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FURTHER DEVELOPMENT OF AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

WEN-JUN LIU, GUO-SHENG CHENG, AND CHUN-CHENG LI

College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China
lwjboy@126.com

gshcheng@sohu.com

lichunchengcxy@126.com

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ABSTRACT. In this paper, we generalize an open problem posed by Q. A. Ngô et al. in the paper Notes on an Integral Inequality, *J. Inequal. in Pure and Appl. Math.*, 7(4)(2006), Art. 120 and give an affirmative answer to it without the differentiable restriction on f.

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1. Introduction

Recently, in the paper [6] Ngô et al. studied some very interesting integral inequalities and proved the following result.

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.1}$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx, \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx, \tag{1.3}$$

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306-07

hold for every positive real number $\alpha > 0$.

Next, they proposed the following open problem:

Problem 1.1. Let $f(x)$ be a continuous function on $[0,1]$ satisfying

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.4}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \tag{1.5}$$

holds for α and β?

We note that, as an open problem, the condition (1.4) may result in an unreasonable restriction on $f(x)$. We remove it herein and propose another more general open problem:

Problem 1.2. Under what conditions does the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^\alpha f^\beta(x)dx, \tag{1.6}$$

hold for $a,\ b,\ \alpha$ and β?

Shortly after the paper [6] was published, Liu et al. [5] gave an affirmative answer to Problem 1.2 for the case $a = 0$ and obtained the following result:

Theorem 1.2. *Let $f(x) \geq 0$ be a continuous function on $[0,b],\ b \geq 0$ satisfying*

$$\int_x^b f^\beta(t)dt \geq \int_x^b t^\beta\,dt, \quad \forall\, x \in [0,b]. \tag{1.7}$$

Then the inequality

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx, \tag{1.8}$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

Almost at the same time, Bougoffa [1] also gave an answer to Problem 1.2 and established the following result (We correct it here according to the presence of the corrigendum in [2]):

Theorem 1.3. *Let $f(x) \geq 0$ be a function, continuous on $[a,b]$ and differentiable in (a,b). If*

$$\int_x^b f(t)dt \geq \int_x^b (t-a)\,dt, \quad \forall\, x \in [a,b] \tag{1.9}$$

and

$$f'(x) \leq 1, \quad \forall\, x \in (a,b),$$

then the inequality (1.6) holds for every positive real number $\alpha > 0$ and $\beta > 0$.

Very recently, Boukerrioua and Guezane-Lakoud [3] obtained the following result:

Theorem 1.4. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.10}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \tag{1.11}$$

holds for $\alpha > 0$ *and* $\beta \geq 1$.

Comparing the above three results, we note that: the condition (1.7) was required in Theorem 1.2, a differentiability condition was restricted on f in Theorem 1.3 while $\beta \geq 1$ was demanded in Theorem 1.4. In this paper, we will give an affirmative answer to Problem 1.2 without the differentiable restriction on f by improving the methods of [5], [6] and [3]. Our main result is Theorem 2.1 which will be proved in Section 2.

2. Main Results and Proofs

Theorem 2.1. *Let* $f(x) \geq 0$ *be a continuous function on* $[a, b]$ *satisfying*

$$\int_x^b f^{\min\{1,\beta\}}(t)dt \geq \int_x^b (t-a)^{\min\{1,\beta\}}\, dt, \quad \forall\, x \in [a, b]. \tag{2.1}$$

Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx, \tag{2.2}$$

holds for every positive real number $\alpha > 0$ *and* $\beta > 0$.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 ([6], General Cauchy inequality)**.** *Let* α *and* β *be positive real numbers satisfying* $\alpha + \beta = 1$. *Then for all positive real numbers* x *and* y, *we always have*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}. \tag{2.3}$$

Lemma 2.3. *Under the conditions of Theorem 2.1, we have*

$$\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx \geq \frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1}. \tag{2.4}$$

Proof. We divide the proof into two steps according to the different intervals of β.
Case of $0 < \beta \leq 1$: Integrating by parts, we have

$$\begin{aligned}
&\int_a^b (x-a)^{\alpha-1}\left(\int_x^b f^{\beta}(t)dt\right)dx \\
&\qquad = \frac{1}{\alpha}\int_a^b \left(\int_x^b f^{\beta}(t)dt\right) d(x-a)^{\alpha} \\
&\qquad = \frac{1}{\alpha}\left[(x-a)^{\alpha}\int_x^b f^{\beta}(t)dt\right]_{x=a}^{x=b} + \frac{1}{\alpha}\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx \\
&\qquad = \frac{1}{\alpha}\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx.
\end{aligned}$$

which yields

$$\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx = \alpha \int_a^b (x-a)^{\alpha-1}\left(\int_x^b f^{\beta}(t)dt\right)dx. \tag{2.5}$$

On the other hand, by (2.1), we get

$$\begin{aligned}\int_a^b (x-a)^{\alpha-1}\left(\int_x^b f^{\beta}(t)dt\right)dx &\geq \int_a^b (x-a)^{\alpha-1}\left(\int_x^b (t-a)^{\beta}\,dt\right)dx \\ &= \frac{1}{\beta+1}\int_a^b (x-a)^{\alpha-1}\left[(b-a)^{\beta+1}-(x-a)^{\beta+1}\right]dx \\ &= \frac{(b-a)^{\alpha+\beta+1}}{\alpha(\alpha+\beta+1)}.\end{aligned}$$

Therefore, (2.4) holds.

Case of $\beta > 1$: We note that the following result has been proved in the first case

$$\int_a^b (x-a)^{\alpha} f(x)dx \geq \frac{(b-a)^{\alpha+2}}{\alpha+2}. \tag{2.6}$$

Using Lemma 2.2, we get

$$\frac{1}{\beta}f^{\beta}(x)+\frac{\beta-1}{\beta}(x-a)^{\beta} \geq f(x)(x-a)^{\beta-1}. \tag{2.7}$$

Multiplying both sides of (2.7) by $(x-a)^{\alpha}$ and integrating the resultant inequality from a to b, we obtain

$$\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx + (\beta-1)\int_a^b (x-a)^{\alpha+\beta}dx \geq \beta\int_a^b (x-a)^{\alpha+\beta-1}f(x)dx, \tag{2.8}$$

which implies

$$\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx + \frac{\beta-1}{\alpha+\beta+1}(b-a)^{\alpha+\beta+1} \geq \beta\int_a^b (x-a)^{\alpha+\beta-1}f(x)dx. \tag{2.9}$$

Moreover, by using (2.6), we get

$$\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx + \frac{\beta-1}{\alpha+\beta+1}(b-a)^{\alpha+\beta+1} \geq \frac{\beta}{\alpha+\beta+1}(b-a)^{\alpha+\beta+1}, \tag{2.10}$$

which implies (2.4). □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.2 again, we obtain

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\alpha}{\alpha+\beta}(x-a)^{\alpha+\beta}dx \geq (x-a)^{\alpha}f^{\beta}(x)dx, \tag{2.11}$$

which gives

$$\beta\int_a^b f^{\alpha+\beta}(x)dx + \alpha\int_a^b (x-a)^{\alpha+\beta}dx \geq (\alpha+\beta)\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx. \tag{2.12}$$

Moreover, by using Lemma 2.3, we get

$$\begin{aligned}(\alpha+\beta)\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx &= \alpha\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx + \beta\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx \\ &\geq \alpha\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx,\end{aligned}$$

that is

$$\beta \int_a^b f^{\alpha+\beta}(x)dx + \alpha \frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \geq \alpha \frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx, \tag{2.13}$$

which completes the proof. □

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