journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 1, Article 21, 4 pp.

PYTHAGOREAN PARAMETERS AND NORMAL STRUCTURE IN BANACH SPACES

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Received 16 August, 2007; accepted 15 February, 2008
Communicated by S.S. Dragomir

ABSTRACT. Recently, Gao introduced some quadratic parameters, such as $E_\epsilon(X)$ and $f_\epsilon(X)$. In this paper, we obtain some sufficient conditions for normal structure in terms of Gao's parameters, improving some known results.

Key words and phrases: Uniform non-squareness; Normal structure.

2000 *Mathematics Subject Classification.* 46B20.

1. INTRODUCTION

There are several parameters and constants which are defined on the unit sphere or the unit ball of a Banach space. These parameters and constants, such as the James and von Neumann-Jordan constants, have been proved to be very useful in the descriptions of the geometric structure of Banach spaces.

Based on a Pythagorean theorem, Gao introduced some quadratic parameters recently [1, 2]. Using these parameters, one can easily distinguish several important classes of spaces such as uniform non-squareness or spaces having normal structure.

In this paper, we are going to continue the study in Gao's parameters. Moreover, we obtain some sufficient conditions for a Banach space to have normal structure.

Let X be a Banach space and X^* its dual. We shall assume throughout this paper that B_X and S_X denote the unit ball and unit sphere of X, respectively.

One of Gao's parameters $E_\epsilon(X)$ is defined by the formula

$$E_\epsilon(X) = \sup\{\|x + \epsilon y\|^2 + \|x - \epsilon y\|^2 : x, y \in S_X\},$$

The author would like to thank the anonymous referees for their helpful suggestions on this paper.
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where ϵ is a nonnegative number. It is worth noting that $E_\epsilon(X)$ was also introduced by Saejung [3] and Yang-Wang [5] recently. Let us now collect some properties related to this parameter (see [1, 4, 5]).

(1) X is uniformly non-square if and only if $E_\epsilon(X) < 2(1+\epsilon)^2$ for some $\epsilon \in (0,1]$.
(2) X has uniform normal structure if $E_\epsilon(X) < 1 + (1+\epsilon)^2$ for some $\epsilon \in (0,1]$.
(3) $E_\epsilon(X) = E_\epsilon(\widetilde{X})$, where $\widetilde{X}$ is the ultrapower of X.
(4) $E_\epsilon(X) = \sup\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 : x, y \in B_X\}$.

It follows from the property (4) that

$$E_\epsilon(X) = \inf\left\{\frac{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2}{\max(\|x\|^2, \|y\|^2)} : x, y \in X, \|x\| + \|y\| \neq 0\right\}.$$

Now let us pay attention to another Gao's parameter $f_\epsilon(X)$, which is defined by the formula

$$f_\epsilon(X) = \inf\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 : x, y \in S_X\},$$

where ϵ is a nonnegative number.

We quote some properties related to this parameter (see [1, 2]).

(1) If $f_\epsilon(X) > 2$ for some $\epsilon \in (0,1]$, then X is uniformly non-square.
(2) X has uniform normal structure if $f_1(X) > 32/9$.

Using a similar method to [4, Theorem 3], we can also deduce that $f_\epsilon(X) = f_\epsilon(\widetilde{X})$, where $\widetilde{X}$ is the ultrapower of X.

2. Main Results

We start this section with some definitions. Recall that X is called *uniformly non-square* if there exists $\delta > 0$, such that if $x, y \in S_X$ then $\|x+y\|/2 \leq 1-\delta$ or $\|x-y\|/2 \leq 1-\delta$. In what follows, we shall show that $f_\epsilon(X)$ also provides a characterization of the uniformly non-square spaces, namely $f_1(X) > 2$.

Theorem 2.1. *X is uniformly non-square if and only if $f_1(X) > 2$.*

Proof. It is convenient for us to assume in this proof that $\dim X < \infty$. The extension of the results to the general case is immediate, depending only on the formula

$$f_\epsilon(X) = \inf\{f_\epsilon(Y) : Y \text{ subspace of } X \text{ and } \dim Y = 2\}.$$

We are going to prove that uniform non-squareness implies $f_1(X) > 2$. Assume on the contrary that $f_1(X) = 2$. It follows from the definition of $f_\epsilon(X)$ that there exist $x, y \in S_X$ so that

$$\|x+y\|^2 + \|x-y\|^2 = 2.$$

Then, since $\|x+y\| + \|x-y\| \geq 2$, we have

$$\|x \pm y\|^2 = 2 - \|x \mp y\|^2 \leq 2 - (2 - \|x \pm y\|)^2,$$

which implies that $\|x \pm y\| = 1$. Now let us put $u = x+y,\ v = x-y$, then $u, v \in S_X$ and $\|u \pm v\| = 2$. This is a contradiction. The converse of this assertion was proved by Gao [2, Theorem 2.8], and thus the proof is complete. □

Consider now the definitions of normal structure. A Banach space X is said to have (*weak*) *normal structure* provided that every (weakly compact) closed bounded convex subset C of X with $\operatorname{diam}(C) > 0$, contains a non-diametral point, i.e., there exists $x_0 \in C$ such that $\sup\{\|x - x_0\| : x \in C\} < \operatorname{diam}(C)$. It is clear that normal structure and weak normal structure coincides when X is reflexive. A Banach space X is said to have *uniform normal structure* if $\inf\{\operatorname{diam}(C)/\operatorname{rad}(C)\} > 1$, where the infimum is taken over all bounded closed convex subsets C of X with $\operatorname{diam}(C) > 0$.

To study the relation between normal structure and Gao's parameter, we need a sufficient condition for normal structure, which was posed by Saejung [4, Lemma 2] recently.

Theorem 2.2. *Let X be a Banach space with*

$$E_\epsilon(X) < 2 + \epsilon^2 + \epsilon\sqrt{4+\epsilon^2}$$

for some $\epsilon \in (0,1]$, then X has uniform normal structure.

Proof. By our hypothesis it is enough to show that X has normal structure. Suppose that X lacks normal structure, then by [4, Lemma 2], there exist $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 \in S_{\widetilde{X}}$ and $\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3 \in S_{\widetilde{X^*}}$ satisfying:

(a) $\|\widetilde{x}_i - \widetilde{x}_j\| = 1$ and $\widetilde{f}_i(\widetilde{x}_j) = 0$ for all $i \neq j$.
(b) $\widetilde{f}_i(\widetilde{x}_i) = 1$ for $i = 1, 2, 3$ and
(c) $\|\widetilde{x}_3 - (\widetilde{x}_2 + \widetilde{x}_1)\| \geq \|\widetilde{x}_2 + \widetilde{x}_1\|$.

Let $2\alpha(\epsilon) = \sqrt{4+\epsilon^2} + 2 - \epsilon$ and consider three possible cases.

CASE 1. $\|\widetilde{x}_1 + \widetilde{x}_2\| \leq \alpha(\epsilon)$. In this case, let us put $\widetilde{x} = \widetilde{x}_1 - \widetilde{x}_2$ and $\widetilde{y} = (\widetilde{x}_1 + \widetilde{x}_2)/\alpha(\epsilon)$. It follows that $\widetilde{x}, \widetilde{y} \in B_{\widetilde{X}}$, and

$$\begin{aligned}
\|\widetilde{x} + \epsilon\widetilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_1 - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_2\| \\
&\geq (1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_1(\widetilde{x}_1) - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_1(\widetilde{x}_2) \\
&= 1 + (\epsilon/\alpha(\epsilon)), \\
\|\widetilde{x} - \epsilon\widetilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_2 - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_1\| \\
&\geq (1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_2(\widetilde{x}_2) - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_2(\widetilde{x}_1) \\
&= 1 + (\epsilon/\alpha(\epsilon)).
\end{aligned}$$

CASE 2. $\|\widetilde{x}_1 + \widetilde{x}_2\| \geq \alpha(\epsilon)$ and $\|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| \leq \alpha(\epsilon)$. In this case, let us put $\widetilde{x} = \widetilde{x}_2 - \widetilde{x}_3$ and $\widetilde{y} = (\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1)/\alpha(\epsilon)$. It follows that $\widetilde{x}, \widetilde{y} \in B_{\widetilde{X}}$, and

$$\begin{aligned}
\|\widetilde{x} + \epsilon\widetilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_2 - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_3 - (\epsilon/\alpha(\epsilon))\widetilde{x}_1\| \\
&\geq (1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_2(\widetilde{x}_2) - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_2(\widetilde{x}_3) - (\epsilon/\alpha(\epsilon))\widetilde{f}_2(\widetilde{x}_1) \\
&= 1 + (\epsilon/\alpha(\epsilon)), \\
\|\widetilde{x} - \epsilon\widetilde{y}\| &= \|(1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_3 - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{x}_2 - (\epsilon/\alpha(\epsilon))\widetilde{x}_1)\| \\
&\geq (1 + (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_3(\widetilde{x}_3) - (1 - (\epsilon/\alpha(\epsilon)))\,\widetilde{f}_3(\widetilde{x}_2) - (\epsilon/\alpha(\epsilon))\widetilde{f}_3(\widetilde{x}_1) \\
&= 1 + (\epsilon/\alpha(\epsilon)).
\end{aligned}$$

CASE 3. $\|\widetilde{x}_1 + \widetilde{x}_2\| \geq \alpha(\epsilon)$ and $\|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| \geq \alpha(\epsilon)$. In this case, let us put $\widetilde{x} = \widetilde{x}_3 - \widetilde{x}_1$ and $\widetilde{y} = \widetilde{x}_2$. It follows that $\widetilde{x}, \widetilde{y} \in S_{\widetilde{X}}$, and

$$\begin{aligned}
\|\widetilde{x} + \epsilon\widetilde{y}\| &= \|\widetilde{x}_3 + \epsilon\widetilde{x}_2 - \widetilde{x}_1\| \\
&\geq \|\widetilde{x}_3 + \widetilde{x}_2 - \widetilde{x}_1\| - (1 - \epsilon) \\
&\geq \alpha(\epsilon) + \epsilon - 1, \\
\|\widetilde{x} - \epsilon\widetilde{y}\| &= \|\widetilde{x}_3 - (\epsilon\widetilde{x}_2 + \widetilde{x}_1)\| \\
&\geq \|\widetilde{x}_3 - (\widetilde{x}_2 + \widetilde{x}_1)\| - (1 - \epsilon) \\
&\geq \alpha(\epsilon) + \epsilon - 1.
\end{aligned}$$

Then, by definition of $E_\epsilon(X)$ and the fact $E_\epsilon(X) = E_\epsilon(\widetilde{X})$,

$$\begin{aligned} E_\epsilon(X) &\geq 2\min\left\{1 + (\epsilon/\alpha(\epsilon)),\ \alpha(\epsilon) + \epsilon - 1\right\}^2 \\ &= 2 + \epsilon^2 + \epsilon\sqrt{4+\epsilon^2}. \end{aligned}$$

This is a contradiction and thus the proof is complete. □

Remark 2.3. It is proved that $E_\epsilon(X) < 1 + (1+\epsilon)^2$ for some $\epsilon \in (0,1]$ implies that X has uniform normal structure. So Theorem 2.2 is an improvement of such a result.

Theorem 2.4. *Let X be a Banach space with*

$$f_\epsilon(X) > ((1+\epsilon^2)^2 + 2\epsilon(1-\epsilon^2))(2 + \epsilon^2 - \epsilon\sqrt{4+\epsilon^2})$$

for some $\epsilon \in (0,1]$, then X has uniform normal structure.

Proof. By our hypothesis it is enough to show that X has normal structure. Assume that X lacks normal structure, then from the proof of Theorem 2.2 we can find $\widetilde{x}, \widetilde{y} \in B_{\widetilde{X}}$ such that

$$\|\widetilde{x} \pm \epsilon\widetilde{y}\| \geq 1 + (\epsilon/\alpha(\epsilon)) = \alpha(\epsilon) + \epsilon - 1 =: \beta(\epsilon).$$

Put $\widetilde{u} = (\widetilde{x} + \epsilon\widetilde{y})/\beta(\epsilon)$ and $\widetilde{v} = (\widetilde{x} - \epsilon\widetilde{y})/\beta(\epsilon)$. It follows that $\|\widetilde{u}\|, \|\widetilde{v}\| \geq 1$, and

$$\begin{aligned} \|\widetilde{u} + \epsilon\widetilde{v}\| &= \left\|\frac{1}{\beta(\epsilon)}((1+\epsilon)\widetilde{x} + \epsilon(1-\epsilon)\widetilde{y})\right\| \\ &\leq \frac{(1+\epsilon) + \epsilon(1-\epsilon)}{\beta(\epsilon)}, \\ \|\widetilde{u} - \epsilon\widetilde{v}\| &= \frac{1}{\beta(\epsilon)}((1-\epsilon)\widetilde{x} + \epsilon(1+\epsilon)\widetilde{y}) \\ &\leq \frac{(1-\epsilon) + \epsilon(1+\epsilon)}{\beta(\epsilon)}. \end{aligned}$$

Hence, by the definition of $f_\epsilon(X)$ and the fact $f_\epsilon(X) = f_\epsilon(\widetilde{X})$, we have

$$\begin{aligned} f_\epsilon(X) &\leq \frac{((1+\epsilon) + \epsilon(1-\epsilon))^2 + ((1-\epsilon) + \epsilon(1+\epsilon))^2}{\beta^2(\epsilon)} \\ &= ((1+\epsilon^2)^2 + 2\epsilon(1-\epsilon^2))(2 + \epsilon^2 - \epsilon\sqrt{4+\epsilon^2}), \end{aligned}$$

which contradicts our hypothesis. □

Remark 2.5. Letting $\epsilon = 1$, one can easily get that if $f_1(X) > 4(3-\sqrt{5})$, then X has uniform normal structure. So this is an extension and an improvement of [2, Theorem 5.3].

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