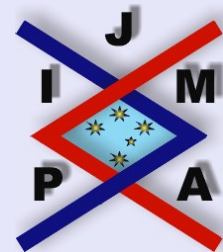


## NECESSARY AND SUFFICIENT CONDITION FOR COMPACTNESS OF THE EMBEDDING OPERATOR

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Abstract

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## Abstract

An improvement of the author's result, proved in 1961, concerning necessary and sufficient conditions for the compactness of an imbedding operator is given.

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*Key words:* Banach spaces, Compactness, Embedding operator.

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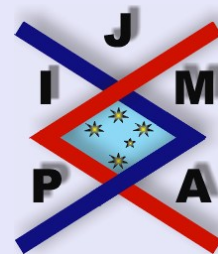
# 1. Introduction

The basic result of this note is:

**Theorem 1.1.** *Let  $X_1 \subset X_2 \subset X_3$  be Banach spaces,  $\|u\|_1 \geq \|u\|_2 \geq \|u\|_3$  (i.e., the norms are comparable) and if  $\|u_n\|_3 \rightarrow 0$  as  $n \rightarrow \infty$  and  $u_n$  is fundamental in  $X_2$ , then  $\|u_n\|_2 \rightarrow 0$ , (i.e., the norms in  $X_2$  and  $X_3$  are compatible). Under the above assumptions the embedding operator  $i : X_1 \rightarrow X_2$  is compact if and only if the following two conditions are valid:*

- a) *The embedding operator  $j : X_1 \rightarrow X_3$  is compact, and the following inequality holds:*
- b)  *$\|u\|_2 \leq s\|u\|_1 + c(s)\|u\|_3, \forall u \in X_1, \forall s \in (0, 1)$ , where  $c(s) > 0$  is a constant.*

This result is an improvement of the author's old result, proved in 1961 (see [1]), where  $X_2$  was assumed to be a Hilbert space. The proof of Theorem 1.1 is simpler than the one in [1].



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## 2. Proof

1. Assume that a) and b) hold and let us prove the compactness of  $i$ . Let  $S = \{u : u \in X_1, \|u\|_1 = 1\}$  be the unit sphere in  $X_1$ . Using assumption a), select a sequence  $u_n$  which converges in  $X_3$ . We claim that this sequence converges also in  $X_2$ . Indeed, since  $\|u_n\|_1 = 1$ , one uses assumption b) to get

$$\|u_n - u_m\|_2 \leq s\|u_n - u_m\|_1 + c(s)\|u_n - u_m\|_3 \leq 2s + c(s)\|u_n - u_m\|_3.$$

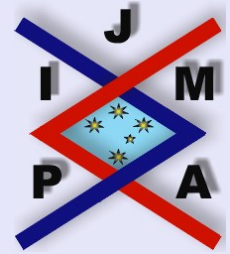
Let  $\eta > 0$  be an arbitrary small given number. Choose  $s > 0$  such that  $2s < \frac{1}{2}\eta$ , and for a fixed  $s$  choose  $n$  and  $m$  so large that  $c(s)\|u_n - u_m\|_3 < \frac{1}{2}\eta$ . This is possible because the sequence  $u_n$  converges in  $X_3$ . Consequently,  $\|u_n - u_m\|_2 \leq \eta$  if  $n$  and  $m$  are sufficiently large. This means that the sequence  $u_n$  converges in  $X_2$ . Thus, the embedding  $i : X_1 \rightarrow X_2$  is compact. In the above argument the compatibility of the norms was not used.

2. Assume now that  $i$  is compact. Let us prove that assumptions a) and b) hold. Assumption a) holds because  $\|u\|_2 \geq \|u\|_3$ . Suppose that assumption b) fails. Then there is a sequence  $u_n$  and a number  $s_0 > 0$  such that  $\|u_n\|_1 = 1$  and

$$(2.1) \quad \|u_n\|_2 \geq s_0 + n\|u_n\|_3.$$

If the embedding operator  $i$  is compact and  $\|u_n\|_1 = 1$ , then one may assume that the sequence  $u_n$  converges in  $X_2$ . Its limit cannot be equal to zero, because, by (2.1),  $\|u_n\|_2 \geq s_0 > 0$ . The sequence  $u_n$  converges in  $X_3$  because  $\|u_n - u_m\|_2 \geq \|u_n - u_m\|_3$ , and its limit in  $X_3$  is not zero, because the norms in  $X_3$  and in  $X_2$  are compatible. Thus, (2.1) implies  $\|u_n\|_3 = O\left(\frac{1}{n}\right) \rightarrow 0$  as  $n \rightarrow \infty$ , while  $\lim_{n \rightarrow \infty} \|u_n\|_3 > 0$ . This is a contradiction, which proves that b) holds.

Theorem 1.1 is proved.  $\square$



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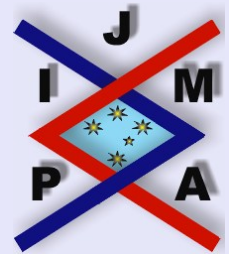
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## References

- [1] A.G. RAMM, A necessary and sufficient condition for compactness of embedding, *Vestnik of Leningrad. Univ., Ser. Math., Mech., Astron.*, **1** (1963), 150–151.



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