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and

$$(1.15) \quad P_3(u, v, w) \\ = (2u^2 - v^2 - vw)(20u^2 + (52v + 28w)u + 53v^2 + 54vw + 7w^2) \\ + (72v^3 + 36vw^2 + 108v^2w)u + 57v^2w^2 + 51v^4 + 6vw^3 + 102v^3w.$$

By (1.8), we immediately obtain $P_i(u, v, w) \geq 0$ for $1 \leq i \leq 3$. Hence, inequality (1.10) is proved.



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2. Some Problems and their Proofs

2.1. The Problems

In 2004-2005, J. Liu [7, 8] posed the following conjectures for the inequality in the acute triangle.

Problem 1. Let $\triangle ABC$ be an acute triangle. Prove the following inequalities

$$(2.1) \quad \sum \left(\frac{\sin 2A}{\sin B + \sin C} \right)^2 \leq \frac{3}{4},$$

and

$$(2.2) \quad \sin \frac{A}{2} \leq \frac{\sqrt{m_b m_c}}{2m_a}.$$

2.2. The Proof of Inequality (2.1)

Proof. Using $\sin 2\alpha = 2 \sin \alpha \cos \alpha$, we find that inequality (2.1) is equivalent to

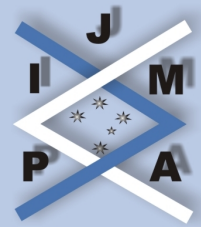
$$(2.3) \quad \begin{aligned} & 4a^{10}b^2 - 10a^5b^5c^2 - 24b^6c^5a + 16a^9b^3 + 4a^8b^4 - 5b^4c^6a^2 \\ & - 8a^{11}b - 5a^4b^6c^2 + 8a^9c^2b + 4a^8b^2c^2 - 16a^{10}cb + 8a^9cb^2 - 5a^6b^4c^2 \\ & + 32a^8b^3c + 8a^7b^4c + 8a^7c^4b - 5a^6c^4b^2 + 32a^8c^3b + 4a^{10}c^2 - 8a^{11}c \\ & - 4a^{12} - 4b^{12} - 4c^{12} + 4a^8c^4 + 16a^9c^3 - 8a^7b^5 - 8a^6b^6 - 8a^6c^6 \\ & - 8a^7c^5 - 8a^5b^7 + 4a^4b^8 - 24a^6b^5c - 24a^5b^6c + 2a^5b^3c^4 + 6a^4b^4c^4 \\ & + 4c^{10}b^2 - 26a^6b^3c^3 + 2a^5b^4c^3 - 24a^5c^6b - 5a^4c^6b^2 - 24a^6c^5b \\ & - 10a^5c^5b^2 + 8a^4b^7c - 8c^{11}b + 32b^8a^3c + 8b^9a^2c + 4b^8a^2c^2 \\ & + 2b^5c^3a^4 - 26b^6c^3a^3 + 2b^5c^4a^3 - 5b^6c^4a^2 - 16b^{10}ca + 8b^9c^2a \end{aligned}$$

$$\begin{aligned}
& + 32b^8c^3a + 8b^7c^4a - 10b^5c^5a^2 + 16b^9a^3 + 4b^{10}a^2 - 8b^{11}a \\
& - 8b^{11}c + 4b^{10}c^2 + 16b^9c^3 + 4b^8c^4 - 8b^7c^5 - 8b^6c^6 + 4a^4c^8 \\
& - 8a^5c^7 - 24b^5c^6a + 8a^4c^7b + 2c^5b^4a^3 - 26c^6b^3a^3 + 2c^5b^3a^4 \\
& + 4c^8b^2a^2 + 8c^9a^2b + 32c^8a^3b + 8b^4c^7a - 8c^{11}a + 32c^8b^3a + 8c^9b^2a \\
& - 16c^{10}ab + 4c^{10}a^2 + 16c^9a^3 - 8b^5c^7 + 4b^4c^8 + 16c^9b^3 \geq 0.
\end{aligned}$$

From (1.6), inequality (2.3) equals

(2.4) $F(x, y, z)$

$$\begin{aligned}
& = -4576x^7z^5 - 5590x^6z^6 - 116x^{10}z^2 - 2453x^4z^8 - 2453x^8z^4 \\
& - 4576x^5z^7 - 788x^3z^9 - 788x^9z^3 - 2453x^8y^4 - 4576y^7z^5 \\
& - 2453y^8z^4 - 2453x^4y^8 - 788x^9y^3 - 788x^3y^9 - 5590x^6y^6 \\
& - 4576x^7y^5 - 4576x^5y^7 - 2453y^4z^8 - 4576y^5z^7 - 5590y^6z^6 \\
& - 116y^{10}z^2 - 788y^9z^3 - 116y^{10}x^2 - 788y^3z^9 - 116y^2z^{10} \\
& + 13448x^6y^5z + 8176x^7yz^4 + 13448x^6yz^5 + 13448x^5yz^6 \\
& + 8176x^4yz^7 + 6448x^2y^3z^7 + 1220xy^9z^2 + 6448x^2y^7z^3 \\
& + 6448x^3y^7z^2 + 1220x^9yz^2 + 10862x^4y^6z^2 + 14288x^2y^5z^5 \\
& + 10862x^2y^4z^6 + 14288x^5y^2z^5 + 10862x^6y^2z^4 + 6448x^7y^2z^3 \\
& - 28248x^3y^5z^4 - 28248x^4y^5z^3 + 14288x^5y^5z^2 - 57474x^4y^4z^4 \\
& - 28248x^3y^4z^5 - 8672x^3y^3z^6 + 6448x^3y^2z^7 + 10862x^2y^6z^4 \\
& - 8672x^3y^6z^3 - 28248x^5y^4z^3 + 10862x^6y^4z^2 - 28248x^4y^3z^5 \\
& - 28248x^5y^3z^4 - 8672x^6y^3z^3 + 6448x^7y^3z^2 + 10862x^4y^2z^6 \\
& + 3420x^8yz^3 + 3420x^8y^3z + 1220x^2yz^9 + 280xy^{10}z + 4066x^2y^2z^8
\end{aligned}$$



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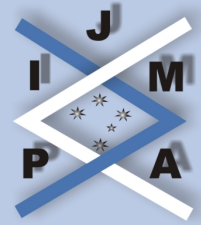
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$$\begin{aligned}
&+ 3420 x^3 y z^8 + 3420 x^3 y^8 z + 4066 x^8 y^2 z^2 + 4066 x^2 y^8 z^2 \\
&+ 1220 x y^2 z^9 + 8176 x^7 y^4 z + 3420 x y^3 z^8 + 3420 x y^8 z^3 \\
&+ 1220 x^2 y^9 z + 280 x y z^{10} + 280 x^{10} y z + 1220 x^9 y^2 z + 8176 x y^4 z^7 \\
&+ 13448 x y^5 z^6 + 13448 x y^6 z^5 + 8176 x^4 y^7 z + 8176 x y^7 z^4 \\
&+ 13448 x^5 y^6 z - 116 x^{10} y^2 - 116 x^2 z^{10} \geq 0.
\end{aligned}$$

Since (2.4) is symmetric for x, y, z , there is no harm in supposing that $x \leq y \leq z$. Using the transformation (1.2), then $F(x, y, z)$ in (2.4) becomes

$$\begin{aligned}
(2.5) \quad &F(x, y, z) \\
&= P(u, v, w) \\
&= (2u^2 - v^2 - vw)[(180224w^2 + 180224v^2 + 180224vw)u^8 \\
&\quad + (1794048v^2w + 1810432vw^2 + 606208w^3 + 1196032v^3)u^7 \\
&\quad + (4360192vw^3 + 7030784v^3w + 771072w^4 + 7875584v^2w^2 \\
&\quad + 3515392v^4)u^6 + (6049280v^5 + 520704w^5 + 19394048v^3w^2 \\
&\quad + 13967872v^2w^3 + 4689152vw^4 + 15123200v^4w)u^5 \\
&\quad + (2838144vw^5 + 6838400v^6 + 12647648v^2w^4 + 30324704v^4w^2 \\
&\quad + 210048w^6 + 26457408v^3w^3 + 20515200v^5w)u^4 + (19291776v^6w \\
&\quad + 52480w^7 + 1074176vw^6 + 5511936v^7 + 32787968v^5w^2 \\
&\quad + 33740480v^4w^3 + 20662912v^3w^4 + 6899776v^2w^5)u^3 \\
&\quad + (32727200v^5w^3 + 7968w^8 + 2528912w^6v^2 + 24395856v^4w^4 \\
&\quad + 27385760v^6w^2 + 14122880v^7w + 268096w^7v + 3530720v^8 \\
&\quad + 10723072v^3w^5)u^2 + (9558576v^8w + 4185944v^3w^6 + 13383144v^4w^5 \\
&\quad + 676240v^2w^7 + 2124128v^9 + 672w^9 + 24737624v^5w^4 + 45200vw^8
\end{aligned}$$



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$$\begin{aligned}
& + 20832112 v^7 w^2 + 28305704 v^6 w^3)u + 15686836 v^5 w^5 + 6092840 v^4 w^6 \\
& + 1326664 v^3 w^7 + 139150 v^2 w^8 + 24651416 v^6 w^4 + 24921352 v^7 w^3 \\
& + 1378920 v^{10} + 6894600 v^9 w + 16572238 v^8 w^2 + 5112 v w^9 + 24 w^{10}] \\
& + (27659640 v^9 w^2 + 10558592 v^{10} w + 689380 v^3 w^8 + 4642800 v^4 w^7 \\
& + 36001700 v^6 w^5 + 45278940 v^8 w^3 + 16715660 v^5 w^6 + 720 v w^{10} \\
& + 49540936 v^7 w^4 + 1919744 v^{11} + 44048 v^2 w^9)u + 5020 v^2 w^{10} \\
& + 49008067 v^8 w^4 + 142314 v^3 w^9 + 23121662 v^{10} w^2 + 8144784 v^{11} w \\
& + 1451049 v^4 w^8 + 40947790 v^9 w^3 + 24 v w^{11} + 7353016 v^5 w^7 \\
& + 1357464 v^{12} + 39938152 v^7 w^5 + 21582818 v^6 w^6.
\end{aligned}$$

This implies $F(x, y, z) = P(u, v, w) \geq 0$ from (1.8). Hence, inequality (2.1) holds. The proof is completed. \square

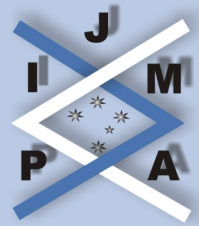
2.3. The Proof of Inequality (2.2)

Proof. Inequality (2.2) is equivalent to

$$(2.6) \quad \sin^4 \frac{A}{2} \leq \frac{m_b^2 m_c^2}{16m_a^4}.$$

By using the formula $\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}$, the law of cosines and the formulas of the medians, we find that (2.6) is simply the following inequality

$$(2.7) \quad -a^8 - 4b^8 + 6a^6c^2 - 34b^2c^6 + 20b^3a^4c + 12a^2c^6 - 32b^5a^2c - 32c^5a^2b - 34b^6c^2 - 51b^4c^4 - 4c^8 - 4a^6bc + 20c^3a^4b - 26a^4b^2c^2 + 54a^2b^4c^2$$



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$$+ 54 a^2 b^2 c^4 - 13 a^4 b^4 - 13 a^4 c^4 + 12 a^2 b^6 + 6 a^6 b^2 + 16 b^7 c + 16 c^7 b \\ - 64 b^3 c^3 a^2 + 48 b^5 c^3 + 48 b^3 c^5 \geq 0.$$

Considering (1.6), inequality (2.7) is transformed into

$$(2.8) \quad F(x, y, z) \\ = x^8 + (4z + 4y)x^7 + (2z^2 + 40yz + 2y^2)x^6 \\ + (-8z^3 + 84yz^2 - 8y^3 + 84y^2z)x^5 \\ + (-20y^2z^2 + 76yz^3 + 76y^3z - 7z^4 - 7y^4)x^4 \\ + (4z^5 + 48y^4z - 248y^3z^2 - 248y^2z^3 + 4y^5 + 48yz^4)x^3 \\ + (4y^6 - 234y^4z^2 - 234y^2z^4 + 28y^5z + 4z^6 + 28yz^5 + 32y^3z^3)x^2 \\ + (-84z^5y^2 + 8z^6y - 84y^5z^2 + 256y^3z^4 + 256y^4z^3 + 8y^6z)x \\ + 84z^5y^3 - 63z^4y^4 - 12y^6z^2 - 12z^6y^2 + 84y^5z^3 \geq 0.$$

It is easy to see that inequality (2.8) is symmetric for y, z . Therefore, we only need to prove that inequality (2.8) holds when $x \leq y \leq z, y \leq x \leq z$ and $y \leq z \leq x$.

Calculating $DS(F)$, it consists of 3 polynomials with $u > 0, v \geq 0, w \geq 0$ as follows

$$(2.9) \quad P_1(u, v, w) \\ = (2u^2 - v^2 - vw)[(192w^2 + 768v^2 + 768vw)u^4 \\ + (256w^3 + 2112vw^2 + 4800v^2w + 3200v^3)u^3 \\ + (5808v^4 + 80w^4 + 7376v^2w^2 + 11616v^3w + 1568vw^3)u^2 \\ + (6336v^5 + 15840v^4w + 13440v^3w^2 + 16w^5 + 4320v^2w^3 + 416vw^4)u \\ + 5112v^6 + 15336v^5w + 48vw^5 + 16560v^4w^2 + 7560v^3w^3$$



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$$\begin{aligned}
& + 1272 v^2 w^4] + (7344 v^7 + 432 w^5 v^2 + 25704 v^6 w + 33912 v^5 w^2 \\
& + 20520 v^4 w^3 + 5400 v^3 w^4)u + 20772 v^7 w + 5193 v^8 + 36 w^6 v^2 \\
& + 1332 w^5 v^3 + 32418 v^6 w^2 + 24552 v^5 w^3 + 9009 v^4 w^4
\end{aligned}$$

for $x \leq y \leq z$,

$$\begin{aligned}
(2.10) \quad & P_2(u, v, w) \\
& = (2u^2 - v^2 - vw)[(-384 vw + 192 v^2 + 192 w^2)u^4 \\
& + (-192 vw^2 + 896 v^3 - 960 v^2 w + 256 w^3)u^3 \\
& + (-976 v^2 w^2 + 1776 v^4 + 80 w^4 + 224 vw^3 - 288 v^3 w)u^2 \\
& + (2032 v^5 - 480 v^3 w^2 + 16 w^5 + 1328 v^4 w + 128 v^2 w^3 + 240 vw^4)u \\
& + 1640 v^6 + 2128 v^5 w + 544 v^4 w^2 + 328 v^3 w^3 + 416 v^2 w^4 + 80 vw^5] \\
& + (2064 v^5 w^2 + 32 w^6 v + 4176 v^6 w + 776 v^4 w^3 + 2320 v^7 \\
& + 416 v^2 w^5 + 968 v^3 w^4)u + 1640 v^8 + 2708 w^2 v^6 + 817 w^4 v^4 \\
& + 524 w^5 v^3 + 956 w^3 v^5 + 84 w^6 v^2 + 3768 wv^7
\end{aligned}$$

for $y \leq x \leq z$, and

$$\begin{aligned}
(2.11) \quad & P_3(u, v, w) \\
& = 384 u^6 v^2 + 11072 w^2 u^2 v^4 + 20992 w^2 u^3 v^3 + 19552 w^2 u^4 v^2 \\
& + 8832 w^2 u^5 v + 2008 w^4 uv^3 + 5296 w^4 u^2 v^2 + 5376 w^4 u^3 v \\
& + 36 w^2 v^6 + 1536 w^2 u^6 + 2792 w^3 uv^4 + 10400 w^3 u^2 v^3 \\
& + 15744 w^3 u^3 v^2 + 10816 w^3 u^4 v + 2368 w^2 uv^5 + 840 w^5 uv^2 \\
& + 1344 w^5 u^2 v + 2816 w^3 u^5 + 132 w^3 v^5 + 1888 w^4 u^4 + 193 w^4 v^4
\end{aligned}$$



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$$\begin{aligned}
&+ 184 w^6 uv + 144 w^5 v^3 + 640 w^5 u^3 + 1200 wuv^6 + 13120 wu^3 v^4 \\
&+ 6256 wu^2 v^5 + 13824 wu^4 v^3 + 58 w^6 v^2 + 128 w^6 u^2 + 7296 wu^5 v^2 \\
&+ 3360 u^4 v^4 + 1792 u^5 v^3 + 288 uv^7 + 1504 u^2 v^6 + 3168 u^3 v^5 \\
&+ 1536 wu^6 v + 12 w^7 v + w^8 + 16 w^7 u
\end{aligned}$$

for $y \leq z \leq x$.

It is not difficult to see that $P_1(u, v, w) \geq 0$ and $P_3(u, v, w) \geq 0$ because $u > 0, v \geq 0, w \geq 0$ and $2u^2 - v^2 - vw > 0$.

In order to prove $P_2(u, v, w) \geq 0$, we only need prove the following inequality

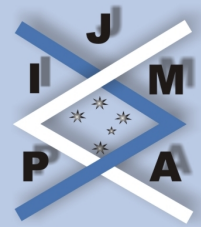
$$\begin{aligned}
(2.12) \quad p(u, v, w) &= (-384 vw + 192 v^2 + 192 w^2)u^4 \\
&+ (-192 vw^2 + 896 v^3 - 960 v^2 w + 256 w^3)u^3 \\
&+ (-976 v^2 w^2 + 1776 v^4 + 80 w^4 + 224 vw^3 - 288 v^3 w)u^2 \\
&+ (2032 v^5 - 480 v^3 w^2 + 16 w^5 + 1328 v^4 w + 128 v^2 w^3 + 240 vw^4)u \\
&+ 1640 v^6 + 2128 v^5 w + 544 v^4 w^2 + 328 v^3 w^3 + 416 v^2 w^4 + 80 vw^5 \\
&\geq 0,
\end{aligned}$$

where $u > 0, v \geq 0$ and $w \geq 0$.

(i) For $u > 0, v \geq w \geq 0$, taking $v = w + t$ with $t \geq 0$, then we have

$$\begin{aligned}
p(u, v, w) &= 192 t^2 u^4 + (576 t w^2 + 1728 t^2 w + 896 t^3)u^3 + (816 w^4 + 4512 w^3 t \\
&+ 8816 w^2 t^2 + 6816 w t^3 + 1776 t^4)u^2 + (2032 t^5 + 11488 w t^4 \\
&+ 3264 w^5 + 14528 w^4 t + 26976 w^3 t^2 + 25152 w^2 t^3)u + 50544 w^4 t^2 \\
&+ 56584 w^3 t^3 + 5136 w^6 + 24552 w^5 t + 1640 t^6 + 35784 w^2 t^4 + 11968 w t^5.
\end{aligned}$$

It obviously follows that $p(u, v, w) \geq 0$, i.e., inequality (2.12) holds.



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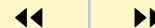
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(ii) When $u > 0, w \geq v \geq 0$, setting $w = v + t$ for $t \geq 0$, we get

$$\begin{aligned} p(u, v, w) &= (2u + 10v)t^5 + (10u^2 + 40uv + 102v^2)t^4 \\ &\quad + (156uv^2 + 32u^3 + 349v^3 + 68u^2v)t^3 \\ &\quad + (24u^4 + 188uv^3 + 22u^2v^2 + 72u^3v + 603v^4)t^2 \\ &\quad + v^2(783v^3 - 72u^3 + 224uv^2 - 156u^2v)t \\ &\quad + 6v^4(17u^2 + 68uv + 107v^2) \\ &= p_1(u, v, t) + p_2(u, v, t), \end{aligned}$$

where

$$\begin{aligned} p_1(u, v, t) &= (2u + 10v)t^5 + (10u^2 + 40uv + 102v^2)t^4 \\ &\quad + (156uv^2 + 32u^3 + 349v^3 + 68u^2v)t^3 \geq 0, \end{aligned}$$

and

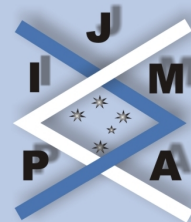
$$\begin{aligned} (2.13) \quad p_2(u, v, t) &= (24u^4 + 188uv^3 + 22u^2v^2 + 72u^3v + 603v^4)t^2 \\ &\quad + v^2(783v^3 - 72u^3 + 224uv^2 - 156u^2v)t \\ &\quad + 6v^4(17u^2 + 68uv + 107v^2). \end{aligned}$$

It is easy to see that $24u^4 + 188uv^3 + 22u^2v^2 + 72u^3v + 603v^4 > 0$, and the discriminant of the quadratic function (2.13) with respect to t is

$$\begin{aligned} (2.14) \quad \Delta(u, v) &= -v^4(935415v^6 + 480144u^3v^3 + 1116096uv^5 \\ &\quad + 803456u^2v^4 + 4608u^6 + 196032u^4v^2 + 46080u^5v) \leq 0. \end{aligned}$$

This is to say that $p_2(u, v, t) \geq 0$.

Hence, $P_2(u, v, w) \geq 0$. From the proof above, the required result (2.6) is proved. \square



2.4. Remarks

Remark 1. By the same argument as above, we also prove the following inequalities conjectures [9, 10, 11] in the acute triangle

$$(2.15) \quad \sum m_a^2 h_a^2 \geq \sum m_a^2 r_a^2,$$

$$(2.16) \quad \sum \sin^8 A \geq \sum \cos^8 \frac{A}{2},$$

$$(2.17) \quad \sum (b - c)^2 \geq \sum \left(\frac{a}{b + c} \right)^2 (r_b - r_c)^2,$$

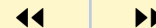
and

$$(2.18) \quad \sum (h_b + h_c - h_a)^3 \geq 3m_a m_b m_c.$$

Remark 2. The operations in this paper were performed using mathematical software Maple 9.0.

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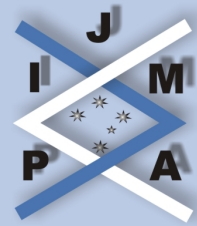
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3. Generalization of the Method

In fact, Difference Substitution can go even further. Now, we consider the following inequality [12]. In $\triangle ABC$, if $\max(A, B, C) \leq \frac{2\pi}{3}$, then

$$(3.1) \quad s^2 \geq R^2 + 10Rr + 3r^2.$$

Utilizing the known formulas $R = \frac{abc}{4S}$, $r = \frac{S}{s}$ and $S = \sqrt{s(s-a)(s-b)(s-c)}$, from (1.6), inequality (3.1) is equivalent to

$$(3.2) \quad 3c^2a^2b^2 - a^2bc^3 - a^3bc^2 - a^3b^2c - a^2b^3c - ab^2c^3 - ab^3c^2 - 2b^3c^3 \\ - 2a^3c^3 + c^4b^2 - 2a^3b^3 + c^5b + a^4c^2 + b^4c^2 + b^5a + a^5c + c^5a \\ - a^6 + a^4b^2 + a^5b - b^6 - c^6 + a^2b^4 + b^5c + a^2c^4 \geq 0,$$

or

$$(3.3) \quad F(x, y, z) \\ = -42x^2y^2z^2 + 14y^4zx + 14xyz^4 + 2xy^2z^3 + 2x^2y^3z + 2xy^3z^2 \\ + 14x^4yz + 2x^3yz^2 + 2x^2yz^3 + 2x^3y^2z - x^4y^2 - x^2z^4 \\ - 2x^3z^3 - x^4z^2 - 2x^3y^3 - y^4z^2 - y^4x^2 - 2y^3z^3 - y^2z^4 \geq 0,$$

where $x > 0, y > 0, z > 0$.

Since inequality (3.3) is symmetric with x, y, z , there is no harm in supposing that $x \leq y \leq z$. From (1.2), $F(x, y, z)$ in (3.3) is transformed into

$$(3.4) \quad F(x, y, z) = P(u, v, w) \\ = (8u^2 + 4uv + 2uw - v^2 - vw)(8uvw^2 + 12uv^2w + 4u^2vw \\ + 4v^4 + 4u^2v^2 + 2uw^3 + 8uv^3 + 8v^3w + 7v^2w^2 + 3vw^3) \\ + 2w^2(v + 2u)^2(v + 2u + w)^2 \geq 0,$$

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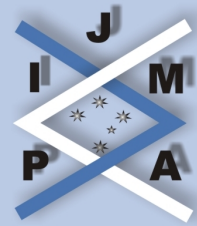
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and for $\max(A, B, C) \leq \frac{2\pi}{3}$ and $y = \cos x$ decreasing in $x \in (0, \pi)$, we have

$$\begin{aligned}(3.5) \quad b^2 + c^2 + bc - a^2 &= b^2 + c^2 - \frac{1}{2}bc \cos \frac{2\pi}{3} - a^2 \\ &= 3x^2 + 3(y+z)x - yz \\ &= 8u^2 + 4uv + 2uw - v^2 - vw \\ &\geq b^2 + c^2 - \frac{1}{2}bc \cos A - a^2 = 0.\end{aligned}$$

Since $F(x, y, z) = P(u, v, w) \geq 0$ for $u > 0, v \geq 0$ and $w \geq 0$, inequality (3.1) is obtained.

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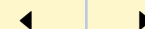
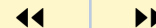
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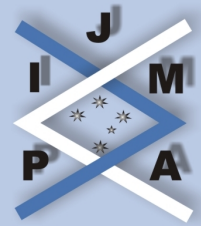
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