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PROPERTIES OF SOME FUNCTIONS CONNECTED TO PRIME NUMBERS

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ABSTRACT. Let θ and ψ be the Chebyshev functions. We denote $\psi_2(x) = \psi(x) - \theta(x)$ and $\rho(x) = \psi(x)/\theta(x)$. We study subadditive and Landau-type properties for θ, ψ, and ψ_2. We show that ρ is subadditive and submultiplicative. Finally, we consider the prime counting function $\pi(x)$ and show that $\pi(x)\pi(y) < \pi(xy)$ for all $x, y \geq \sqrt{53}$.

Key words and phrases: Arithmetic functions, Inequalities, Landau's inequality, Additivity, Multiplicativity.

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1. INTRODUCTION

Throughout this paper, p will always denote a prime number. We will also use the following notations (most of them classic):

- $p_n =$ the n^{th} prime (in increasing order);
- $\pi(x) =$ the number of prime numbers that do not exceed x;
- $\theta(x) = \sum\limits_{p \leq x} \log p$ (the Chebyshev theta function);
- $\psi(x) = \sum\limits_{p^k \leq x} \log p$ (the Chebyshev psi function);
- $\psi_2(x) = \psi(x) - \theta(x) = \sum\limits_{\substack{p^k \leq x \\ k \geq 2}} \log p$;
- $\psi_t(x) = \sum\limits_{\substack{p^k \leq x \\ k \geq t}} \log p$;
- $\rho(x) = \frac{\psi(x)}{\theta(x)}$.

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One of the most studied problems in number theory is the Hardy-Littlewood conjecture [2], which states that

$$\pi(x) + \pi(y) \geq \pi(x+y) \quad \text{for all integers } x, y \geq 2.$$

It is not known at this moment whether this statement is true or false. However, its particular case $\pi(2x) \leq 2\pi(x)$, also known as Landau's inequality, was proved by E. Landau [5] for big enough x. Later, J. B. Rosser and L. Schoenfeld [7] managed to prove this inequality for all $x \geq 2$.

We ask whether other functions related to prime numbers have similar properties. Namely, we will answer such questions for the functions $\psi_2 = \theta - \psi$, and $\rho = \psi/\theta$. Since we did not manage to find bibliographic references for the mentioned properties for θ and ψ, we will supply proofs for these cases as well.

Note that, since $\psi_2(x) \sim \sqrt{x}$, the answers to our questions for the function ψ_2 seem to be affirmative. Such an approach, however, would only give us the required inequalities for "large enough" (but unspecified) values of x. This would prevent us from currently using these inequalities for specified values of the variables. On the other hand, using suitable inequalities, we will prove in Section 2 that $\psi_2(2x) < \psi_2(2x)$ for all $x, y \geq 25$. This is an example of how inequalities with specified "starting points" will enrich the information obtained from the asymptotic equivalences.

On the other hand, the asymptotic behaviour of θ and ψ does not even suggest an "asymptotic" answer to the questions posed, so we will have to use another approach in order to deal with this case.

For the function ρ, the multiplicative point of view seems to be more significant, so we will also study some multiplicative properties of ρ as well. We will then consider the prime counting function π from this point of view and prove the inequality $\pi(x)\pi(y) < \pi(xy)$ for all $x, y \geq \sqrt{53}$.

We will try as a general principle to prove the required properties for values greater than a specified margin, and then use computer checking in order to lower that margin as much as possible. To this end, we will make use of some already known inequalities that we list below:

- **I1**: $|\theta(x) - x| \leq 0.006788 \frac{x}{\log x}$ for all $x \geq 10544111$ (see [1]),
- **I2**: $|\theta(x) - x| \leq 0.2 \frac{x}{\log^2 x}$ for all $x \geq 3594641$ (see [1]),
- **I3**: $\psi_2(x) \geq 0.998684\sqrt{x}$ for all $x \geq 121$ (see [8]),
- **I4**: $\psi_2(x) \leq 1.4262\sqrt{x}$ for all $x \geq 1$ (see [6]),
- **I5**: $\pi(x) \leq \frac{x}{\log x - 1.1}$ for all $x \geq 60184$ (see [1]),
- **I6**: $\pi(x) > \frac{x}{\log x - 1}$ for all $x \geq 5393$ (see [1]).

We will also use some inequalities derived from the above ones. Our approach will be based on the following ideas: If a sharp inequality in x is valid for x greater than a large value M, if we want to use that inequality for, say, $\sqrt[3]{x}$, the inequality we derive will only be valid (without further arguments) for x larger than M^3. It is likely that M^3 is a very large number, sometimes being out of reach for computer checking of various relations. One way of dealing with this problem is to weaken a little bit the initial sharp inequality, and try to balance this loss by a smaller "starting point". This approach might lead us to inequalities which better fit the particular problems we are facing.

We will apply this kind of treatment to inequalities **I1** and **I2**. We will use some of the derived inequalities in the proofs of the properties in the next section. The good "balance" between the strength of an inequality and its "starting point" changes from problem to problem, and we picked the most suitable inequalities for our purposes from a list that we obtained by gradually weakening the mentioned inequalities. We will supply this list in the Appendix along with the way we obtained them; some of these inequalities might also be useful in other applications.

2. SUBADDITIVE AND LANDAU-TYPE PROPERTIES

When we discuss for a given function such properties as subadditivity, we may ask if the property holds for all possible values of the variables, or, if the answer to this first type of problem turns out to be negative, we may ask if the properties hold "asymptotically", i.e., for values of the variable which are greater than a given value M (specified, if possible, or unspecified, if we do not have a choice).

Let us start with

Proposition 2.1. *Let f be one of the functions θ, ψ or $\psi_k,\ k \geq 2$. There is no $M > 0$ such that $f(x+y) > f(x) + f(y)$ for all $x, y > M$ or $f(x+y) < f(x) + f(y)$ for all $x, y > M$.*

Proof. For $f = \psi$ or $f = \psi_k,\ k \geq 2$, since between $(2n)!$ and $(2n)! + n$ there are no prime powers, we have $f(x+y) < f(x) + f(y)$ for all $x = (2n)! - 1$ and $4 \leq y < n + 2$, so the first statement is true.

If, on the other hand, we consider an integer $x > 2$ and a prime power (of the suitable exponent) $y > x!$, then $f(x+y-1) > f(x) + f(y-1)$. Since we may take x as large as we please, the second statement follows.

For $f = \theta$, we consider in the above primes instead of prime powers. □

We may still ask if the considered functions have Landau-type properties (for all x if possible, or at least for large enough x).

We first show that θ and ψ fail to have such a property:

Proposition 2.2. *Let f be θ or ψ. There is no $M > 0$ such that $f(2x) \geq 2f(x)$ for all $x > M$ or $f(2x) \leq 2f(x)$ for all $x > M$.*

Proof. Suppose, for instance, that $\theta(2x) > 2\theta(x)$ for all x greater than a certain M. Ingham [3] proved that

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{1/2} \log\log\log x} \geq \frac{1}{2} \qquad \text{and} \qquad \liminf_{x\to\infty} \frac{\psi(x) - x}{x^{1/2} \log\log\log x} \leq -\frac{1}{2},$$

so the expression $\psi(x) - x$ changes sign infinitely many times. Using $\psi(x) - \theta(x) = O\sqrt{x}$ in the above relations, we find that $\theta(x) - x$ also changes sign infinitely many times. We can therefore find $a > M$ such that $\theta(a) > a$. Let $\alpha = \theta(a) - a > 0$. Our hypothesis leads to $\theta(2^n a) > 2^n \theta(a)$ for all $n \in \mathbb{N}^*$. We obtain

$$2^n \alpha = 2^n(\theta(a) - a) < \theta(2^n a) - 2^n a < 1.3 \frac{2^n a}{\log(2^n a)} = 1.3 \frac{2^n a}{\log a + n \log 2},$$

the last inequality being due to (4.17). We derive that

$$\alpha < \frac{1.3a}{\log a + n \log 2} \qquad \text{for all } n \geq 2.$$

Taking limits when $n \longrightarrow \infty$, we obtain the contradiction $\alpha \leq 0$.

Consequently, there is no M such that $\theta(2x) < 2\theta(x)$ for all $x > M$.

In order to prove that the inequality $\theta(2x) > 2\theta(x)$ cannot hold for all x greater than a value M, we repeat the above reasoning for $a > M$ such that $\theta(a) < a$.

As shown above, the expression $\psi(x) - x$ also changes sign infinitely many times. Inequalities **I4** and (4.17) give $|\psi(x) - x| < 2.7 \frac{x}{\log x}$ for all $x > 1$. Therefore, we may repeat the above reasoning to prove our claims for ψ. □

Let us now turn to the functions $\psi_k,\ k \geq 2$. We will show that these functions have Landau-type properties for x greater than a certain value (that we will actually specify in the case $k = 2$).

Since inequality **I4** is not sharp enough for the results we want to establish, we will first prove a few inequalities for ψ_2.

Taking into account the relation

$$\psi_2(x) = \psi(x) - \theta(x) = \theta(\sqrt{x}) + \theta(\sqrt[3]{x}) + \cdots + \theta(\sqrt[k]{x}), \quad \text{with } k = \left[\frac{\log x}{\log 2}\right],$$

we may write for every $m = \overline{1, k-1}$

$$\psi_2(x) \leq \theta(\sqrt{x}) + \theta(\sqrt[3]{x}) + \cdots + \theta(\sqrt[m]{x}) + \theta(\sqrt[m+1]{x}) \left(\frac{\log x}{\log 2} - m\right).$$

We use inequalities (4.27) and (4.30) to derive

$$\theta(\sqrt{x}) \leq \sqrt{x}\left(1 + \frac{8}{\log^2 x}\right) \quad \text{for all } x \geq 11950849 \text{ and}$$

$$\theta(\sqrt[3]{x}) \leq \sqrt[3]{x}\left(1 + \frac{31.5}{\log^2 x}\right) \quad \text{for all } x \geq 11697083.$$

As mentioned above, we would like to use sharper inequalities from the given table, or even the one of Dusart, but the derived inequalities would then only be valid (without further argument) for very large values of x, so they would be out of reach for computer checking.

For $a = \overline{4, m+1}$ we will use (4.32) to obtain

$$\theta(\sqrt[a]{x}) \leq \sqrt[a]{x}\left(1 + \frac{4a^2}{\log^2 x}\right) \quad \text{for all } x \geq 1.$$

Therefore, for all $x \geq 11950849$ and all $m \leq [\log x/\log 2] - 1$ we may write

$$\begin{aligned} \frac{\psi_2(x)}{\sqrt{x}} \leq 1 + \frac{8}{\log^2 x} + \frac{1}{\sqrt[6]{x}}\left(1 + \frac{31.5}{\log^2 x}\right) + \sum_{a=4}^{m} \frac{1}{\sqrt[2a]{x^{a-2}}}\left(1 + \frac{4a^2}{\log^2 x}\right) \\ + \frac{1}{\sqrt[2m+2]{x^{m-1}}}\left(1 + \frac{4(m+1)^2}{\log^2 x}\right)\left(\frac{\log x}{\log 2} - m\right). \end{aligned} \tag{2.1}$$

For all integers $a \geq 3$ the functions

$$x \mapsto \frac{1}{\sqrt[2a]{x^{a-2}}}\left(1 + \frac{4a^2}{\log^2 x}\right)$$

are monotonically decreasing, and $x \mapsto 8/\log^2 x$ is monotonically decreasing also. As far as

$$\frac{1}{\sqrt[2m+2]{x^{m-1}}}\left(1 + \frac{4(m+1)^2}{\log^2 x}\right)\left(\frac{\log x}{\log 2} - m\right)$$

is concerned, if $m \geq 4$ it is decreasing for $x \geq 2e^{2m}$. Therefore, the expression on the right hand side of the above inequality is in its turn monotonically decreasing for $x \geq 2e^{2m}$. Let us write (2.1) for $m = 11$. The value of the right hand side at $x = 168210000$ is less than $1.09999905 < 1.1$. Therefore, $\psi_2(x) < 1.1\sqrt{x}$ for all $x > 223230000$. Computer checking now gives

$$\psi_2(x) < 1.1\sqrt{x} \quad \text{for all } x > 2890319.61. \tag{2.2}$$

Now, using this inequality, further computer checking gives:

$$\psi_2(x) < 1.2\sqrt{x} \quad \text{for all } x > 80489.724, \tag{2.3}$$

(2.4) $$\psi_2(x) < 1.3\sqrt{x} \quad \text{for all } x > 2481.97, \text{ and}$$

(2.5) $$\psi_2(x) < 1.4\sqrt{x} \quad \text{for all } x > 374.6354.$$

Let us note that if we tried to prove inequality (2.2) using the inequality $\psi_2(x) < 1.001102\sqrt{x} + 3\sqrt[3]{x}$, valid for all $x > 0$ (see [8]), we would have faced a larger amount of computer checking.

We can now prove

Theorem 2.3. $\psi_2(2x) \leq 2\psi_2(x)$ *for all* $x \geq 25$.

Proof. Using **I3** and (2.4), we may for all $x > 1240.985$ write $\psi_2(2x) < 1.3\sqrt{2x} < 2 \cdot 0.998684\sqrt{x} < 2\psi_2(x)$. Computer checking for the remaining values completes the proof. □

Remark 2.4. For every integer $k \geq 3$ there exists $M_k > 0$ such that

$$\psi_k(2x) < 2\psi_k(x) \quad \text{for all } x > M_k.$$

Proof. Since $\psi_k(x) = \theta(\sqrt[k]{x}) + \theta(\sqrt[k+1]{x}) + \cdots + \theta(\sqrt[t]{x})$, $t = [\log x/\log 2]$, using (4.32) we derive inequalities of the type $\alpha\sqrt[k]{x} < \psi_k(x) < \beta\sqrt[k]{x}$ for any $\alpha < 1, \beta > 1$ and any x greater than a certain value M_k (for which we do not have a general formula, but which might be actually computed for specific values of k, α and β). Now, if we choose α and β such that $\beta\sqrt[k]{2} < \alpha$, the proof is similar to that of Theorem 2.3. □

Let us now turn to the function $\rho(x) = \psi(x)/\theta(x)$. This function is subadditive:

Proposition 2.5. $\rho(x+y) \leq \rho(x) + \rho(y)$ *for all* $x, y \geq 2$

Proof. Let $x, y \geq 2$. According to **I3**,**I4** and (4.17),

$$1 + \frac{0.998684\sqrt{t}}{t\left(1 + \frac{1.3}{\log t}\right)} < \rho(t) < 1 + \frac{1.43\sqrt{t}}{t\left(1 - \frac{1.3}{\log t}\right)} \quad \text{for all } t > e^{1.3} > 3.67.$$

Therefore,

$$\rho(x+y) < 1 + \frac{1.43}{\sqrt{x+y}\left(1 - \frac{1.3}{\log(x+y)}\right)}.$$

Since the function h that maps t to $\frac{1.43\sqrt{t}}{t(1-1.3/\log t)}$ is monotonically decreasing for $t > e^{1.3}$ and $h(5) < 1.49 < 2$, if $x + y \geq 5$ we obtain

$$\begin{aligned}\rho(x+y) &< 1 + \frac{1.43}{\sqrt{x+y}\left(1 - \frac{1.3}{\log(x+y)}\right)} \\ &< 1 + \frac{0.998684\sqrt{x}}{x\left(1 + \frac{1.3}{\log x}\right)} + 1 + \frac{0.998684\sqrt{y}}{y\left(1 + \frac{1.3}{\log y}\right)} \\ &< \rho(x) + \rho(y).\end{aligned}$$

If $x + y < 5$, then $x, y \in [2, 3)$. Therefore,

$$\rho(x+y) = \frac{2\log 2 + \log 3}{\log 2 + \log 3} < 2 = \rho(x) + \rho(y),$$

and the proof is complete. □

3. SUBMULTIPLICATIVITY-TYPE PROPERTIES

Let us start with

Proposition 3.1. $\rho(xy) < \rho(x) + \rho(y)$ *for all* $x, y \geq 2$.

Proof. Let $x, y \geq 2$. Using **I4** and (4.17), we derive that

$$\rho(xy) < 1 + \frac{1.43}{\sqrt{xy}\left(1 - \frac{1.23228}{\log(xy)}\right)}.$$

The function

$$h(t) = \frac{1.43}{1 - \frac{1.23228}{\log t}}$$

being monotonically decreasing for $t > e^{1.23228} = 3.4\dots$ and taking at $t = 11$ the value $2.94\dots < 3$, we may write for $xy \geq 11$

$$1 + \frac{1.43}{\sqrt{xy}\left(1 - \frac{1.23228}{\log(xy)}\right)} \leq 1 + \frac{2.95}{\sqrt{xy}} < 2 < \rho(x) + \rho(y).$$

Therefore, our claim is true for $xy \geq 11$.
Since the largest value of $\rho(t)$ for $t \in [2, 11)$ is $\rho(9) = 1.4\dots < 2$, we obtain $\rho(xy) < \rho(x) + \rho(y)$ for $xy < 11$ as well. □

A more meaningful property of ρ seems to be submultiplicativity:

Proposition 3.2. $\rho(xy) < \rho(x)\rho(y)$ *for all* $x, y \geq 4$.

Proof. Inequality **I3** and direct computation for $t < 121$ show that $\psi_2(x) \geq 0.635\sqrt{x}$ for all $x \geq 16$. Using **I4** and (4.17), we derive

$$1 + \frac{0.635}{\sqrt{x}\left(1 + \frac{1.23228}{\log x}\right)} \leq \rho(x) \leq 1 + \frac{1.43}{\sqrt{x}\left(1 - \frac{1.23228}{\log x}\right)}. \tag{3.1}$$

The function

$$x \mapsto 1 + \frac{0.635}{\left(1 + \frac{1.23228}{\log x}\right)}$$

is monotonically increasing, while

$$x \mapsto 1 + \frac{1.43}{\left(1 - \frac{1.23228}{\log x}\right)}$$

is monotonically decreasing. We derive

$$1 + \frac{0.4396}{\sqrt{x}} \leq \rho(x) \leq 1 + \frac{2.6}{\sqrt{x}} \quad \text{for all } x \geq 16. \tag{3.2}$$

Therefore, we obtain for all $x, y \geq 16$

$$\rho(xy) < 1 + \frac{2.6}{\sqrt{xy}} < \left(1 + \frac{0.4396}{\sqrt{x}}\right)\left(1 + \frac{0.4396}{\sqrt{y}}\right) < \rho(x)\rho(y).$$

Now let $x < 16$ or $y < 16$. Symmetry allows us to only consider the case $x < 16$. If $xy \geq 2482$ and $y \geq 1241$, we use (2.4) and (4.13) to get

$$\frac{1.3}{\sqrt{xy}\left(1 - \frac{0.3}{\log(xy)}\right)} \geq \rho(xy).$$

Let us consider the functions

$$f(x) = \frac{0.998684}{\left(1 + \frac{0.3}{\log x}\right)} \quad \text{and} \quad g(x) = 1 + \frac{1.3}{\sqrt{2}\left(1 - \frac{0.3}{\log(2x)}\right)}.$$

f is monotonically increasing, while g is monotonically decreasing. Therefore,

$$\begin{aligned}\rho(x)\rho(y) \geq \rho(y) \geq 1 + \frac{f(1241)}{\sqrt{y}} \geq 1 + \frac{0.958}{\sqrt{y}} \\ > 1 + \frac{0.953}{\sqrt{y}} \geq 1 + \frac{g(2482)}{\sqrt{y}} \geq 1 + \frac{1.3}{\sqrt{2y}\left(1 - \frac{0.3}{\log(2y)}\right)} \\ \geq 1 + \frac{1.3}{\sqrt{xy}\left(1 - \frac{0.3}{\log(xy)}\right)} \geq \rho(xy).\end{aligned}$$

Computer checking for the remaining cases completes the proof. □

Remark 3.3.

(a) If $x, y \in [2, 4)$, $\rho(x)\rho(y) = 1 < \rho(xy)$.
(b) $\rho(2)\rho(x) \geq \rho(2x)$ for all $x \geq 25$.
(c) $\rho(3)\rho(x) \geq \rho(3x)$ for all $x \geq 23/3$.

Let us finish by investigating a similar property for $\pi(x)$. Ishikawa [4] proved that $\pi(x+y) < \pi(x)\pi(y)$ for all integers $x, y \geq 5$. We prove here

Theorem 3.4. *For all* $x, y \geq \sqrt{53}$, $\pi(x)\pi(y) \leq \pi(xy)$.

Proof. We weaken **I5** by means of computer checking to

$$\pi(x) < \frac{x}{\log x - 1.12} \quad \text{for all } x \geq 5.$$

Weakening also **I6**, we obtain

$$\pi(x) > \frac{x}{\log x - 0.145} \quad \text{for all } x \geq 17.$$

We derive that for $x, y \geq e^{2.12+\sqrt{3.095}} = 48.38845\ldots$

$$(\log x - 2.12)(\log y - 2.12) \geq 3.095 = 3.24 - 0.145,$$

so

$$\log x + \log y - 0.145 \leq (\log x - 1.12)(\log y - 1.12).$$

Consequently,

$$\pi(x)\pi(y) \leq \frac{x}{\log x - 1.12}\frac{y}{\log y - 1.12} \leq \frac{xy}{\log xy - 0.145} \leq \pi(xy).$$

Now, if $x < 48.38845\ldots$ or $y < 48.38845\ldots$, the symmetry of the required relation allows us to only consider the case $x < 48.38845\ldots$. We will consider the cases $x \in [p_n, p_{n+1})$, $n = \overline{1, 15}$. Computation shows that for these values of n we have

$$1 + \frac{n \log p_{n+1} + 0.12 p_n}{p_n - n} \leq 4.579.$$

Therefore, for $y \geq e^{4.579} = 97.4\ldots$ we have the inequality $(p_n - n)\log y \geq n \log p_{n+1} + 1.12 p_n - n$, otherwise written as $p_n(\log y - 1.12) \geq n(\log p_{n+1} + \log y - 1)$. Using this relation and **I6** we derive for $y \geq 97.5$ and $xy \geq 5393$

$$\pi(x)\pi(y) \leq \frac{ny}{\log y - 1.12} \leq \frac{p_n y}{\log p_{n+1} + \log y - 1} \leq \frac{xy}{\log(xy) - 1} \leq \pi(xy).$$

Computer checking for the remaining cases completes the proof. □

Remark 3.5. In fact, computer checking shows that for $x, y > 0$ we only have three "small" regions where $\pi(xy) < \pi(x)\pi(y)$:

- $x \in [5, 7)$, $y \in [7, 37/5)$, $xy < 37$;
- $x \in [7, 37/5)$, $y \in [5, 7)$, $xy < 37$, and
- $x, y \in [7, 11)$, $xy < 53$.

Remark 3.6. The relation $\pi(xy) \geq \pi(x)\pi(y)$ holds for all positive integers x, y with the following three exceptions: $x = 5, y = 7$; $x = 7, y = 5$ and $x = y = 7$.

4. APPENDIX: USEFUL INEQUALITIES

Proposition 4.1.

$$|\theta(x) - x| \leq 0.007\frac{x}{\log x} \quad \text{for all } x \geq 10443773 \tag{4.1}$$

Proof. According to **I1**, relation (4.1) holds for all $x \geq 10544111$, but it may also be valid for some smaller values of x.

Let us consider the functions

$$\alpha(x) = x + 0.007\frac{\log x}{x} - \theta(x) \quad \text{and} \quad \beta(x) = x - 0.007\frac{\log x}{x} - \theta(x).$$

Relation (4.1) is then equivalent to

$$\alpha(x) \geq 0 \tag{4.2}$$

and

$$\beta(x) \leq 0. \tag{4.3}$$

Since the function $x + 0.007x/\log x$ is monotonically increasing for $x > 1$, the only opportunities for α to decrease are the prime numbers, and its local minima have the shape $\alpha(p_n)$. Therefore, relation (4.2) holds for $x \geq 2$ if and only if it holds for $p_{\pi(x)}$. Consequently, if p_n is the greatest prime for which (4.2) fails, (4.2) will be valid for all $x \geq p_{n+1}$.
As far as β is concerned, the function $x - 0.007x/\log x$ being in its turn monotonically increasing for $x > 1$, the only reasons for β to decrease are also the occurrences of prime numbers. Since, according to **I2**, β eventually settles to negative values, the last positive value of

$$p_{n+1} - 0.007\frac{\log p_{n+1}}{p_{n+1}} - \theta(p_n)$$

will show that relation (4.3) is valid for all $x \geq p_{n+1}$.

Performing the computer checking as suggested by the above considerations, we obtain the claim of the proposition. □

Let us note that for the particular values of x in the above proof, the result of Schoenfeld $\theta(x) < x$ for all $x < 10^{11}$ [9] allows us to only consider the inequalities involving the function β.

Similar reasoning and computation lead us to the inequalities (4.4) – (4.32) below:

$$|\theta(x) - x| \leq 0.008\frac{x}{\log x} \quad \text{for all } x \geq 10358041; \tag{4.4}$$

$$|\theta(x) - x| \leq 0.009\frac{x}{\log x} \quad \text{for all } x \geq 6695617; \tag{4.5}$$

$$|\theta(x) - x| \le 0.01\frac{x}{\log x} \quad \text{for all } \ge 5880037; \tag{4.6}$$

$$|\theta(x) - x| \le 0.02\frac{x}{\log x} \quad \text{for all } x \ge 1099247; \tag{4.7}$$

$$|\theta(x) - x| \le 0.03\frac{x}{\log x} \quad \text{for all } x \ge 467867; \tag{4.8}$$

$$|\theta(x) - x| \le 0.04\frac{x}{\log x} \quad \text{for all } x \ge 302969; \tag{4.9}$$

$$|\theta(x) - x| \le 0.05\frac{x}{\log x} \quad \text{for all } x \ge 175829; \tag{4.10}$$

$$|\theta(x) - x| \le 0.1\frac{x}{\log x} \quad \text{for all } x \ge 32297; \tag{4.11}$$

$$|\theta(x) - x| \le 0.2\frac{x}{\log x} \quad \text{for all } x \ge 5407; \tag{4.12}$$

$$|\theta(x) - x| \le 0.3\frac{x}{\log x} \quad \text{for all } x \ge 1973; \tag{4.13}$$

$$|\theta(x) - x| \le 0.4\frac{x}{\log x} \quad \text{for all } x \ge 809; \tag{4.14}$$

$$|\theta(x) - x| \le 0.5\frac{x}{\log x} \quad \text{for all } x \ge 563; \tag{4.15}$$

$$|\theta(x) - x| \le \frac{x}{\log x} \quad \text{for all } x \ge 41; \tag{4.16}$$

$$|\theta(x) - x| \le 1.23227674\frac{x}{\log x} \quad \text{for all } x > 1; \tag{4.17}$$

$$|\theta(x) - x| \le 0.3\frac{x}{\log^2 x} \quad \text{for all } x \ge 1091021; \tag{4.18}$$

$$|\theta(x) - x| \le 0.4\frac{x}{\log^2 x} \quad \text{for all } x \ge 467629; \tag{4.19}$$

$$|\theta(x) - x| \le 0.5\frac{x}{\log^2 x} \quad \text{for all } x \ge 303283; \tag{4.20}$$

$$|\theta(x) - x| \le 0.6\frac{x}{\log^2 x} \quad \text{for all } x \ge 175837; \tag{4.21}$$

$$|\theta(x) - x| \le 0.7\frac{x}{\log^2 x} \quad \text{for all } x \ge 88807; \tag{4.22}$$

$$|\theta(x) - x| \leq 0.8\frac{x}{\log^2 x} \quad \text{for all } x \geq 70111; \tag{4.23}$$

$$|\theta(x) - x| \leq 0.9\frac{x}{\log^2 x} \quad \text{for all } x \geq 32363; \tag{4.24}$$

$$|\theta(x) - x| \leq \frac{x}{\log^2 x} \quad \text{for all } x \geq 32299; \tag{4.25}$$

$$|\theta(x) - x| \leq 1.5\frac{x}{\log^2 x} \quad \text{for all } x \geq 11779; \tag{4.26}$$

$$|\theta(x) - x| \leq 2\frac{x}{\log^2 x} \quad \text{for all } x \geq 3457; \tag{4.27}$$

$$|\theta(x) - x| \leq 2.5\frac{x}{\log^2 x} \quad \text{for all } x \geq 1429; \tag{4.28}$$

$$|\theta(x) - x| \leq 3\frac{x}{\log^2 x} \quad \text{for all } x \geq 569; \tag{4.29}$$

$$|\theta(x) - x| \leq 3.5\frac{x}{\log^2 x} \quad \text{for all } x \geq 227; \tag{4.30}$$

$$|\theta(x) - x| \leq 3.9\frac{x}{\log^2 x} \quad \text{for all } x \geq 59; \tag{4.31}$$

$$|\theta(x) - x| \leq 4\frac{x}{\log^2 x} \quad \text{for all } x > 1. \tag{4.32}$$

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