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# ON SOME NEW NONLINEAR DISCRETE INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT. In this paper, some new discrete inequalities in two independent variables which provide explicit bounds on unknown functions are established. The inequalities given here can be used as handy tools in qualitative theory of certain finite difference equations.

*Key words and phrases:* Discrete inequalities; Two independent variables; Difference equation.

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## 1. INTRODUCTION

The finite difference inequalities involving functions of one and more than one independent variables which provide explicit bounds for unknown functions play a fundamental role in the development of the theory of differential equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain applications. For example, see [1] – [8] and the references therein. In the qualitative analysis of some classes of finite difference equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. In this paper, we establish some new discrete inequalities involving functions of two independent variables. Our results generalize some results in [6, 8].

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## 2. MAIN RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}_0 = 0, 1, 2, ...$ are the given subsets of $\mathbb{R}$. We use the usual conventions that empty sums and products are taken to be $0$ and $1$ respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the sums involved exist on the respective domains of their definitions.

The following lemmas are useful in our main results.

**Lemma 2.1** ([6]). *Let $u(n),\ a(n),\ b(n)$ be nonnegative and continuous functions defined for $n \in \mathbb{N}_0$.*

i) *Assume that $a(n)$ is nondecreasing for $n \in \mathbb{N}_0$. If*

$$u(n) \le a(n) + \sum_{s=0}^{n-1} b(s)u(s),$$

*for $n \in \mathbb{N}_0$, then*

$$u(n) \le a(n) \prod_{s=0}^{n-1} [1 + b(s)],$$

*for $n \in \mathbb{N}_0$.*

ii) *Assume that $a(n)$ is nonincreasing for $n \in \mathbb{N}_0$. If*

$$u(n) \le a(n) + \sum_{s=n+1}^{\infty} b(s)u(s),$$

*for $n \in \mathbb{N}_0$, then*

$$u(n) \le a(n) \prod_{s=n+1}^{\infty} [1 + b(s)],$$

*for $n \in \mathbb{N}_0$.*

**Lemma 2.2.** *Assume that $p \ge q > 0,\ a \ge 0$, then*

$$a^{\frac{q}{p}} \le \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}},$$

*for any $k > 0$.*

*Proof.* Let $b = \frac{p}{q}$, then $b \ge 1$, by [8, Lemma 1], we have:

$$a^{\frac{q}{p}} \le \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}},$$

for any $k > 0$. □

**Theorem 2.3.** *Let $u(m, n)$, $a(m, n)$, $b(m, n)$, $e(m, n)$, $c_i(m, n)$ $(i = 1, 2, ..., l)$, be nonnegative continuous functions defined for $m, n, l \in \mathbb{N}_0$ and $p \ge q_i > 0$, $p$, $q_i$ $(i = 1, 2, ..., l)$ are constants. If*

$$[u(m,n)]^p \le a(m,n) + b(m,n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} \left[ \left( \sum_{i=1}^{l} c_i(s,t)(u(s,t))^{q_i} \right) + e(s,t) \right], \tag{2.1}$$

*for* $m, n, \in \mathbb{N}_0$ *then*

$$u(m,n) \le \left[a(m,n) + b(m,n)f(m,n)\prod_{s=0}^{m-1}\left(1 + \sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)b(s,t)\right)\right]^{\frac{1}{p}}, \tag{2.2}$$

*for any* $k > 0,\ \ m, n, \in \mathbb{N}_0$*, where*

$$f(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)\left[\frac{p-q_i}{p}k^{\frac{q_i}{p}} + a(s,t)\frac{q_i}{p}k^{\frac{q_i-p}{p}}\right]\right) + e(s,t)\right], \tag{2.3}$$

*for* $m, n \in \mathbb{N}_0$.

*Proof.* Define a function $z(m,n)$ by

$$z(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(u(s,t))^{q_i}\right) + e(s,t)\right], \tag{2.4}$$

Then (2.1) can be restated as

$$[u(m,n)]^p \le a(m,n) + b(m,n)z(m,n). \tag{2.5}$$

By (2.5) we have

$$u(m,n) \le (a(m,n) + b(m,n)z(m,n))^{\frac{1}{p}}. \tag{2.6}$$

Thus, from (2.4), (2.6) we obtain

$$z(m,n) \le \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(a(s,t) + b(s,t)z(s,t))^{\frac{q_i}{p}}\right) + e(s,t)\right]. \tag{2.7}$$

By Lemma 2.2, we have

$$\begin{aligned} z(m,n) &\le \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)\left(\frac{q_i}{p}k^{\frac{q_i-p}{p}}(a(s,t) + b(s,t)z(s,t))\right.\right.\right. \\ &\qquad\left.\left.\left. + \frac{p-q_i}{p}k^{\frac{q_i}{p}}\right)\right) + e(s,t)\right] \\ &= f(m,n) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left(\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)b(s,t)\right)z(s,t), \end{aligned} \tag{2.8}$$

where $f(m,n)$ is defined by (2.3). It is easy to see that $f(m,n)$ is nonnegative, continuous, nondecreasing in $m$ and nonincreasing in $n$ for $m, n, \in \mathbb{N}_0$.

Firstly, we assume that $f(m,n) > 0$ for $m, n, l \in \mathbb{N}_0$. From (2.8) we easily observe that

$$\frac{z(m,n)}{f(m,n)} \le 1 + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left(\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)b(s,t)\right), \tag{2.9}$$

set:

$$v(m,n) = 1 + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left(\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)b(s,t))\frac{z(s,t)}{f(s,t)}\right), \tag{2.10}$$

then

$$\frac{z(m,n)}{f(m,n)} \le v(m,n). \tag{2.11}$$

From (2.10), we get

$$[v(m+1,n)-v(m,n)]-[v(m+1,n+1)-v(m,n+1)]$$

$$=\left(\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,n+1)b(m,n+1)\right)\frac{z(m,n+1)}{f(m,n+1)}$$

$$\leq\left(\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,n+1)b(m,n+1)\right)v(m,n+1). \tag{2.12}$$

From (2.11) and using the fact $v(m,n)>0$, $v(m,n+1)\leq v(m,n)$ for $m,n\in\mathbb{N}_0$, we obtain

$$\frac{v(m+1,n)-v(m,n)}{v(m,n)}-\frac{v(m+1,n+1)-v(m,n+1)}{v(m,n+1)}\leq\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,n+1)b(m,n+1). \tag{2.13}$$

Keeping $m$ fixed in (2.13), setting $n=t$ and summing over $t=n,n+1,...,r-1$, where $r\geq n+1$ is arbitrary in $\mathbb{N}_0$, to obtain

$$\frac{v(m+1,n)-v(m,n)}{v(m,n)}-\frac{v(m+1,n+1)-v(m,n+1)}{v(m,n+1)}\leq\sum_{t=n+1}^{r}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,t)b(m,t). \tag{2.14}$$

Noting that

$$\lim_{r\to\infty}v(m,r)=\lim_{r\to 0}v(m+1,\infty)=1,$$

and letting $r\to\infty$ in (2.14), we get

$$\frac{v(m+1,n)-v(m,n)}{v(m,n)}\leq\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,t)b(m,t), \tag{2.15}$$

i.e.,

$$v(m+1,n)\leq\left[1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,t)b(m,t)\right]v(m,n). \tag{2.16}$$

Now by keeping $n$ fixed in (2.16) and setting $m=s$ and substituting $s=0,1,2,...,m-1$ successively and using the fact that $v(0,n)=1$, we have

$$v(m,n)\leq\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)b(s,t)\right]. \tag{2.17}$$

From (2.11) and (2.17), we obtain

$$z(m,n)\leq f(m,n)\prod_{s=0}^{m-1}\left[1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)b(s,t)\right]. \tag{2.18}$$

The desired inequality (2.2) follows from (2.6) and (2.18).

If $f(m,n)$ is nonnegative, we carry out the above procedure with $f(m,n)+\varepsilon$ instead of $f(m,n)$ where $\varepsilon>0$ is an arbitrary small constant and subsequently pass to the limit as $\varepsilon\to 0$ to obtain (2.2). This completes the proof. □

**Theorem 2.4.** *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$, $e(m,n)$, $c_i(m,n)$ $(i=1,2,...,l)$ *be nonnegative continuous functions defined for* $m,n,l \in \mathbb{N}_0$*, and* $p \geq q_i > 0,\ p,\ q_i\ (i=1,2,...,l)$ *are constants. If*

$$[u(m,n)]^p \leq a(m,n) + b(m,n)\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l} c_i(s,t)(u(s,t))^{q_i}\right) + e(s,t)\right], \tag{2.19}$$

*for* $m,n,\in\mathbb{N}_0$ *then*

$$u(m,n) \leq \left[a(m,n) + b(m,n)\bar{f}(m,n)\prod_{s=m+1}^{\infty}\left(1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(m,t)b(m,t)\right)\right]^{\frac{1}{p}}, \tag{2.20}$$

*for any* $k>0,\ \ m,n,\in\mathbb{N}_0$, *where*

$$\bar{f}(m,n) = \sum_{s=m+1}^{\infty}\left[\sum_{t=n+1}^{\infty}\left(\sum_{i=1}^{l}c_i(s,t)\left(\frac{p-q_i}{p}k^{\frac{q_i}{p}} + a(s,t)\frac{q_i}{p}k^{\frac{q_i-p}{p}}\right)\right) + e(s,t)\right], \tag{2.21}$$

*for* $m,n\in\mathbb{N}_0$.

The proof of Theorem 2.4 can be completed by following the proof of Theorem 2.3 with suitable changes, we omit it here.

**Remark 2.5.** If we take $l=1, q_1=1$, then the inequalities established in Theorems 2.3 and 2.4 reduce to the inequalities established in [8, Theorems 1 and 2].

**Remark 2.6.** If we take $l=1, q_1=1$, and $p=1$, $e(x,y)=0$, then the inequalities established in Theorem 2.3 and 2.4 reduce to the inequalities established in [6, Theorem 2.2 $(\alpha_1)$ and $(\alpha_2)$].

**Theorem 2.7.** *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$, $e(m,n)$, $c_i(m,n)$ $(i=1,2,...,l)$, *be nonnegative continuous functions defined for* $m,n,l\in\mathbb{N}_0$. *Assume that* $a(m,n)$ *are nondecreasing in* $m\in\mathbb{N}_0$*, and* $p\geq q_i>0,\ p,\ q_i\ (i=1,2,...,l)$ *are constants. If*

$$[u(m,n)]^p \leq a(m,n) + \sum_{s=0}^{m-1} b(s,n)(u(s,n))^p + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(u(s,t))^{q_i}\right)+e(s,t)\right], \tag{2.22}$$

*for* $m,n,\in\mathbb{N}_0$, *then*

$$u(m,n) \leq (B(m,n))^{\frac{1}{p}}\left[a(m,n) + F(m,n)\prod_{s=0}^{m-1}\left(1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)(B(s,t))^{\frac{q_i}{p}}\right)\right]^{\frac{1}{p}} \tag{2.23}$$

*for any* $n>0,\ \ m,n\in\mathbb{N}_0$, *where*

$$F(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(B(s,t))^{\frac{q_i}{p}}\left(\frac{p-q_i}{p}k^{\frac{q_i}{p}} + a(s,t)\frac{q_i}{p}k^{\frac{q_i-p}{p}}\right)\right)+e(s,t)\right], \tag{2.24}$$

$$B(m,n)=\prod_{s=0}^{m-1} b(s,n), \tag{2.25}$$

*for* $m,n\in\mathbb{N}_0$.

*Proof.* Define a function $z(m,n)$ by

$$z(m,n)=a(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(u(s,t))^{q_i}\right)+e(s,t)\right], \tag{2.26}$$

Then (2.22) can be restated as

$$[u(m,n)]^p\le z(m,n)+\sum_{s=0}^{m-1} b(s,n)[u(s,n)]^p. \tag{2.27}$$

Clearly, $z(m,n)$ is a nonnegative continuous and nondecreasing function in $m$, $m\in\mathbb{N}_0$. Treating $n$, $n\in\mathbb{N}_0$ fixed in (2.27), and using Lemma 2.1 (i) to (2.27) we have:

$$[u(m,n)]^p\le z(m,n)B(m,n), \tag{2.28}$$

where $B(m,n)$ is defined by (2.25). From (2.28) and (2.26) we obtain

$$[u(m,n)]^p\le B(m,n)(a(m,n)+v(m,n)), \tag{2.29}$$

where

$$v(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(u(s,t))^{q_i}\right)+e(s,t)\right]. \tag{2.30}$$

From (2.29), we have:

$$u(m,n)\le (B(m,n))^{\frac{1}{p}}(a(m,n)+v(m,n))^{\frac{1}{p}}, \tag{2.31}$$

for $m,n\in\mathbb{N}_0$. From (2.30), (2.31) and Lemma 2.2, we get

$$\begin{aligned}
v(m,n)&\le \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(B(s,t))^{\frac{q_i}{p}}(a(s,t)+v(s,t))^{\frac{q_i}{p}}\right)+e(s,t)\right]dtds\\
&\le \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l}c_i(s,t)(B(s,t))^{\frac{q_i}{p}}\left(\frac{p-q_i}{p}k^{\frac{q_i}{p}}\right.\right.\right.\\
&\qquad\left.\left.\left.+\ \frac{q_i}{p}k^{\frac{q_i-p}{p}}(a(s,t)+v(s,t))\right)\right)+e(s,t))\right]dtds\\
&=F(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}c_i(s,t)(B(s,t))^{\frac{q_i}{p}}\frac{q_i}{p}k^{\frac{q_i-p}{p}}v(s,t),
\end{aligned} \tag{2.32}$$

for $m,n\in\mathbb{N}_0,\ k>0$, where $F(m,n)$ is defined by (2.24). The rest of the proof of (2.23) can be completed by the proof of Theorem 2.3, we omit the details. □

**Theorem 2.8.** *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$, $e(m,n)$, $c_i(m,n)$ $(i=1,2,...,l)$, *be nonnegative continuous functions defined for* $m,n,l\in\mathbb{N}_0$. *Assume that* $a(m,n)$ *are nonincreasing in* $m\in\mathbb{N}_0$, *and* $p\geq q_i>0$, $p,\ q_i\ (i=1,2,...,l)$ *are constants. If*

$$[u(m,n)]^p \leq a(m,n)+\sum_{s=m+1}^{\infty} b(s,n)(u(s,n))^p + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l} c_i(s,t)(u(s,t))^{q_i}\right)+e(s,t)\right], \tag{2.33}$$

*for* $m,n,\in\mathbb{N}_0$, *then*

$$u(m,n) \leq (\bar{B}(m,n))^{\frac{1}{p}}\left[a(m,n)+\bar{F}(m,n)\prod_{s=m+1}^{\infty}\left(1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{l}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c_i(s,t)(\bar{B}(s,t))^{\frac{q_i}{p}}\right)\right]^{\frac{1}{p}} \tag{2.34}$$

*for any* $k>0,\ \ m,n\in\mathbb{N}_0$, *where*

$$\bar{F}(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[\left(\sum_{i=1}^{l} c_i(s,t)(\bar{B}(s,t))^{\frac{q_i}{p}}\left(\frac{p-q_i}{p}k^{\frac{q_i}{p}}+a(s,t)\frac{q_i}{p}k^{\frac{q_i-p}{p}}\right)\right)+e(s,t)\right], \tag{2.35}$$

$$\bar{B}(m,n)=\prod_{s=m+1}^{\infty}[1+b(s,n)], \tag{2.36}$$

*for* $m,n\in\mathbb{N}_0$.

The proof of Theorem 2.8 can be completed by following the proof of Theorem 2.7 with suitable changes, we omit it here.

**Remark 2.9.** If we take $l=1$, $q=1$, then the inequalities established in Theorems 2.7 and 2.8 reduce to the inequalities established in [8, Theorems 3 and 4].

**Remark 2.10.** If we take $l=1$, $q=1$, and $p=1$, $e(x,y)=0$, then the inequalities established in Theorems 2.7 and 2.8 reduce to the inequalities established in [6, Theorem 2.3].

## 3. Some Applications

**Example 3.1.** Consider the finite difference equation:

$$[u(m,n)^p]=a(m,n)+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} h(s,t,u(s,t)), \tag{3.1}$$

where $h:\mathbb{N}_0^2\times\mathbb{R}\to\mathbb{R}$, $a:\mathbb{N}_0^2\to\mathbb{R}$,.

Suppose that

$$|h(m,n,u)|\leq\sum_{i=1}^{3} c_i(m,n)|u|^{q_i}, \tag{3.2}$$

where $c_i(m,n)$, $(i=1,2,3)$ are nonnegative continuous functions for $m,n,\in\mathbb{N}_0$, $p\geq q_i>0$, $(i=1,2,3)$ $p,\ q_i$, are constants. If $u(m,n)$ is any solution of (3.1) – (3.2), then

$$|u(m,n)|\leq\left[a(m,n)+\bar{f}(m,n)\prod_{s=m+1}^{\infty}\left(1+\sum_{t=n+1}^{\infty}\sum_{i=1}^{3}\frac{q_i}{p}k^{\frac{q_i-p}{p}}c(s,t)\right)\right]^{\frac{1}{p}},\tag{3.3}$$

for $m,n\in\mathbb{N}_0,\ k>0$, where

$$\bar{f}(m,n)=\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[\sum_{i=1}^{3}c_i(s,t)\left(\frac{p-q_i}{p}k^{\frac{q_i}{p}}+\frac{q_i}{p}k^{\frac{q_i-p}{p}}a(s,t)\right)\right].\tag{3.4}$$

In fact, if $u(m,n)$ is any solution of (3.1) – (3.2), then it satisfies the equivalent integral equation:

$$[u(m,n)]^p\leq\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\sum_{i=1}^{3}c_i(m,n)|u|^{q_i},\tag{3.5}$$

Now a suitable application of Theorem 2.4 to (3.5) yields the required estimate in (3.3).

**Example 3.2.** Consider the finite differential equation:

$$\begin{aligned}u(m+1,n+1)-u(m+1,n)-u(m,n+1)+u(m,n)&\\=h(m,n,u(m,n))+r(m,n),&\end{aligned}\tag{3.6}$$

$$u(m,\infty)=\sigma(m),\quad u(\infty,n)=\tau(n),\quad u(\infty,\infty)=d,\tag{3.7}$$

where $h:N_0^2\times\mathbb{R}\to\mathbb{R}$, $r:\mathbb{N}_0^2\to\mathbb{R}$, $\sigma,\tau:\mathbb{N}_0\to\mathbb{R}$, $d$ is a real constant.

Suppose that

$$|h(m,n,u)-h(m,n,v)|\leq c(m,n)|u-v|^q,\tag{3.8}$$

where $c(m,n)$ is defined as in Theorem 2.4, $q\leq 1$, $q$ is a constant.

If $u(m,n)$, $v(m,n)$ are two solutions of (3.6) – (3.7), then

$$|u(m,n)-v(m,n)|\leq\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}(c(s,t)(1-q)k^q)\prod_{s=m+1}^{\infty}\left(1+\sum_{t=n+1}^{\infty}qk^{q-1}c(s,t)\right),\tag{3.9}$$

for $m,n\in\mathbb{N}_0,k>0$. In fact, if $u(m,n)$ is a solution of (3.6) – (3.7), then it can be written as

$$u(m,n)=\sigma(m)+\tau(n)-d+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}[h(s,t,u(s,t))+r(s,t)].\tag{3.10}$$

Let $u(m,n),v(m,n)$ be two solutions of (3.6) – (3.7), we have

$$|u(m,n)-v(m,n)|\leq\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}c(s,t)|u(s,t)-v(s,t)|^q,\tag{3.11}$$

for $m,n\in\mathbb{N}_0$.

Now a suitable application of the inequality in Theorem 2.8 to (3.11) yields (3.9).

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