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ON A QUESTION OF MILMAN AND ALESKER CONCERNING THE MONOTONICITY OF A DOMAIN FUNCTIONAL

G. KEADY
School of Mathematics and Statistics,
University of Western Australia
6009, Australia
keady@maths.uwa.edu.au

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Abstract. We answer affirmatively the special case of $q = 1$, $n = 2$, $j = 2$ of Question 3 on page 1004 of Alesker, *Annals of Mathematics*, **149** (1999).

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1. Introduction

Let D_0 be a bounded convex domain in $\mathbb{R}^2$ containing the origin. Let $B(0, \rho_1)$ be the disk, centred at 0, of radius ρ_1, and consider the Minkowski sum

$$D(t) = (1 - t)D_0 + tB(0, \rho_1).$$

Define $I_O(\cdot, q)$ by

$$I_O(D(t), q) = \int_{D(t)} |x|^{2q},$$

and note that at $q = 1$ this is the polar moment of inertia about the origin of the domain $D(t)$. When $q = 1$ we will simply write $I_O(D(t))$ and omit the second argument. The derivatives of $I_O(D(t), q)$ with respect to ρ_1 are shown to be positive: see Alesker [1, Theorem 6.1]. Alesker [1, p. 1004, Question 3] asks about the domain-monotonicity of the derivatives with respect to ρ_1 evaluated at $\rho_1 = 0$. Our answer to the special case of this question will be given in Theorem 3.1 in §3.

Alesker considers one-dimensional sets. These suffice to show that domain monotonicity will not hold true in general unless the origin is in the domain.

The notation in this paper is the same as in [7].

When the convex sets are centrally-symmetric, several of the proofs in [7] simplify, and there are additional results. Alesker asked the question for centrally-symmetric convex sets, but here we can, for $n = 2$, $q = 1$, answer it more generally, merely requiring the set to contain the origin.

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2. SUPPORT FUNCTIONS

We use the letter p for the support function as in [4] and Santalo's book [9]. An adequate description of p is as 'the perpendicular distance from the origin to the tangent'. The radius of curvature is then

$$\rho = p + \ddot{p}, \quad \text{where } \dot{f} = \frac{df}{d\varphi},\ ds = \rho d\varphi\ .$$

For a diagram, see page 2 of [9]. Santalo's ϕ is the angle between a line *normal* to the tangent and the x-axis. Following Santalo's notation, let H be a point on the tangent line such that OH is perpendicular to the tangent line. $|OH| = p$. The boundaries of our convex sets D can be determined from the functions $p(\phi)$ through formulae (1.3) of [9]. Our φ is the angle between the *tangent* line (through H) and the x-axis. We have

$$\varphi = \phi + \frac{\pi}{2}.$$

Then the area and perimeter are given by

$$A = \text{Area}(D) = \frac{1}{2}\int_0^{2\pi} p\rho\, d\varphi = \frac{1}{2}\int_0^{2\pi} (p^2 - \dot{p}^2) d\varphi, \tag{2.1}$$

$$L = \int_0^{2\pi} \rho\, d\varphi = \int_0^{2\pi} p\, d\varphi\ . \tag{2.2}$$

D is convex iff $\rho \geq 0$. In the case of a polygon, for example, we might interpret ρ as a nonnegative measure. The set $\mathcal{S}$ of support functions forms a cone: $\mathcal{S}$ is convex, and if $t > 0$ and $p \in \mathcal{S}$, then $tp \in \mathcal{S}$.

We now suppose that we have two convex domains D_0 and D_1. We denote $\text{Area}(D_0) = A_0$ and $\text{Area}(D_1) = A_1$. We have the following pretty, and very well-known, result:

Lemma 2.1. *For convex sets D_0, D_1, the support function for $D(t)$ is given by*

$$p_t = (1-t)p_0 + tp_1. \tag{2.3}$$

In particular, the preceding lemma yields that

$$L(t) := L(D(t)) = (1-t)L_0 + tL_1\ , \tag{2.4}$$

$$A(t) := \text{Area}(D(t)) = (1-t)^2 A_0 + 2t(1-t)A_{0,1} + t^2 A_1\ , \tag{2.5}$$

where the *mixed-area* $A_{0,1}$ satisfies

$$A_{0,1} := A(D_0, D_1) = \frac{1}{2}\int_0^{2\pi} (p_0 p_1 - \dot{p}_0 \dot{p}_1)\, d\varphi\ . \tag{2.6}$$

(All of these functionals are monotonic under domain inclusion for convex sets.)

There are many nice properties of support functions. Here is one. See [10, p. 37], or the first page of [8].

Theorem 2.2. *If $0 \in D \subseteq \hat{D}$ then $0 \leq p \leq \hat{p}$.*

We do not use, but state:

Theorem 2.3 ([3, p. 56])**.** *Let C denote the convex hull of the union of the convex domains D_0 and D_1. Then, the support functions satisfy*

$$p_C = \max(p_{D_0}, p_{D_1}).$$

(Further general references on convex domains and their support functions include [2], [4], [6], [8], [10].)

Starting from the expressions for $x(\varphi)$ and $y(\varphi)$ for boundary points, expressing the coordinates in terms of p and $\dot{p} = dp/d\varphi$, in [7], using Lemma 2.1, the following expression for $I_O(D(t))$ when $D_1 = B(0, \rho_1)$ is derived:

$$I_O(D(t)) = (1-t)^4 I_O(D_0) + \rho_1 t(1-t)^3 I(\partial D_0) + (\rho_1 t(1-t))^2 Z + (\rho_1 t)^3(1-t)L + \frac{\pi}{2}(\rho_1 t)^4.$$

Here

$$Z := \frac{1}{2}\int_0^{2\pi} (3p^2 - \dot{p}^2)\, d\varphi. \tag{2.7}$$

In the two preceding equations, p is the support function for D_0, and L and Z are evaluated for D_0.

3. A PARTIAL ANSWER TO ALESKER'S QUESTION 3

Theorem 3.1. *Let D, $\hat{D}$ be convex domains containing the origin with smooth boundaries. Define*

$$\alpha(j) = \frac{d^j}{d\rho_1^j} I_O(D(t))|_{\rho_1=0}$$

and define $\hat{\alpha}$ similarly. Then (for $n = 2$, $q = 1$),

$$\alpha(2) = 2t^2(1-t)^2 Z,$$

and

$$Z = A + \int_0^{2\pi} p(\varphi)^2\, d\varphi.$$

The function Z is nonnegative and is monotonic under domain inclusion, i.e. if $0 \in D \subseteq \hat{D}$, then $0 \le Z(D) \le Z(\hat{D})$.

Proof. The area is monotonic under domain inclusion. Using Theorem 2.2, so is $\int_0^{2\pi} p(\varphi)^2\, d\varphi$. Hence we have the required monotonicity of Z.

The restriction that the boundaries be smooth can be removed by taking limits.

Alesker states that $\alpha(1)$ can be shown to be monotonic under domain inclusion. Returning to $n = 2$, $q = 1$, as $\alpha(3) = 6t^3(1-t)L$, we also have that $\alpha(3)$ is monotonic under domain inclusion. $\alpha(4)$ is independent of domain. □

For centrally symmetric domains, other properties of the second derivatives can be deduced from the results in [5].

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