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NECESSARY AND SUFFICIENT CONDITION FOR COMPACTNESS OF THE EMBEDDING OPERATOR

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Abstract

An improvement of the author's result, proved in 1961, concerning necessary and sufficient conditions for the compactness of an imbedding operator is given.

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1. Introduction

The basic result of this note is:

Theorem 1.1. *Let* $X_1 \subset X_2 \subset X_3$ *be Banach spaces,* $||u||_1 \geq ||u||_2 \geq ||u||_3$ *(i.e., the norms are comparable) and if* $||u_n||_3 \to 0$ *as* $n \to \infty$ *and* u_n *is fundamental in* X_2*, then* $||u_n||_2 \to 0$*, (i.e., the norms in* X_2 *and* X_3 *are compatible). Under the above assumptions the embedding operator* $i : X_1 \to X_2$ *is compact if and only if the following two conditions are valid:*

a) The embedding operator $j : X_1 \to X_3$ *is compact,*

and the following inequality holds:

b) $||u||_2 \leq s||u||_1 + c(s)||u||_3, \quad \forall u \in X_1, \forall s \in (0, 1)$*, where* $c(s) > 0$ *is a constant.*

This result is an improvement of the author's old result, proved in 1961 (see [1]), where X_2 was assumed to be a Hilbert space. The proof of Theorem 1.1 is simpler than the one in [1].

2. Proof

1. Assume that a) and b) hold and let us prove the compactness of i. Let $S = \{u : u \in X_1, ||u||_1 = 1\}$ be the unit sphere in X_1. Using assumption a), select a sequence u_n which converges in X_3. We claim that this sequence converges also in X_2. Indeed, since $||u_n||_1 = 1$, one uses assumption b) to get

$$||u_n - u_m||_2 \leq s||u_n - u_m||_1 + c(s)||u_n - u_m||_3 \leq 2s + c(s)||u_n - u_m||_3.$$

Let $\eta > 0$ be an arbitrary small given number. Choose $s > 0$ such that $2s < \frac{1}{2}\eta$, and for a fixed s choose n and m so large that $c(s)||u_n - u_m||_3 < \frac{1}{2}\eta$. This is possible because the sequence u_n converges in X_3. Consequently, $||u_n - u_m||_2 \leq \eta$ if n and m are sufficiently large. This means that the sequence u_n converges in X_2. Thus, the embedding $i : X_1 \to X_2$ is compact. In the above argument the compatibility of the norms was not used.

2. Assume now that i is compact. Let us prove that assumptions a) and b) hold. Assumption a) holds because $||u||_2 \geq ||u||_3$. Suppose that assumption b) fails. Then there is a sequence u_n and a number $s_0 > 0$ such that $||u_n||_1 = 1$ and

$$||u_n||_2 \geq s_0 + n||u_n||_3. \tag{2.1}$$

If the embedding operator i is compact and $||u_n||_1 = 1$, then one may assume that the sequence u_n converges in X_2. Its limit cannot be equal to zero, because, by (2.1), $||u_n||_2 \geq s_0 > 0$. The sequence u_n converges in X_3 because $||u_n - u_m||_2 \geq ||u_n - u_m||_3$, and its limit in X_3 is not zero, because the norms in X_3 and in X_2 are compatible. Thus, (2.1) implies $||u_n||_3 = O\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$, while $\lim_{n\to\infty} ||u_n||_3 > 0$. This is a contradiction, which proves that b) holds.

Theorem 1.1 is proved. □

References

[1] A.G. RAMM, A necessary and sufficient condition for compactness of embedding, *Vestnik of Leningrad. Univ., Ser. Math., Mech., Astron.*, **1** (1963), 150–151.