

A GENERAL INEQUALITY OF NGÔ-THANG-DAT-TUAN TYPE

TAMÁS F. MÓRI

Department of Probability Theory and Statistics
Loránd Eötvös University
Pázmány P. s. 1/C, H-1117 Budapest, Hungary
EMail: moritamas@ludens.elte.hu

Received: 04 November, 2008

Accepted: 14 January, 2009

Communicated by: [S.S. Dragomir](#)

2000 AMS Sub. Class.: 26D15

Key words: Integral inequality, Young inequality.

Abstract: In the present note a general integral inequality is proved in the direction that was initiated by Q. A. Ngô et al [Note on an integral inequality, *J. Inequal. Pure and Appl. Math.*, 7(4) (2006), Art.120].

Acknowledgements: This research has been supported by the Hungarian National Foundation for Scientific Research, Grant No. K 67961.



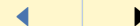
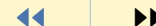
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-Dat-Tuan Type

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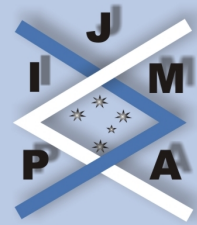
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1. Introduction

In their paper [7] Ngô, Tang, Dat, and Tuan proved the following inequalities. If f is a nonnegative, continuous function on $[0, 1]$ satisfying

$$\int_x^1 f(t) dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1],$$

then

$$\int_0^1 f(x)^{\alpha+1} dx \geq \int_0^1 x^\alpha f(x) dx, \quad \int_0^1 f(x)^{\alpha+1} dx \geq \int_0^1 x f(x)^\alpha dx$$

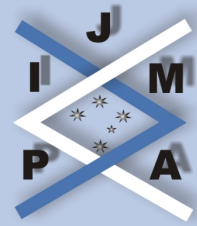
for every positive number α .

This result has initiated a series of papers containing various extensions and generalizations [1, 2, 3, 5, 6]. Among others, it turns out that the conditions above imply

$$\int_0^1 f(x)^{\alpha+\beta} dx \geq \int_0^1 x^\alpha f(x)^\beta dx$$

for every $\alpha > 0$, $\beta \geq 1$, which answered an open question of Ngô et al. in the positive [3].

The aim of this note is to formulate and prove a further generalization. It is presented in Section 2. Section 3 contains corollaries, which are immediate extensions of a couple of known results.



2. Main Result

Theorem 2.1. Let $u, v : [0, +\infty) \rightarrow \mathbb{R}$ be nonnegative, differentiable, increasing functions. Suppose that $u'(t)$ is positive and increasing, and $\frac{v'(t)u(t)}{u'(t)}$ is increasing for $t > 0$. Let f and g be nonnegative, integrable functions defined on the interval $[a, b]$. Suppose g is increasing, and

$$(2.1) \quad \int_x^b g(t) dt \leq \int_x^b f(t) dt$$

holds for every $x \in [a, b]$. Then

$$(2.2) \quad \int_a^b u(g(t))v(g(t)) dt \leq \int_a^b u(f(t))v(g(t)) dt \leq \int_a^b u(f(t))v(f(t)) dt,$$

$$(2.3) \quad \int_a^b u(g(t))v(f(t)) dt \leq \int_0^1 u(f(t))v(f(t)) dt,$$

provided the integrals are finite.

Remark 1.

1. Here and throughout, by *increasing* we always mean *nondecreasing*.
2. Note that continuity of f or g is not required.
3. Unfortunately, the other inequality

$$(2.4) \quad \int_0^1 u(g(t))v(g(t)) dt \leq \int_0^1 u(g(t))v(f(t)) dt,$$

which seems to be missing from (2.3), is not necessarily valid. Set $[a, b] = [0, 1]$, $u(t) = t^\beta$, $v(t) = t^\alpha$, with $\alpha > 0$, $\beta > 1$. Let $g(t) = t$, and $f(t) = 1$, if

$1/2 \leq t \leq 1$, and zero otherwise. Then all the conditions of Theorem 2.1 are satisfied, and

$$\int_a^b u(g(t))v(g(t)) dt = \int_0^1 t^{\alpha+\beta} dt = \frac{1}{\alpha + \beta + 1},$$

$$\int_a^b u(g(t))v(f(t)) dt = \int_{1/2}^1 t^\beta dt = \frac{1}{\beta + 1} \left(1 - \frac{1}{2^{\beta+1}} \right).$$

It is easy to see that (2.4) does not hold if $\alpha < \frac{\beta+1}{2^{\beta+1}-1}$.

Although f is discontinuous in this counterexample, it is not continuity that can help, for f can be approximated in L_1 with continuous (piecewise linear) functions.

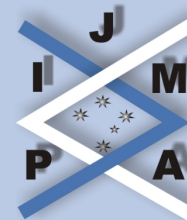
For the proof we shall need the following lemmas of independent interest.

Lemma 2.2. *Let f and g be nonnegative integrable functions on $[a, b]$ that satisfy (2.1). Let $h : [a, b] \rightarrow \mathbb{R}$ be nonnegative and increasing. Then*

$$(2.5) \quad \int_a^b h(t)g(t) dt \leq \int_a^b h(t)f(t) dt.$$

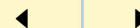
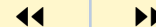
Proof. We can suppose that u is right continuous, because it can only have countably many discontinuities, so replacing $u(t)$ with $u(t+)$ in these points does not affect the integrals. Clearly, $h(t) = h(a) + \int_{(a,t]} dh(s)$, hence

$$\begin{aligned} \int_a^b h(t)g(t) dt &= \int_a^b \left(h(a) + \int_{a+}^{t+} dh(s) \right) g(t) dt \\ &= h(a) \int_a^b g(t) dt + \int_a^b \int_a^b I(s \leq t)g(t) dh(s) dt, \end{aligned}$$



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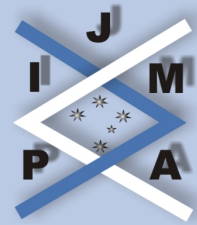


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where $I(\cdot)$ stands for the characteristic function of the set in brackets. By Fubini's theorem we can interchange the order of the integration, obtaining

$$\begin{aligned}\int_a^b h(t)g(t) dt &= h(a) \int_a^b g(t) dt + \int_a^b \int_a^b I(s \leq t)g(t) dt dh(s) \\ &= h(a) \int_a^b g(t) dt + \int_a^b \left(\int_t^b g(s) ds \right) dh(s).\end{aligned}$$

Remembering condition (2.1), we can write

$$\begin{aligned}\int_a^b h(t)g(t) dt &\leq h(a) \int_a^b f(t) dt + \int_a^b \left(\int_t^b f(s) ds \right) dh(s) \\ &= \int_a^b h(t)f(t) dt,\end{aligned}$$

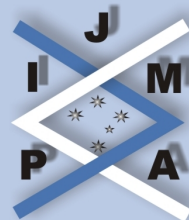
as required. □

Lemma 2.3. *Let f and g be as in Theorem 2.1, and let $v : [0, +\infty) \rightarrow \mathbb{R}$ be a nonnegative increasing function. Define $V(x) = \int_0^x v(t) dt$, $x \geq 0$. Then*

$$(2.6) \quad \int_a^b V(g(t)) dt \leq \int_a^b V(f(t)) dt.$$

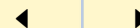
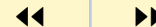
Equivalently, we can say that inequality (2.6) is valid for all increasing convex functions $V : [0, +\infty) \rightarrow \mathbb{R}$.

Proof. We can suppose that the right-hand side is finite, for the integrand on the left-hand side is bounded. Let V^* denote the Legendre transform of V , that is, $V^*(x) = \int_0^x v^{-1}(t) dt$, where $v^{-1}(t) = \inf\{s : v(s) \geq t\}$ is the (right continuous) generalized inverse of v . Then by the Young inequality [4] we have that $xy \leq V(x) + V^*(y)$



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holds for every $x, y \geq 0$, with equality if and only if $v(x-) \leq y \leq v(x+)$. Hence, by substituting $x = f(t)$ and $y = v(g(t))$ we obtain

$$(2.7) \quad f(t)v(g(t)) \leq V(f(t)) + V^*(v(g(t))) = V(f(t)) + g(t)v(g(t)) - V(g(t)).$$

By integrating this we get that

$$(2.8) \quad \int_a^b f(t)v(g(t)) dt \leq \int_a^b V(f(t)) dt + \int_a^b g(t)v(g(t)) dt - \int_a^b V(g(t)) dt.$$

With $h(t) = v(g(t))$ Lemma 2.2 yields

$$(2.9) \quad \int_a^b g(t)v(g(t)) dt \leq \int_0^1 f(t)v(g(t)) dt.$$

Combining (2.8) with (2.9) we arrive at (2.6). \square

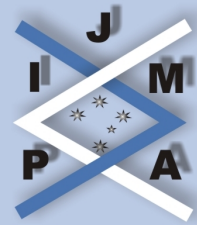
Proof of Theorem 2.1. First we prove for the case where $u(t) = t$. Then $t v'(t)$ has to be increasing.

The first inequality of (2.2) has already been proved in (2.9). On the other hand, from the Young inequality, similarly to (2.7) we can derive that

$$\begin{aligned} f(t)v(g(t)) &\leq V(f(t)) + V^*(v(g(t))) \\ &= V^*(v(g(t))) + f(t)v(f(t)) - V^*(v(f(t))). \end{aligned}$$

Therefore,

$$(2.10) \quad \int_a^b f(t)v(g(t)) dt \leq \int_a^b V^*(v(g(t))) dt + \int_a^b f(t)v(f(t)) dt - \int_a^b V^*(v(f(t))) dt.$$



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Here

$$V^*(v(x)) = xv(x) - V(x) = \int_0^x [(tv(t))' - v(t)] dt = \int_0^x tv'(t) dt,$$

thus Lemma 2.3 can be applied with $V^*(v(x))$ in place of $V(x)$.

$$(2.11) \quad \int_a^b V^*(v(g(t))) dt \leq \int_a^b V^*(v(f(t))) dt.$$

Now we can complete the proof of the second inequality of (2.2) by plugging (2.11) back into (2.10).

Next, since $[f(t) - g(t)][v(f(t)) - v(g(t))] \geq 0$, we obtain that

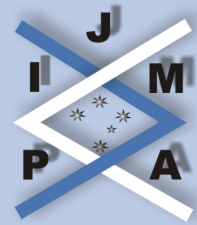
$$\int_0^1 f(t)v(f(t)) dt - \int_0^1 g(t)v(f(t)) dt \geq \int_0^1 f(t)v(g(t)) dt - \int_0^1 g(t)v(g(t)) \geq 0,$$

by (2.2). This proves (2.3).

For the general case, we first apply Lemma 2.3 on the interval $[x, b]$, with $u(t)$ in place of $V(t)$. We can see that $u(f(t))$ and $u(g(t))$ satisfy condition (2.1). Now, u is invertable. Let $w(t) = v(u^{-1}(t))$, then

$$w'(t) = \frac{v'(u^{-1}(t))}{u'(u^{-1}(t))},$$

hence, by the conditions of Theorem 2.1, $tw'(t)$ is increasing. The proof can be completed by applying the particular case just proved to the functions $u(f(t))$ and $u(g(t))$, with w in place of v . \square



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3. Corollaries, Particular Cases

In this section we specialize Theorem 2.1 to obtain some well known results that were mentioned in the Introduction. First, let $u(x) = x^\beta$ and $v(x) = x^\alpha$ with $\alpha > 0$ and $\beta \geq 1$. They clearly satisfy the conditions of Theorem 2.1.

Corollary 3.1. *Let f and g be nonnegative, integrable functions defined on the interval $[a, b]$. Suppose g is increasing, and*

$$(3.1) \quad \int_x^b g(t) dt \leq \int_x^b f(t) dt$$

holds for every $x \in [a, b]$. Then, for arbitrary $\alpha > 0$ and $\beta \geq 1$ we have

$$(3.2) \quad \int_a^b g(t)^{\alpha+\beta} dt \leq \int_a^b g(t)^\alpha f(t)^\beta dt \leq \int_a^b f(t)^{\alpha+\beta} dt,$$

$$(3.3) \quad \int_a^b f(t)^\alpha g(t)^\beta dt \leq \int_a^b f(t)^{\alpha+\beta} dt.$$

Next, change α , β , $f(t)$, and $g(t)$ in Corollary 3.1 to α/β , 1, $f(t)^\beta$ and $g(t)^\beta$, respectively.

Corollary 3.2. *Let α and β be arbitrary positive numbers. Let f and g satisfy the conditions of Corollary 3.1, but, instead of (3.1) suppose that*

$$(3.4) \quad \int_x^b g(t)^\beta dt \leq \int_x^b f(t)^\beta dt$$

holds for every $x \in [a, b]$. Then inequalities (3.2) and (3.3) remain valid.

In particular, for the case of $[a, b] = [0, 1]$, $g(t) = t$ Corollary 3.1 yields Theorem 2.3 of [3], and Corollary 3.2 implies Theorem 2.1 of [5]. If, in addition, we set $\beta = 1$, Corollary 3.1 gives Theorems 3.2 and 3.3 of [7].

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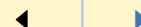
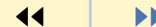
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