PROVING INEQUALITIES IN ACUTE TRIANGLE WITH DIFFERENCE SUBSTITUTION

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Abstract: In this paper, we prove several inequalities in the acute triangle by means of so-called Difference Substitution. As generalization of the method, we also consider an example that the greatest interior angle is less than or equal to 120° in the triangle.

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1. Introduction

In [1, 2], L. Yang suggested the use of Difference Substitution to prove asymmetric polynomial inequalities, as it had been used previously to deal with symmetric ones.

If $x_1 \le x_2 \le x_3 \le \cdots \le x_n$ with $n \in N^*$, then we set

$$(1.1)\qquad \begin{cases} x_1 = t_1, \\ x_2 = t_1 + t_2, \\ x_3 = t_1 + t_2 + t_3, \\ \dots\dots\dots\dots\dots\dots \\ x_n = t_1 + t_2 + t_3 + \cdots + t_n, \end{cases}$$

where $t_i \ge 0$ for $2 \le i \le n$ and $i \in N^*$.

The expansion (1.1) is so-called a "splitting" transformation, and $\{t_1, t_2, \dots, t_n\}$ is simply the difference sequence of $\{x_1, x_2, \dots, x_n\}$.

In general, for the n-variant polynomials, there are $n!$ different orders of $\{x_1, x_2, \dots, x_n\}$, sorting by size. In the instance of $n = 3$, we let $x \le y \le z$, and take

$$(1.2)\qquad \begin{cases} x = u, \\ y = u + v, \\ z = u + v + w, \end{cases}$$

where $v \ge 0$, $w \ge 0$.

Analogously, if $y \le x \le z$, then its "splitting" transformation is

$$(1.3)\qquad \begin{cases} y = u, \\ x = u + v, \\ z = u + v + w, \end{cases}$$

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where $v \geq 0$, $w \geq 0$.

Sequentially, for $y \leq z \leq x$ or $z \leq x \leq y$ or $z \leq y \leq x$ or $x \leq z \leq y$, we set four similar linear transformations.

For a 3-variant polynomial $F(x, y, z)$, by using the six linear transformations above, we obtain 6 members $P_i(u, v, w)$ with $1 \leq i \leq 6$, and call the set $\{P_1, P_2, \dots, P_6\}$ the Difference Substitution of $F(x, y, z)$ and denote this by $DS(F)$. If all the coefficients of these members $DS(F)$ are nonnegative, then $F \geq 0$ whenever x, y, z all are nonnegative. In other words, F is positive semi-definite on $\mathbb{R}^3_+$. Difference substitution is a very valid method for proving inequalities. For more information on Difference Substitution, please refer to [3] and [4].

In this paper, by using Difference Substitution, the authors prove several inequalities in acute triangles.

Throughout the paper we denote A, B, C as the interior angles, a, b, c as the side-lengths, S as the area, s as the semi-perimeter, R as the circumradius, r as the in-radius, h_a, h_b, h_c as the altitudes, m_a, m_b, m_c as the medians, and r_a, r_b, r_c as the radii of the described circles of triangle ABC respectively. Moreover, we will customarily use the cyclic sum symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, and $\sum f(a, b) = f(a, b) + f(b, c) + f(c, a)$, etc.

Let us begin with the well-known Walker's inequality [5]. In the acute triangle, show that

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{1.4}$$

or

$$\begin{aligned} -2\,a^3b^3 + a^4b^2 - a^4bc + a^5b + ab^5 + b^5c + b^4c^2 \\ -2\,b^3c^3 + b^2c^4 - 2\,c^3a^3 + c^4a^2 + c^5a + c^5b + c^2a^4 \\ + a^5c - ab^4c + a^2b^4 - b^6 - c^6 - a^6 - abc^4 \geq 0. \end{aligned} \tag{1.5}$$

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Let

$$(1.6)\qquad \begin{cases} x = \frac{b+c-a}{2} > 0, \\ y = \frac{c+a-b}{2} > 0, \\ z = \frac{a+b-c}{2} > 0. \end{cases}$$

Then inequality (1.4) or (1.5) is equivalent to

$$(1.7)\quad F(x,y,z) = 6\,xyz^4 + 2\,xy^2z^3 + 2\,xy^3z^2 + 6\,xy^4z + 2\,x^2yz^3 + 2\,x^2y^3z + 2\,x^3yz^2 + 2\,x^3y^2z + 6\,x^4yz - x^4y^2 - x^2z^4 - 2\,x^3z^3 - x^4z^2 - 2\,x^3y^3 - y^4z^2 - y^4x^2 - 2\,y^3z^3 - y^2z^4 - 18\,x^2y^2z^2 \geq 0.$$

There is no harm in supposing $x \leq y \leq z$ since inequality (1.7) is symmetric for x, y, z. Then, by using (1.2), for the acute triangle, it follows that

$$(1.8)\quad \begin{aligned} b^2 + c^2 - a^2 &= (z+x)^2 + (x+y)^2 - (y+z)^2 = 2[x^2 + (y+z)x - yz] \\ &= 2\{u^2 + [(u+v) + (u+v+w)]u - (u+v)(u+v+w)\} \\ &= 2(2u^2 - v^2 - vw) > 0, \end{aligned}$$

and $F(x,y,z)$ in (1.7) is transformed into

$$(1.9)\quad \begin{aligned} &F(x,y,z) \\ &= P(u,v,w) \\ &= \left(2\,u^2 - v^2 - vw\right)\Big[\left(4\,v^2 + 4\,w^2 + 4\,vw\right)u^2 \\ &\quad + \left(8\,v^3 + 20\,vw^2 + 12\,v^2w + 8\,w^3\right)u + 4\,v^4 + 8\,v^3w + 2\,w^4 \\ &\quad + 18\,vw^3 + 22\,v^2w^2\Big] + \left(24\,v^3w^2 + 36\,v^2w^3 + 12\,vw^4\right)u \\ &\quad + 34\,v^3w^3 + 19\,v^2w^4 + 2\,vw^5 + 17\,v^4w^2. \end{aligned}$$

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Obviously $F(x,y,z) = P(u,v,w) \geq 0$ from (1.8) and $u > 0, v \geq 0, w \geq 0$, i.e., inequality (1.4) or (1.5) is true.

Now, let us consider another semi-symmetric inequality [6] in the acute triangle

$$\cos(B-C) \leq \frac{h_a}{m_a}. \tag{1.10}$$

It is equivalent to

$$-a^4 + \left(3\,b^2 + 3\,c^2\right)a^2 - 2\,(b-c)^2\,(b+c)^2 \geq 0, \tag{1.11}$$

and from (1.6), this equals

$$\begin{aligned} &F(x,y,z) \\ &\quad = \left(-y^2 - z^2 + 14\,yz\right)x^2 - (y+z)\left(z^2 - 14\,yz + y^2\right)x + yz\,(y+z)^2 \geq 0. \end{aligned} \tag{1.12}$$

Calculating $DS(F)$, it consists of 3 polynomials with $u > 0, v \geq 0, w \geq 0$ as follows

$$\begin{aligned} &P_1(u,v,w) \\ &\quad = 40\,u^4 + 112\,u^3v + 108\,u^2v^2 + 56\,u^3w + 14\,u^2w^2 + 40\,uv^3 + 20\,uvw^2 \\ &\qquad + 60\,uv^2w + 108\,u^2vw + 8\,v^3w + 5\,v^2w^2 + vw^3 + 4\,v^4, \end{aligned} \tag{1.13}$$

$$\begin{aligned} &P_2(u,v,w) \\ &\quad = \left(2\,u^2 - v^2 - vw\right)\left(20\,u^2 + (24\,w + 52\,v)\,u + 53\,v^2 + 6\,w^2 + 52\,vw\right) \\ &\qquad + \left(72\,v^3 + 36\,vw^2 + 108\,v^2w\right)u + 57\,v^2w^2 + 51\,v^4 + 6\,vw^3 + 102\,v^3w, \end{aligned} \tag{1.14}$$

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and

$$P_3(u,v,w) \tag{1.15}$$
$$= \left(2\,u^2 - v^2 - vw\right)\left(20\,u^2 + (52\,v + 28\,w)\,u + 53\,v^2 + 54\,vw + 7\,w^2\right)$$
$$+ \left(72\,v^3 + 36\,vw^2 + 108\,v^2w\right)u + 57\,v^2w^2 + 51\,v^4 + 6\,vw^3 + 102\,v^3w.$$

By (1.8), we immediately obtain $P_i(u,v,w) \geq 0$ for $1 \leq i \leq 3$. Hence, inequality (1.10) is proved.

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2. Some Problems and their Proofs

2.1. The Problems

In 2004-2005, J. Liu [7, 8] posed the following conjectures for the inequality in the acute triangle.

Problem 1. Let $\triangle ABC$ be an acute triangle. Prove the following inequalities

$$\sum \left(\frac{\sin 2A}{\sin B + \sin C}\right)^2 \le \frac{3}{4}, \tag{2.1}$$

and

$$\sin \frac{A}{2} \le \frac{\sqrt{m_b m_c}}{2m_a}. \tag{2.2}$$

2.2. The Proof of Inequality (2.1)

Proof. Using $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we find that inequality (2.1) is equivalent to

$$\begin{aligned}
(2.3)\quad & 4\,a^{10}b^2 - 10\,a^5b^5c^2 - 24\,b^6c^5a + 16\,a^9b^3 + 4\,a^8b^4 - 5\,b^4c^6a^2 \\
& - 8\,a^{11}b - 5\,a^4b^6c^2 + 8\,a^9c^2b + 4\,a^8b^2c^2 - 16\,a^{10}cb + 8\,a^9cb^2 - 5\,a^6b^4c^2 \\
& + 32\,a^8b^3c + 8\,a^7b^4c + 8\,a^7c^4b - 5\,a^6c^4b^2 + 32\,a^8c^3b + 4\,a^{10}c^2 - 8\,a^{11}c \\
& - 4\,a^{12} - 4\,b^{12} - 4\,c^{12} + 4\,a^8c^4 + 16\,a^9c^3 - 8\,a^7b^5 - 8\,a^6b^6 - 8\,a^6c^6 \\
& - 8\,a^7c^5 - 8\,a^5b^7 + 4\,a^4b^8 - 24\,a^6b^5c - 24\,a^5b^6c + 2\,a^5b^3c^4 + 6\,a^4b^4c^4 \\
& + 4\,c^{10}b^2 - 26\,a^6b^3c^3 + 2\,a^5b^4c^3 - 24\,a^5c^6b - 5\,a^4c^6b^2 - 24\,a^6c^5b \\
& - 10\,a^5c^5b^2 + 8\,a^4b^7c - 8\,c^{11}b + 32\,b^8a^3c + 8\,b^9a^2c + 4\,b^8a^2c^2 \\
& + 2\,b^5c^3a^4 - 26\,b^6c^3a^3 + 2\,b^5c^4a^3 - 5\,b^6c^4a^2 - 16\,b^{10}ca + 8\,b^9c^2a
\end{aligned}$$

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$$+ 32\, b^8c^3a + 8\, b^7c^4a - 10\, b^5c^5a^2 + 16\, b^9a^3 + 4\, b^{10}a^2 - 8\, b^{11}a$$
$$- 8\, b^{11}c + 4\, b^{10}c^2 + 16\, b^9c^3 + 4\, b^8c^4 - 8\, b^7c^5 - 8\, b^6c^6 + 4\, a^4c^8$$
$$- 8\, a^5c^7 - 24\, b^5c^6a + 8\, a^4c^7b + 2\, c^5b^4a^3 - 26\, c^6b^3a^3 + 2\, c^5b^3a^4$$
$$+ 4\, c^8b^2a^2 + 8\, c^9a^2b + 32\, c^8a^3b + 8\, b^4c^7a - 8\, c^{11}a + 32\, c^8b^3a + 8\, c^9b^2a$$
$$- 16\, c^{10}ab + 4\, c^{10}a^2 + 16\, c^9a^3 - 8\, b^5c^7 + 4\, b^4c^8 + 16\, c^9b^3 \geq 0.$$

From (1.6), inequality (2.3) equals

$$\begin{aligned}(2.4)\quad F(x,y,z) &= - 4576\, x^7z^5 - 5590\, x^6z^6 - 116\, x^{10}z^2 - 2453\, x^4z^8 - 2453\, x^8z^4\\
&- 4576\, x^5z^7 - 788\, x^3z^9 - 788\, x^9z^3 - 2453\, x^8y^4 - 4576\, y^7z^5\\
&- 2453\, y^8z^4 - 2453\, x^4y^8 - 788\, x^9y^3 - 788\, x^3y^9 - 5590\, x^6y^6\\
&- 4576\, x^7y^5 - 4576\, x^5y^7 - 2453\, y^4z^8 - 4576\, y^5z^7 - 5590\, y^6z^6\\
&- 116\, y^{10}z^2 - 788\, y^9z^3 - 116\, y^{10}x^2 - 788\, y^3z^9 - 116\, y^2z^{10}\\
&+ 13448\, x^6y^5z + 8176\, x^7yz^4 + 13448\, x^6yz^5 + 13448\, x^5yz^6\\
&+ 8176\, x^4yz^7 + 6448\, x^2y^3z^7 + 1220\, xy^9z^2 + 6448\, x^2y^7z^3\\
&+ 6448\, x^3y^7z^2 + 1220\, x^9yz^2 + 10862\, x^4y^6z^2 + 14288\, x^2y^5z^5\\
&+ 10862\, x^2y^4z^6 + 14288\, x^5y^2z^5 + 10862\, x^6y^2z^4 + 6448\, x^7y^2z^3\\
&- 28248\, x^3y^5z^4 - 28248\, x^4y^5z^3 + 14288\, x^5y^5z^2 - 57474\, x^4y^4z^4\\
&- 28248\, x^3y^4z^5 - 8672\, x^3y^3z^6 + 6448\, x^3y^2z^7 + 10862\, x^2y^6z^4\\
&- 8672\, x^3y^6z^3 - 28248\, x^5y^4z^3 + 10862\, x^6y^4z^2 - 28248\, x^4y^3z^5\\
&- 28248\, x^5y^3z^4 - 8672\, x^6y^3z^3 + 6448\, x^7y^3z^2 + 10862\, x^4y^2z^6\\
&+ 3420\, x^8yz^3 + 3420\, x^8y^3z + 1220\, x^2yz^9 + 280\, xy^{10}z + 4066\, x^2y^2z^8\end{aligned}$$

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$$+\,3420\,x^3yz^8+3420\,x^3y^8z+4066\,x^8y^2z^2+4066\,x^2y^8z^2$$
$$+\,1220\,xy^2z^9+8176\,x^7y^4z+3420\,xy^3z^8+3420\,xy^8z^3$$
$$+\,1220\,x^2y^9z+280\,xyz^{10}+280\,x^{10}yz+1220\,x^9y^2z+8176\,xy^4z^7$$
$$+\,13448\,xy^5z^6+13448\,xy^6z^5+8176\,x^4y^7z+8176\,xy^7z^4$$
$$+\,13448\,x^5y^6z-116\,x^{10}y^2-116\,x^2z^{10}\geq 0.$$

Since (2.4) is symmetric for x, y, z, there is no harm in supposing that $x \leq y \leq z$. Using the transformation (1.2), then $F(x, y, z)$ in (2.4) becomes

$$
\begin{aligned}
(2.5)\quad & F(x,y,z)\\
&=P(u,v,w)\\
&=(2u^2-v^2-vw)[(180224\,w^2+180224\,v^2+180224\,vw)u^8\\
&\quad+(1794048\,v^2w+1810432\,vw^2+606208\,w^3+1196032\,v^3)u^7\\
&\quad+(4360192\,vw^3+7030784\,v^3w+771072\,w^4+7875584\,v^2w^2\\
&\quad+3515392\,v^4)u^6+(6049280\,v^5+520704\,w^5+19394048\,v^3w^2\\
&\quad+13967872\,v^2w^3+4689152\,vw^4+15123200\,v^4w)u^5\\
&\quad+(2838144\,vw^5+6838400\,v^6+12647648\,v^2w^4+30324704\,v^4w^2\\
&\quad+210048\,w^6+26457408\,v^3w^3+20515200\,v^5w)u^4+(19291776\,v^6w\\
&\quad+52480\,w^7+1074176\,vw^6+5511936\,v^7+32787968\,v^5w^2\\
&\quad+33740480\,v^4w^3+20662912\,v^3w^4+6899776\,v^2w^5)u^3\\
&\quad+(32727200\,v^5w^3+7968\,w^8+2528912\,w^6v^2+24395856\,v^4w^4\\
&\quad+27385760\,v^6w^2+14122880\,v^7w+268096\,w^7v+3530720\,v^8\\
&\quad+10723072\,v^3w^5)u^2+(9558576\,v^8w+4185944\,v^3w^6+13383144\,v^4w^5\\
&\quad+676240\,v^2w^7+2124128\,v^9+672\,w^9+24737624\,v^5w^4+45200\,vw^8
\end{aligned}
$$

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$$+ 20832112\, v^7w^2 + 28305704\, v^6w^3)u + 15686836\, v^5w^5 + 6092840\, v^4w^6$$
$$+ 1326664\, v^3w^7 + 139150\, v^2w^8 + 24651416\, v^6w^4 + 24921352\, v^7w^3$$
$$+ 1378920\, v^{10} + 6894600\, v^9w + 16572238\, v^8w^2 + 5112\, vw^9 + 24\, w^{10}]$$
$$+ (27659640\, v^9w^2 + 10558592\, v^{10}w + 689380\, v^3w^8 + 4642800\, v^4w^7$$
$$+ 36001700\, v^6w^5 + 45278940\, v^8w^3 + 16715660\, v^5w^6 + 720\, vw^{10}$$
$$+ 49540936\, v^7w^4 + 1919744\, v^{11} + 44048\, v^2w^9)u + 5020\, v^2w^{10}$$
$$+ 49008067\, v^8w^4 + 142314\, v^3w^9 + 23121662\, v^{10}w^2 + 8144784\, v^{11}w$$
$$+ 1451049\, v^4w^8 + 40947790\, v^9w^3 + 24\, vw^{11} + 7353016\, v^5w^7$$
$$+ 1357464\, v^{12} + 39938152\, v^7w^5 + 21582818\, v^6w^6.$$

This implies $F(x,y,z) = P(u,v,w) \geq 0$ from (1.8). Hence, inequality (2.1) holds. The proof is completed. □

2.3. The Proof of Inequality (2.2)

Proof. Inequality (2.2) is equivalent to

$$\sin^4 \frac{A}{2} \leq \frac{m_b^2 m_c^2}{16 m_a^4}. \tag{2.6}$$

By using the formula $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, the law of cosines and the formulas of the medians, we find that (2.6) is simply the following inequality

$$(2.7)\quad - a^8 - 4\, b^8 + 6\, a^6c^2 - 34\, b^2c^6 + 20\, b^3a^4c + 12\, a^2c^6 - 32\, b^5a^2c - 32\, c^5a^2b$$
$$- 34\, b^6c^2 - 51\, b^4c^4 - 4\, c^8 - 4\, a^6bc + 20\, c^3a^4b - 26\, a^4b^2c^2 + 54\, a^2b^4c^2$$

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$$+54\,a^2b^2c^4-13\,a^4b^4-13\,a^4c^4+12\,a^2b^6+6\,a^6b^2+16\,b^7c+16\,c^7b \\ -64\,b^3c^3a^2+48\,b^5c^3+48\,b^3c^5\geq 0.$$

Considering (1.6), inequality (2.7) is transformed into

$$\begin{aligned}(2.8)\quad & F(x,y,z)\\ &=x^8+(4\,z+4\,y)x^7+(2\,z^2+40\,yz+2\,y^2)x^6\\ &\quad+(-8\,z^3+84\,yz^2-8\,y^3+84\,y^2z)x^5\\ &\quad+(-20\,y^2z^2+76\,yz^3+76\,y^3z-7\,z^4-7\,y^4)x^4\\ &\quad+(4\,z^5+48\,y^4z-248\,y^3z^2-248\,y^2z^3+4\,y^5+48\,yz^4)x^3\\ &\quad+(4\,y^6-234\,y^4z^2-234\,y^2z^4+28\,y^5z+4\,z^6+28\,yz^5+32\,y^3z^3)x^2\\ &\quad+(-84\,z^5y^2+8\,z^6y-84\,y^5z^2+256\,y^3z^4+256\,y^4z^3+8\,y^6z)x\\ &\quad+84\,z^5y^3-63\,z^4y^4-12\,y^6z^2-12\,z^6y^2+84\,y^5z^3\geq 0.\end{aligned}$$

It it easy to see that inequality (2.8) is symmetric for y,z. Therefore, we only need to prove that inequality (2.8) holds when $x\leq y\leq z$, $y\leq x\leq z$ and $y\leq z\leq x$.

Calculating $DS(F)$, it consists of 3 polynomials with $u>0, v\geq 0, w\geq 0$ as follows

$$\begin{aligned}(2.9)\quad & P_1(u,v,w)\\ &=(2u^2-v^2-vw)[(192\,w^2+768\,v^2+768\,vw)u^4\\ &\quad+(256\,w^3+2112\,vw^2+4800\,v^2w+3200\,v^3)u^3\\ &\quad+(5808\,v^4+80\,w^4+7376\,v^2w^2+11616\,v^3w+1568\,vw^3)u^2\\ &\quad+(6336\,v^5+15840\,v^4w+13440\,v^3w^2+16\,w^5+4320\,v^2w^3+416\,vw^4)u\\ &\quad+5112\,v^6+15336\,v^5w+48\,vw^5+16560\,v^4w^2+7560\,v^3w^3\end{aligned}$$

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$$+ 1272\, v^2w^4] + (7344\, v^7 + 432\, w^5v^2 + 25704\, v^6w + 33912\, v^5w^2$$
$$+ 20520\, v^4w^3 + 5400\, v^3w^4)u + 20772\, v^7w + 5193\, v^8 + 36\, w^6v^2$$
$$+ 1332\, w^5v^3 + 32418\, v^6w^2 + 24552\, v^5w^3 + 9009\, v^4w^4$$

for $x \le y \le z$,

(2.10)
$$P_2(u, v, w)$$
$$= (2u^2 - v^2 - vw)[(-384\, vw + 192\, v^2 + 192\, w^2)u^4$$
$$+ (-192\, vw^2 + 896\, v^3 - 960\, v^2w + 256\, w^3)u^3$$
$$+ (-976\, v^2w^2 + 1776\, v^4 + 80\, w^4 + 224\, vw^3 - 288\, v^3w)u^2$$
$$+ (2032\, v^5 - 480\, v^3w^2 + 16\, w^5 + 1328\, v^4w + 128\, v^2w^3 + 240\, vw^4)u$$
$$+ 1640\, v^6 + 2128\, v^5w + 544\, v^4w^2 + 328\, v^3w^3 + 416\, v^2w^4 + 80\, vw^5]$$
$$+ (2064\, v^5w^2 + 32\, w^6v + 4176\, v^6w + 776\, v^4w^3 + 2320\, v^7$$
$$+ 416\, v^2w^5 + 968\, v^3w^4)u + 1640\, v^8 + 2708\, w^2v^6 + 817\, w^4v^4$$
$$+ 524\, w^5v^3 + 956\, w^3v^5 + 84\, w^6v^2 + 3768\, wv^7$$

for $y \le x \le z$, and

(2.11)
$$P_3(u,v,w)$$
$$= 384\, u^6v^2 + 11072\, w^2u^2v^4 + 20992\, w^2u^3v^3 + 19552\, w^2u^4v^2$$
$$+ 8832\, w^2u^5v + 2008\, w^4uv^3 + 5296\, w^4u^2v^2 + 5376\, w^4u^3v$$
$$+ 36\, w^2v^6 + 1536\, w^2u^6 + 2792\, w^3uv^4 + 10400\, w^3u^2v^3$$
$$+ 15744\, w^3u^3v^2 + 10816\, w^3u^4v + 2368\, w^2uv^5 + 840\, w^5uv^2$$
$$+ 1344\, w^5u^2v + 2816\, w^3u^5 + 132\, w^3v^5 + 1888\, w^4u^4 + 193\, w^4v^4$$

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$$+184\,w^6uv+144\,w^5v^3+640\,w^5u^3+1200\,wuv^6+13120\,wu^3v^4$$
$$+6256\,wu^2v^5+13824\,wu^4v^3+58\,w^6v^2+128\,w^6u^2+7296\,wu^5v^2$$
$$+3360\,u^4v^4+1792\,u^5v^3+288\,uv^7+1504\,u^2v^6+3168\,u^3v^5$$
$$+1536\,wu^6v+12\,w^7v+w^8+16\,w^7u$$

for $y \le z \le x$.

It is not difficult to see that $P_1(u,v,w) \ge 0$ and $P_3(u,v,w) \ge 0$ because $u > 0, v \ge 0, w \ge 0$ and $2u^2 - v^2 - vw > 0$.

In order to prove $P_2(u,v,w) \ge 0$, we only need prove the following inequality

$$\begin{aligned}(2.12)\quad p(u,v,w) &= (-384\,vw+192\,v^2+192\,w^2)u^4\\ &\quad+(-192\,vw^2+896\,v^3-960\,v^2w+256\,w^3)u^3\\ &\quad+(-976\,v^2w^2+1776\,v^4+80\,w^4+224\,vw^3-288\,v^3w)u^2\\ &\quad+(2032\,v^5-480\,v^3w^2+16\,w^5+1328\,v^4w+128\,v^2w^3+240\,vw^4)u\\ &\quad+1640\,v^6+2128\,v^5w+544\,v^4w^2+328\,v^3w^3+416\,v^2w^4+80\,vw^5\\ &\ge 0,\end{aligned}$$

where $u > 0$, $v \ge 0$ and $w \ge 0$.

(i) For $u > 0, v \ge w \ge 0$, taking $v = w + t$ with $t \ge 0$, then we have

$$\begin{aligned}p(u,v,w) ={}&192\,t^2u^4+(576\,tw^2+1728\,t^2w+896\,t^3)u^3+(816\,w^4+4512\,w^3t\\ &+8816\,w^2t^2+6816\,wt^3+1776\,t^4)u^2+(2032\,t^5+11488\,wt^4\\ &+3264\,w^5+14528\,w^4t+26976\,w^3t^2+25152\,w^2t^3)u+50544\,w^4t^2\\ &+56584\,w^3t^3+5136\,w^6+24552\,w^5t+1640\,t^6+35784\,w^2t^4+11968\,wt^5.\end{aligned}$$

It obviously follows that $p(u,v,w) \ge 0$, i.e., inequality (2.12) holds.

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(*ii*) When $u > 0, w \geq v \geq 0$, setting $w = v + t$ for $t \geq 0$, we get

$$\begin{aligned} p(u,v,w) =&(2\,u + 10\,v)t^5 + (10\,u^2 + 40\,uv + 102\,v^2)t^4 \\ &+ (156\,uv^2 + 32\,u^3 + 349\,v^3 + 68\,u^2v)t^3 \\ &+ (24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4)t^2 \\ &+ v^2(783\,v^3 - 72\,u^3 + 224\,uv^2 - 156\,u^2v)t \\ &+ 6\,v^4(17\,u^2 + 68\,uv + 107\,v^2) \\ =&p_1(u,v,t) + p_2(u,v,t), \end{aligned}$$

where

$$\begin{aligned} p_1(u,v,t) = (2\,u + 10\,v)t^5 + (10\,u^2 + 40\,uv + 102\,v^2)t^4 \\ + (156\,uv^2 + 32\,u^3 + 349\,v^3 + 68\,u^2v)t^3 \geq 0, \end{aligned}$$

and

$$\begin{aligned} p_2(u,v,t) = (24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4)t^2 \\ + v^2(783\,v^3 - 72\,u^3 + 224\,uv^2 - 156\,u^2v)t \\ + 6\,v^4(17\,u^2 + 68\,uv + 107\,v^2). \end{aligned} \tag{2.13}$$

It is easy to see that $24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4 > 0$, and the discriminant of the quadratic function (2.13) with respect to t is

$$\begin{aligned} \Delta(u,v) = -v^4(935415\,v^6 + 480144\,u^3v^3 + 1116096\,uv^5 \\ + 803456\,u^2v^4 + 4608\,u^6 + 196032\,u^4v^2 + 46080\,u^5v) \leq 0. \end{aligned} \tag{2.14}$$

This is to say that $p_2(u,v,t) \geq 0$.

Hence, $P_2(u,v,w) \geq 0$. From the proof above, the required result (2.6) is proved. □

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2.4. Remarks

Remark 1. By the same argument as above, we also prove the following inequalities conjectures [9, 10, 11] in the acute triangle

$$\sum m_a^2 h_a^2 \geq \sum m_a^2 r_a^2, \tag{2.15}$$

$$\sum \sin^8 A \geq \sum \cos^8 \frac{A}{2}, \tag{2.16}$$

$$\sum (b-c)^2 \geq \sum \left(\frac{a}{b+c}\right)^2 (r_b - r_c)^2, \tag{2.17}$$

and

$$\sum (h_b + h_c - h_a)^3 \geq 3 m_a m_b m_c. \tag{2.18}$$

Remark 2. The operations in this paper were performed using mathematical software Maple 9.0.

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3. Generalization of the Method

In fact, Difference Substitution can go even further. Now, we consider the following inequality [12]. In $\triangle ABC$, if $\max(A, B, C) \leq \frac{2\pi}{3}$, then

$$s^2 \geq R^2 + 10Rr + 3r^2. \tag{3.1}$$

Utilizing the known formulas $R = \frac{abc}{4S}$, $r = \frac{S}{s}$ and $S = \sqrt{s(s-a)(s-b)(s-c)}$, from (1.6), inequality (3.1) is equivalent to

$$\begin{aligned} 3\,c^2a^2b^2 - a^2bc^3 - a^3bc^2 - a^3b^2c - a^2b^3c - ab^2c^3 - ab^3c^2 - 2\,b^3c^3 \\ - 2\,a^3c^3 + c^4b^2 - 2\,a^3b^3 + c^5b + a^4c^2 + b^4c^2 + b^5a + a^5c + c^5a \\ - a^6 + a^4b^2 + a^5b - b^6 - c^6 + a^2b^4 + b^5c + a^2c^4 \geq 0, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} &F(x,y,z) \\ &= -42\,x^2y^2z^2 + 14\,y^4zx + 14\,xyz^4 + 2\,xy^2z^3 + 2\,x^2y^3z + 2\,xy^3z^2 \\ &\quad + 14\,x^4yz + 2\,x^3yz^2 + 2\,x^2yz^3 + 2\,x^3y^2z - x^4y^2 - x^2z^4 \\ &\quad - 2\,x^3z^3 - x^4z^2 - 2\,x^3y^3 - y^4z^2 - y^4x^2 - 2\,y^3z^3 - y^2z^4 \geq 0, \end{aligned} \tag{3.3}$$

where $x > 0, y > 0, z > 0$.

Since inequality (3.3) is symmetric with x, y, z, there is no harm in supposing that $x \leq y \leq z$. From (1.2), $F(x,y,z)$ in (3.3) is transformed into

$$\begin{aligned} F(x,y,z) =& P(u,v,w) \\ =& (8\,u^2 + 4\,uv + 2\,uw - v^2 - vw)(8\,uvw^2 + 12\,uv^2w + 4\,u^2vw \\ &+ 4\,v^4 + 4\,u^2v^2 + 2\,uw^3 + 8\,uv^3 + 8\,v^3w + 7\,v^2w^2 + 3\,vw^3) \\ &+ 2\,w^2(v + 2\,u)^2(v + 2\,u + w)^2 \geq 0, \end{aligned} \tag{3.4}$$

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and for $\max(A, B, C) \le \frac{2\pi}{3}$ and $y = \cos x$ decreasing in $x \in (0, \pi)$, we have

$$\begin{aligned} b^2 + c^2 + bc - a^2 &= b^2 + c^2 - \frac{1}{2}bc\cos\frac{2\pi}{3} - a^2 \\ &= 3x^2 + 3(y+z)x - yz \\ &= 8u^2 + 4uv + 2uw - v^2 - vw \\ &\ge b^2 + c^2 - \frac{1}{2}bc\cos A - a^2 = 0. \end{aligned} \tag{3.5}$$

Since $F(x, y, z) = P(u, v, w) \ge 0$ for $u > 0, v \ge 0$ and $w \ge 0$, inequality (3.1) is obtained.

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