

# ON JENSEN TYPE INEQUALITIES WITH ORDERED VARIABLES

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*Abstract:* In this paper, we present some basic results concerning an extension of Jensen type inequalities with ordered variables to functions with inflection points, and then give several relevant applications of these results.



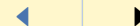
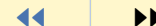
Jensen Type Inequalities  
With Ordered Variables

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## 1. Basic Results

An  $n$ -tuple of real numbers  $X = (x_1, x_2, \dots, x_n)$  is said to be increasingly ordered if  $x_1 \leq x_2 \leq \dots \leq x_n$ . If  $x_1 \geq x_2 \geq \dots \geq x_n$ , then  $X$  is decreasingly ordered.

In addition, a set  $X = (x_1, x_2, \dots, x_n)$  with  $\frac{x_1 + x_2 + \dots + x_n}{n} = s$  is said to be  $k$ -arithmetic ordered if  $k$  of the numbers  $x_1, x_2, \dots, x_n$  are smaller than or equal to  $s$ , and the other  $n - k$  are greater than or equal to  $s$ . On the assumption that  $x_1 \leq x_2 \leq \dots \leq x_n$ ,  $X$  is  $k$ -arithmetic ordered if

$$x_1 \leq \dots \leq x_k \leq s \leq x_{k+1} \leq \dots \leq x_n.$$

It is easily seen that

$$X_1 = (s - x_1 + x_{k+1}, s - x_2 + x_{k+2}, \dots, s - x_n + x_k)$$

is a  $k$ -arithmetic ordered set if  $X$  is increasingly ordered, and is an  $(n - k)$ -arithmetic ordered set if  $X$  is decreasingly ordered.

Similarly, an  $n$ -tuple of positive real numbers  $A = (a_1, a_2, \dots, a_n)$  with  $\sqrt[k]{a_1 a_2 \dots a_n} = r$  is said to be  $k$ -geometric ordered if  $k$  of the numbers  $a_1, a_2, \dots, a_n$  are smaller than or equal to  $r$ , and the other  $n - k$  are greater than or equal to  $r$ . Notice that

$$A_1 = \left( \frac{a_{k+1}}{a_1}, \frac{a_{k+2}}{a_2}, \dots, \frac{a_k}{a_n} \right)$$

is a  $k$ -geometric ordered set if  $A$  is increasingly ordered, and is an  $(n - k)$ -geometric ordered set if  $A$  is decreasingly ordered.

**Theorem 1.1.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $f(u)$  be a function on a real interval  $I$ , which is convex for  $u \geq s$ ,  $s \in I$ , and satisfies

$$f(x) + kf(y) \geq (1 + k)f(s)$$



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for any  $x, y \in I$  such that  $x \leq y$  and  $x + ky = (1 + k)s$ . If  $x_1, x_2, \dots, x_n \in I$  such that

$$\frac{x_1 + x_2 + \dots + x_n}{n} = S \geq s$$

and at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are smaller than or equal to  $S$ , then

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(S).$$

*Proof.* We will consider two cases:  $S = s$  and  $S > s$ .

**A.** Case  $S = s$ . Without loss of generality, assume that  $x_1 \leq x_2 \leq \dots \leq x_n$ . Since  $x_1 + x_2 + \dots + x_n = ns$ , and at least  $n - k$  of the numbers  $x_1, x_2, \dots, x_n$  are smaller than or equal to  $s$ , there exists an integer  $n - k \leq i \leq n - 1$  such that  $(x_1, x_2, \dots, x_n)$  is an  $i$ -arithmetic ordered set, i.e.

$$x_1 \leq \dots \leq x_i \leq s \leq x_{i+1} \leq \dots \leq x_n.$$

By Jensen's inequality for convex functions,

$$f(x_{i+1}) + f(x_{i+2}) + \dots + f(x_n) \geq (n - i)f(z),$$

where

$$z = \frac{x_{i+1} + x_{i+2} + \dots + x_n}{n - i}, \quad z \geq s, \quad z \in I.$$

Thus, it suffices to prove that

$$f(x_1) + \dots + f(x_i) + (n - i)f(z) \geq nf(s).$$

Let  $y_1, y_2, \dots, y_i \in I$  be defined by

$$x_1 + ky_1 = (1 + k)s, \quad x_2 + ky_2 = (1 + k)s, \quad \dots, \quad x_i + ky_i = (1 + k)s.$$



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We will show that  $z \geq y_1 \geq y_2 \geq \cdots \geq y_i \geq s$ . Indeed, we have

$$y_1 \geq y_2 \geq \cdots \geq y_i,$$
$$y_i - s = \frac{s - x_i}{k} \geq 0,$$

and

$$\begin{aligned} ky_1 &= (1+k)s - x_1 \\ &= (1+k-n)s + x_2 + \cdots + x_n \\ &\leq (k+i-n)s + x_{i+1} + \cdots + x_n \\ &= (k+i-n)s + (n-i)z \leq kz. \end{aligned}$$

Since  $z \geq y_1 \geq y_2 \geq \cdots \geq y_i \geq s$  implies  $y_1, y_2, \dots, y_i \in I$ , by hypothesis we have

$$\begin{aligned} f(x_1) + kf(y_1) &\geq (1+k)f(s), \\ f(x_2) + kf(y_2) &\geq (1+k)f(s), \\ &\dots\dots\dots \\ f(x_i) + kf(y_i) &\geq (1+k)f(s). \end{aligned}$$

Adding all these inequalities, we get

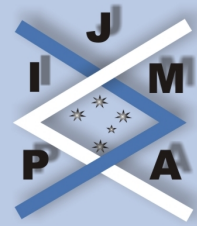
$$f(x_1) + f(x_2) + \cdots + f(x_i) + k[f(y_1) + f(y_2) + \cdots + f(y_i)] \geq i(1+k)f(s).$$

Consequently, it suffices to show that

$$pf(z) + (i-p)f(s) \geq f(y_1) + f(y_2) + \cdots + f(y_i),$$

where  $p = \frac{n-i}{k} \leq 1$ . Let  $t = pz + (1-p)s$ ,  $s \leq t \leq z$ . Since the decreasingly ordered vector  $\vec{A}_i = (t, s, \dots, s)$  majorizes the decreasingly ordered vector  $\vec{B}_i = (y_1, y_2, \dots, y_i)$ , by Karamata's inequality for convex functions we have

$$f(t) + (i-1)f(s) \geq f(y_1) + f(y_2) + \cdots + f(y_i).$$



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Adding this inequality to Jensen's inequality for the convex function

$$pf(z) + (1 - p)f(s) \geq f(t),$$

the conclusion follows.

**B.** Case  $S > s$ . The function  $f(u)$  is convex for  $u \geq S$ ,  $u \in I$ . According to the result from Case A, it suffices to show that

$$f(x) + kf(y) \geq (1 + k)f(S),$$

for any  $x, y \in I$  such that  $x < S < y$  and  $x + ky = (1 + k)S$ .

For  $x \geq s$ , this inequality follows by Jensen's inequality for convex function.

For  $x < s$ , let  $z$  be defined by  $x + kz = (1 + k)s$ . Since  $k(z - s) = s - x > 0$  and  $k(y - z) = (1 + k)(S - s) > 0$ , we have

$$x < s < z < y, \quad s < S < y.$$

Since  $x + kz = (1 + k)s$  and  $x < z$ , we have by hypothesis

$$f(x) + kf(z) \geq (1 + k)f(s).$$

Therefore, it suffices to show that

$$k[f(y) - f(z)] \geq (1 + k)[f(S) - f(s)],$$

which is equivalent to

$$\frac{f(y) - f(z)}{y - z} \geq \frac{f(S) - f(s)}{S - s}.$$

This inequality is true if

$$\frac{f(y) - f(z)}{y - z} \geq \frac{f(y) - f(s)}{y - s} \geq \frac{f(S) - f(s)}{S - s}.$$



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The left inequality and the right inequality can be reduced to Jensen's inequalities for convex functions,

$$(y - z)f(s) + (z - s)f(y) \geq (y - s)f(z)$$

and

$$(S - s)f(y) + (y - S)f(s) \geq (y - s)f(S),$$

respectively. □

*Remark 1.* In the particular case  $k = n - 1$ , if  $f(x) + (n - 1)f(y) \geq nf(s)$  for any  $x, y \in I$  such that  $x \leq y$  and  $x + (n - 1)y = ns$ , then the inequality in Theorem 1.1,

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(S),$$

holds for any  $x_1, x_2, \dots, x_n \in I$  which satisfy  $\frac{x_1 + x_2 + \cdots + x_n}{n} = S \geq s$ . This result has been established in [1, p. 143] and [2].

*Remark 2.* In the particular case  $k = 1$  (when  $n - 1$  of  $x_1, x_2, \dots, x_n$  are smaller than or equal to  $S$ ), the hypothesis  $f(x) + kf(y) \geq (1 + k)f(s)$  in Theorem 1.1 has a symmetric form:

$$f(x) + f(y) \geq 2f(s)$$

for any  $x, y \in I$  such that  $x + y = 2s$ .

*Remark 3.* Let  $g(u) = \frac{f(u) - f(s)}{u - s}$ . In some applications it is useful to replace the hypothesis  $f(x) + kf(y) \geq (1 + k)f(s)$  in Theorem 1.1 by the equivalent condition:

$$g(x) \leq g(y) \text{ for any } x, y \in I \text{ such that } x < s < y \text{ and } x + ky = (1 + k)s.$$

Their equivalence follows from the following observation:

$$\begin{aligned} f(x) + kf(y) - (1 + k)f(s) &= f(x) - f(s) + k(f(y) - f(s)) \\ &= (x - s)g(x) + k(y - s)g(y) \\ &= (x - s)(g(x) - g(y)). \end{aligned}$$



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**Remark 4.** If  $f$  is differentiable on  $I$ , then Theorem 1.1 holds true by replacing the hypothesis  $f(x) + kf(y) \geq (1+k)f(s)$  with the more restrictive condition:

$$f'(x) \leq f'(y) \text{ for any } x, y \in I \text{ such that } x \leq s \leq y \text{ and } x + ky = (1+k)s.$$

To prove this assertion, we have to show that this condition implies  $f(x) + kf(y) \geq (1+k)f(s)$  for any  $x, y \in I$  such that  $x \leq s \leq y$  and  $x + ky = (1+k)s$ . Let us denote

$$F(x) = f(x) + kf(y) - (1+k)f(s) = f(x) + kf\left(\frac{s + ks - x}{k}\right) - (1+k)f(s).$$

Since  $F'(x) = f'(x) - f'(y) \leq 0$ ,  $F(x)$  is decreasing for  $x \in I$ ,  $x \leq s$ , and hence  $F(x) \geq F(s) = 0$ .

**Remark 5.** The inequality in Theorem 1.1 becomes equality for  $x_1 = x_2 = \dots = x_n = S$ . In the particular case  $S = s$ , if there are  $x, y \in I$  such that  $x < s < y$ ,  $x + ky = (k+1)s$  and  $f(x) + kf(y) = (1+k)f(s)$ , then equality holds again for  $x_1 = x$ ,  $x_2 = \dots = x_{n-k} = s$  and  $x_{n-k+1} = \dots = x_n = y$ .

**Remark 6.** Let  $i$  be an integer such that  $n - k \leq i \leq n - 1$ . We may rewrite the inequality in Theorem 1.1 as either

$$f(S - a_1 + a_{n-i+1}) + f(S - a_2 + a_{n-i+2}) + \dots + f(S - a_n + a_{n-i}) \geq nf(S)$$

with  $a_1 \geq a_2 \geq \dots \geq a_n$ , or

$$f(S - a_1 + a_{i+1}) + f(S - a_2 + a_{i+2}) + \dots + f(S - a_n + a_i) \geq nf(S)$$

with  $a_1 \leq a_2 \leq \dots \leq a_n$ .

**Corollary 1.2.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $g$  be a function on  $(0, \infty)$  such that  $f(u) = g(e^u)$  is convex for  $u \geq 0$ , and

$$g(x) + kg(y) \geq (1+k)g(1)$$





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for any positive real numbers  $x$  and  $y$  with  $x \leq y$  and  $xy^k = 1$ . If  $a_1, a_2, \dots, a_n$  are positive real numbers such that  $\sqrt[n]{a_1 a_2 \cdots a_n} = r \geq 1$  and at least  $n - k$  of  $a_1, a_2, \dots, a_n$  are smaller than or equal to  $r$ , then

$$g(a_1) + g(a_2) + \cdots + g(a_n) \geq ng(r).$$

*Proof.* We apply Theorem 1.1 to the function  $f(u) = g(e^u)$ . In addition, we set  $s = 0$ ,  $S = \ln r$ , and replace  $x$  with  $\ln x$ ,  $y$  with  $\ln y$ , and each  $x_i$  with  $\ln a_i$ .  $\square$

*Remark 7.* If  $f$  is differentiable on  $(0, \infty)$ , then Corollary 1.2 holds true by replacing the hypothesis  $g(x) + kg(y) \geq (1 + k)g(1)$  with the more restrictive condition:

$$xg'(x) \leq yg'(y) \text{ for all } x, y > 0 \text{ such that } x \leq 1 \leq y \text{ and } xy^k = 1.$$

To prove this claim, it suffices to show that this condition implies  $g(x) + kg(y) \geq (1 + k)g(1)$  for all  $x, y > 0$  with  $x \leq 1 \leq y$  and  $xy^k = 1$ . Let us define the function  $G$  by

$$G(x) = g(x) + kg(y) - (1 + k)g(1) = g(x) + kg\left(\sqrt[k]{\frac{1}{x}}\right) - (1 + k)g(1).$$

Since

$$G'(x) = g'(x) - \frac{1}{x\sqrt[k]{x}}g'(y) = \frac{xg'(x) - yg'(y)}{x} \leq 0,$$

$G(x)$  is decreasing for  $x \leq 1$ . Therefore,  $G(x) \geq G(1) = 0$  for  $x \leq 1$ , and hence  $g(x) + kg(y) \geq (1 + k)g(1)$ .

*Remark 8.* Let  $i$  be an integer such that  $n - k \leq i \leq n - 1$ . We may rewrite the inequality for  $r = 1$  in Corollary 1.2 as either

$$g\left(\frac{x_{n-i+1}}{x_1}\right) + g\left(\frac{x_{n-i+2}}{x_2}\right) + \cdots + g\left(\frac{x_{n-i}}{x_n}\right) \geq ng(1)$$



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for  $x_1 \geq x_2 \geq \dots \geq x_n > 0$ , or

$$g\left(\frac{x_{i+1}}{x_1}\right) + g\left(\frac{x_{i+2}}{x_2}\right) + \dots + g\left(\frac{x_i}{x_n}\right) \geq ng(1)$$

for  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ .

**Theorem 1.3.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $f(u)$  be a function on a real interval  $I$ , which is concave for  $u \leq s$ ,  $s \in I$ , and satisfies

$$kf(x) + f(y) \leq (k + 1)f(s)$$

for any  $x, y \in I$  such that  $x \leq y$  and  $kx + y = (k + 1)s$ . If  $x_1, x_2, \dots, x_n \in I$  such that  $\frac{x_1 + x_2 + \dots + x_n}{n} = S \leq s$  and at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to  $S$ , then

$$f(x_1) + f(x_2) + \dots + f(x_n) \leq nf(S).$$

*Proof.* This theorem follows from Theorem 1.1 by replacing  $f(u)$  by  $-f(-u)$ ,  $s$  by  $-s$ ,  $S$  by  $-S$ ,  $x$  by  $-y$ ,  $y$  by  $-x$ , and each  $x_i$  by  $-x_{n-i+1}$  for all  $i$ .  $\square$

**Remark 9.** In the particular case  $k = n - 1$ , if  $(n - 1)f(x) + f(y) \leq nf(s)$  for any  $x, y \in I$  such that  $x \leq y$  and  $(n - 1)x + y = ns$ , then the inequality in Theorem 1.3,

$$f(x_1) + f(x_2) + \dots + f(x_n) \leq nf(S),$$

holds for any  $x_1, x_2, \dots, x_n \in I$  which satisfy  $\frac{x_1 + x_2 + \dots + x_n}{n} = S \leq s$ . This result has been established in [1, p. 147] and [2].

**Remark 10.** In the particular case  $k = 1$  (when  $n - 1$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to  $S$ ), the hypothesis  $kf(x) + f(y) \leq (k + 1)f(s)$  in Theorem 1.3 has a symmetric form:  $f(x) + f(y) \leq 2f(s)$  for any  $x, y \in I$  such that  $x + y = 2s$ .



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**Remark 11.** Let  $g(u) = \frac{f(u)-f(s)}{u-s}$ . The hypothesis  $kf(x) + f(y) \leq (k+1)f(s)$  in Theorem 1.3 is equivalent to

$$g(x) \geq g(y) \text{ for any } x, y \in I \text{ such that } x < s < y \text{ and } kx + y = (k+1)s.$$

**Remark 12.** If  $f$  is differentiable on  $I$ , then Theorem 1.3 holds true if we replace the hypothesis  $kf(x) + f(y) \leq (k+1)f(s)$  with the more restrictive condition

$$f'(x) \geq f'(y) \text{ for any } x, y \in I \text{ such that } x \leq s \leq y \text{ and } kx + y = (k+1)s.$$

**Remark 13.** The inequality in Theorem 1.3 becomes equality for  $x_1 = x_2 = \dots = x_n = S$ . In the particular case  $S = s$ , if there are  $x, y \in I$  such that  $x < s < y$ ,  $kx + y = (k+1)s$  and  $kf(x) + f(y) = (1+k)f(s)$ , then equality holds again for  $x_1 = \dots = x_k = x$ ,  $x_{k+1} = \dots = x_{n-1} = s$  and  $x_n = y$ .

**Remark 14.** Let  $i$  be an integer such that  $1 \leq i \leq k$ . We may rewrite the inequality in Theorem 1.3 as either

$$f(S - a_1 + a_{i+1}) + f(S - a_2 + a_{i+2}) + \dots + f(S - a_n + a_i) \leq nf(S)$$

with  $a_1 \leq a_2 \leq \dots \leq a_n$ , or

$$f(S - a_1 + a_{n-i+1}) + f(S - a_2 + a_{n-i+2}) + \dots + f(S - a_n + a_{n-i}) \leq nf(S)$$

with  $a_1 \geq a_2 \geq \dots \geq a_n$ .

**Corollary 1.4.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $g$  be a function on  $(0, \infty)$  such that  $f(u) = g(e^u)$  is concave for  $u \leq 0$ , and

$$kg(x) + g(y) \leq (k+1)g(1)$$

for any positive real numbers  $x$  and  $y$  with  $x \leq y$  and  $x^k y = 1$ . If  $a_1, a_2, \dots, a_n$  are positive real numbers such that  $\sqrt[k]{a_1 a_2 \dots a_n} = r \leq 1$  and at least  $n - k$  of  $a_1, a_2, \dots, a_n$  are greater than or equal to  $r$ , then

$$g(a_1) + g(a_2) + \dots + g(a_n) \leq ng(r).$$



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*Proof.* We apply Theorem 1.3 to the function  $f(u) = g(e^u)$ . In addition, we set  $s = 0$ ,  $S = \ln r$ , and replace  $x$  with  $\ln x$ ,  $y$  with  $\ln y$ , and each  $x_i$  with  $\ln a_i$ .  $\square$

*Remark 15.* If  $f$  is differentiable on  $(0, \infty)$ , then Corollary 1.4 holds true by replacing the hypothesis  $kg(x) + g(y) \leq (k+1)g(1)$  with the more restrictive condition:

$$xg'(x) \geq yg'(y) \text{ for all } x, y > 0 \text{ such that } x \leq 1 \leq y \text{ and } x^k y = 1.$$

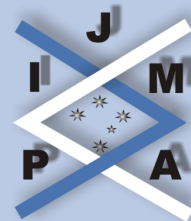
*Remark 16.* Let  $i$  be an integer such that  $1 \leq i \leq k$ . We may rewrite the inequality for  $r = 1$  in Corollary 1.4 as either

$$g\left(\frac{x_{i+1}}{x_1}\right) + g\left(\frac{x_{i+2}}{x_2}\right) + \cdots + g\left(\frac{x_i}{x_n}\right) \leq ng(1)$$

for  $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ , or

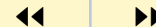
$$g\left(\frac{x_{n-i+1}}{x_1}\right) + g\left(\frac{x_{n-i+2}}{x_2}\right) + \cdots + g\left(\frac{x_{n-i}}{x_n}\right) \leq ng(1)$$

for  $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ .



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## 2. Applications

**Proposition 2.1.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $x_1, x_2, \dots, x_n$  be nonnegative real numbers such that  $x_1 + x_2 + \dots + x_n = n$ .

(a) If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are smaller than or equal to 1, then

$$k(x_1^3 + x_2^3 + \dots + x_n^3) + (1 + k)n \geq (1 + 2k)(x_1^2 + x_2^2 + \dots + x_n^2);$$

(b) If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to 1, then

$$x_1^3 + x_2^3 + \dots + x_n^3 + (k + 1)n \leq (k + 2)(x_1^2 + x_2^2 + \dots + x_n^2).$$

*Proof.* (a) The inequality is equivalent to  $f(x_1) + f(x_2) + \dots + f(x_n) \geq nf(S)$ , where  $S = \frac{x_1 + x_2 + \dots + x_n}{n} = 1$  and  $f(u) = ku^3 - (1 + 2k)u^2$ . For  $u \geq 1$ ,

$$f''(u) = 2(3ku - 1 - 2k) \geq 2(k - 1) \geq 0.$$

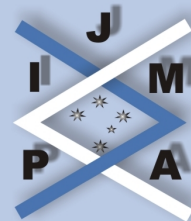
Therefore,  $f$  is convex for  $u \geq s = 1$ . According to Theorem 1.1 and Remark 3, we have to show that  $g(x) \leq g(y)$  for any nonnegative real numbers  $x < y$  such that  $x + ky = 1 + k$ , where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = ku^2 - (1 + k)u - 1 - k.$$

Indeed,

$$g(y) - g(x) = (k - 1)x(y - x) \geq 0.$$

Equality occurs for  $x_1 = x_2 = \dots = x_n = 1$ . On the assumption that  $x_1 \leq x_2 \leq \dots \leq x_n$ , equality holds again for  $x_1 = 0$ ,  $x_2 = \dots = x_{n-k} = 1$  and  $x_{n-k+1} = \dots = x_n = 1 + \frac{1}{k}$ .



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(b) Write the inequality as  $f(x_1) + f(x_2) + \cdots + f(x_n) \leq nf(S)$ , where  $S = \frac{x_1 + x_2 + \cdots + x_n}{n} = 1$  and  $f(u) = u^3 - (k+2)u^2$ . From the second derivative,

$$f''(u) = 2(3u - k - 2),$$

it follows that  $f$  is concave for  $u \leq s = 1$ . According to Theorem 1.3 and Remark 11, we have to show that  $g(x) \geq g(y)$  for any nonnegative real numbers  $x < y$  such that  $kx + y = k + 1$ , where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = u^2 - (k+1)u - k - 1.$$

It is easy to see that

$$g(x) - g(y) = (k-1)x(y-x) \geq 0.$$

Equality occurs for  $x_1 = x_2 = \cdots = x_n = 1$ . On the assumption that  $x_1 \leq x_2 \leq \cdots \leq x_n$ , equality holds again for  $x_1 = \cdots = x_k = 0$ ,  $x_{k+1} = \cdots = x_{n-1} = 1$  and  $x_n = k + 1$ .  $\square$

*Remark 17.* For  $k = n - 1$ , the inequalities above become as follows

$$(n-1)(x_1^3 + x_2^3 + \cdots + x_n^3) + n^2 \geq (2n-1)(x_1^2 + x_2^2 + \cdots + x_n^2)$$

and

$$x_1^3 + x_2^3 + \cdots + x_n^3 + n^2 \leq (n+1)(x_1^2 + x_2^2 + \cdots + x_n^2),$$

respectively. By Remark 1 and Remark 9, these inequalities hold for any nonnegative real numbers  $x_1, x_2, \dots, x_n$  which satisfy  $x_1 + x_2 + \cdots + x_n = n$  (Problems 3.4.1 and 3.4.2 from [1, p. 154]).

*Remark 18.* For  $k = 1$ , we get the following statement:

Let  $x_1, x_2, \dots, x_n$  be nonnegative real numbers such that  $x_1 + x_2 + \cdots + x_n = n$ .



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(a) If  $x_1 \leq \dots \leq x_{n-1} \leq 1 \leq x_n$ , then

$$x_1^3 + x_2^3 + \dots + x_n^3 + 2n \geq 3(x_1^2 + x_2^2 + \dots + x_n^2);$$

(b) If  $x_1 \leq 1 \leq x_2 \leq \dots \leq x_n$ , then

$$x_1^3 + x_2^3 + \dots + x_n^3 + 2n \leq 3(x_1^2 + x_2^2 + \dots + x_n^2).$$

**Proposition 2.2.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $x_1, x_2, \dots, x_n$  be positive real numbers such that  $x_1 + x_2 + \dots + x_n = n$ . If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to 1, then

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - n \geq \frac{4k}{(k+1)^2}(x_1^2 + x_2^2 + \dots + x_n^2 - n).$$

*Proof.* Rewrite the inequality as  $f(x_1) + f(x_2) + \dots + f(x_n) \leq nf(S)$ , where  $S = \frac{x_1 + x_2 + \dots + x_n}{n} = 1$  and  $f(u) = \frac{4ku^2}{(k+1)^2} - \frac{1}{u}$ . For  $0 < u \leq s = 1$ , we have

$$f''(u) = \frac{8k}{(k+1)^2} - \frac{2}{u^3} \leq \frac{8k}{(k+1)^2} - 2 = \frac{-2(k-1)^2}{(k+1)^2} \leq 0;$$

therefore,  $f$  is concave on  $(0, 1]$ . By Theorem 1.3 and Remark 11, we have to show that  $g(x) \geq g(y)$  for any positive real numbers  $x < y$  such that  $kx + y = k + 1$ , where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{4k(u+1)}{(k+1)^2} + \frac{1}{u}.$$

Indeed,

$$g(x) - g(y) = (y - x) \left[ \frac{1}{xy} - \frac{4k}{(k+1)^2} \right] = \frac{(y-x)(2kx - k - 1)^2}{(k+1)^2 xy} \geq 0.$$



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Equality occurs for  $x_1 = x_2 = \dots = x_n = 1$ . Under the assumption that  $x_1 \leq x_2 \leq \dots \leq x_n$ , equality holds again for  $x_1 = \dots = x_k = \frac{k+1}{2k}$ ,  $x_{k+1} = \dots = x_{n-1} = 1$  and  $x_n = \frac{k+1}{2}$ .  $\square$

**Remark 19.** For  $k = n - 1$ , the inequality in Proposition 2.2 becomes as follows:

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - n \geq \frac{4(n-1)}{n^2}(x_1^2 + x_2^2 + \dots + x_n^2 - n).$$

By Remark 9, this inequality holds for any positive real numbers  $x_1, x_2, \dots, x_n$  which satisfy  $x_1 + x_2 + \dots + x_n = n$  (Problems 3.4.5 from [1, p. 158]).

**Remark 20.** For  $k = 1$ , the following nice statement follows:

If  $x_1, x_2, \dots, x_n$  are positive real numbers such that  $x_1 \leq 1 \leq x_2 \leq \dots \leq x_n$  and  $x_1 + x_2 + \dots + x_n = n$ , then

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \geq x_1^2 + x_2^2 + \dots + x_n^2.$$

**Proposition 2.3.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $x_1, x_2, \dots, x_n$  be nonnegative real numbers such that  $x_1 + x_2 + \dots + x_n = n$ .

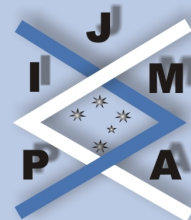
(a) If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are smaller than or equal to 1, then

$$\frac{1}{k+1+kx_1^2} + \frac{1}{k+1+kx_2^2} + \dots + \frac{1}{k+1+kx_n^2} \geq \frac{n}{2k+1};$$

(b) If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to 1, then

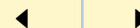
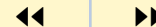
$$\frac{1}{k^2+k+1+kx_1^2} + \frac{1}{k^2+k+1+kx_2^2} + \dots + \frac{1}{k^2+k+1+kx_n^2} \leq \frac{n}{(k+1)^2}.$$





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*Proof.* (a) We may write the inequality as  $f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(S)$ , where  $S = \frac{x_1 + x_2 + \cdots + x_n}{n} = 1$  and  $f(u) = \frac{1}{k+1+ku^2}$ . Since the second derivative,

$$f''(u) = \frac{2k(3ku^2 - k - 1)}{(k + 1 + ku^2)^3},$$

is positive for  $u \geq 1$ ,  $f$  is convex for  $u \geq s = 1$ . According to Theorem 1.1 and Remark 3, we have to show that  $g(x) \leq g(y)$  for any nonnegative real numbers  $x < y$  such that  $x + ky = 1 + k$ , where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u + 1)}{(2k + 1)(k + 1 + ku^2)}.$$

Indeed, we have

$$g(y) - g(x) = \frac{k^2(y - x)}{(2k + 1)(k + 1 + kx^2)(k + 1 + ky^2)} \left( xy + x + y - 1 - \frac{1}{k} \right) \geq 0,$$

since

$$xy + x + y - 1 - \frac{1}{k} = \frac{x(2k - 1 + y)}{k} \geq 0.$$

Equality occurs for  $x_1 = x_2 = \cdots = x_n = 1$ . On the assumption that  $x_1 \leq x_2 \leq \cdots \leq x_n$ , equality holds again for  $x_1 = 0$ ,  $x_2 = \cdots = x_{n-k} = 1$  and  $x_{n-k+1} = \cdots = x_n = 1 + \frac{1}{k}$ .

(b) We will apply Theorem 1.3 to the function  $f(u) = \frac{1}{k^2 + k + 1 + ku^2}$ , for  $s = S = 1$ . Since the second derivative,

$$f''(u) = \frac{2k(3ku^2 - k^2 - k - 1)}{(k^2 + k + 1 + ku^2)^3},$$



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is negative for  $0 \leq u < 1$ ,  $f$  is concave for  $0 \leq u \leq 1$ . According to Remark 11, we have to show that  $g(x) \geq g(y)$  for any nonnegative real numbers  $x < y$  such that  $kx + y = k + 1$ , where

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u + 1)}{(k + 1)^2(k^2 + k + 1 + ku^2)}.$$

We have

$$g(x) - g(y) = \frac{k^2(y - x)}{(k + 1)^2(k^2 + k + 1 + kx^2)(k^2 + k + 1 + ky^2)} \times \left( k + \frac{1}{k} + 1 - xy - x - y \right) \geq 0,$$

since

$$k + \frac{1}{k} + 1 - xy - x - y = k \left( x - \frac{1}{k} \right)^2 \geq 0.$$

Equality occurs for  $x_1 = x_2 = \dots = x_n = 1$ . On the assumption that  $x_1 \leq x_2 \leq \dots \leq x_n$ , equality holds again for  $x_1 = \dots = x_k = \frac{1}{k}$ ,  $x_{k+1} = \dots = x_{n-1} = 1$  and  $x_n = k$ .  $\square$

**Remark 21.** For  $k = n - 1$ , the inequalities in Proposition 2.3 become as follows:

$$\frac{1}{n + (n - 1)x_1^2} + \frac{1}{n + (n - 1)x_2^2} + \dots + \frac{1}{n + (n - 1)x_n^2} \geq \frac{n}{2n - 1}$$

and

$$\frac{1}{n^2 - n + 1 + (n - 1)x_1^2} + \frac{1}{n^2 - n + 1 + (n - 1)x_2^2} + \dots + \frac{1}{n^2 - n + 1 + (n - 1)x_n^2} \leq \frac{1}{n},$$



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respectively. By Remark 1 and Remark 9, these inequalities hold for any nonnegative numbers  $x_1, x_2, \dots, x_n$  which satisfy  $x_1 + x_2 + \dots + x_n = n$  (Problems 3.4.3 and 3.4.4 from [1, p. 156]).

*Remark 22.* For  $k = 1$ , we get the following statement:

Let  $x_1, x_2, \dots, x_n$  be nonnegative real numbers such that  $x_1 + x_2 + \dots + x_n = n$ .

(a) If  $x_1 \leq \dots \leq x_{n-1} \leq 1 \leq x_n$ , then

$$\frac{1}{2 + x_1^2} + \frac{1}{2 + x_2^2} + \dots + \frac{1}{2 + x_n^2} \geq \frac{n}{3};$$

(b) If  $x_1 \leq 1 \leq x_2 \leq \dots \leq x_n$ , then

$$\frac{1}{3 + x_1^2} + \frac{1}{3 + x_2^2} + \dots + \frac{1}{3 + x_n^2} \leq \frac{n}{4}.$$

*Remark 23.* By Theorem 1.1 and Theorem 1.3, the following more general statement holds:

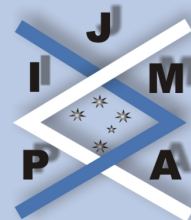
Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $x_1, x_2, \dots, x_n$  be nonnegative real numbers such that  $x_1 + x_2 + \dots + x_n = nS$ .

(a) If  $S \geq 1$  and at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are smaller than or equal to  $S$ , then

$$\frac{1}{k + 1 + kx_1^2} + \frac{1}{k + 1 + kx_2^2} + \dots + \frac{1}{k + 1 + kx_n^2} \geq \frac{n}{k + 1 + kS^2};$$

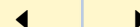
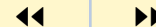
(b) If  $S \leq 1$  and at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to  $S$ , then

$$\frac{1}{k^2 + k + 1 + kx_1^2} + \frac{1}{k^2 + k + 1 + kx_2^2} + \dots + \frac{1}{k^2 + k + 1 + kx_n^2} \leq \frac{n}{k^2 + k + 1 + kS^2}.$$



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**Proposition 2.4.** Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $a_1 a_2 \cdots a_n = 1$ .

(a) If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are smaller than or equal to 1, then

$$\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \cdots + \frac{1}{1 + ka_n} \geq \frac{n}{1 + k};$$

(b) If at least  $n - k$  of  $x_1, x_2, \dots, x_n$  are greater than or equal to 1, then

$$\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n}{1 + k}.$$

*Proof.* (a) We will apply Corollary 1.2 to the function  $g(x) = \frac{1}{1+kx}$ , for  $r = 1$ . The function  $f(u) = g(e^u) = \frac{1}{1+ke^u}$  has the second derivative

$$f''(u) = \frac{ke^u(ke^u - 1)}{(1 + ke^u)^3},$$

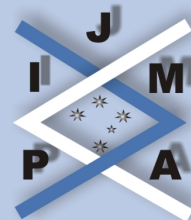
which is positive for  $u > 0$ . Therefore,  $f$  is convex for  $u \geq 0$ . Thus, it suffices to show that  $g(x) + kg(y) \geq (1 + k)g(1)$  for any  $x, y > 0$  such that  $xy^k = 1$ . The inequality  $g(x) + kg(y) \geq (1 + k)g(1)$  is equivalent to

$$\frac{y^k}{y^k + k} + \frac{k}{1 + ky} \geq 1,$$

or, equivalently,

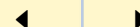
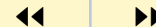
$$y^k + k - 1 \geq ky.$$

The last inequality immediately follows from the AM-GM inequality applied to the positive numbers  $y^k, 1, \dots, 1$ . Equality occurs for  $a_1 = a_2 = \cdots = a_n = 1$ .



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(b) We can obtain the required inequality either by replacing each number  $a_i$  with its reverse  $\frac{1}{a_i}$  in the inequality in part (a), or by means of Corollary 1.4. Equality occurs for  $a_1 = a_2 = \dots = a_n = 1$ .  $\square$

*Remark 24.* For  $k = n - 1$ , we get the known inequalities

$$\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \dots + \frac{1}{1 + (n-1)a_n} \geq 1$$

and

$$\frac{1}{a_1 + n - 1} + \frac{1}{a_2 + n - 1} + \dots + \frac{1}{a_n + n - 1} \leq 1,$$

which hold for any positive numbers  $a_1, a_2, \dots, a_n$  such that  $a_1 a_2 \dots a_n = 1$ .

*Remark 25.* Using the substitution  $a_1 = \frac{x_{k+1}}{x_1}$ ,  $a_2 = \frac{x_{k+2}}{x_2}$ ,  $\dots$ ,  $a_n = \frac{x_k}{x_n}$ , we get the following statement:

Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $x_1, x_2, \dots, x_n$  be positive real numbers.

(a) If  $x_1 \geq x_2 \geq \dots \geq x_n$ , then

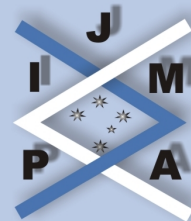
$$\frac{x_1}{x_1 + kx_{k+1}} + \frac{x_2}{x_2 + kx_{k+2}} + \dots + \frac{x_n}{x_n + kx_k} \geq \frac{n}{1 + k};$$

(b) If  $x_1 \leq x_2 \leq \dots \leq x_n$ , then

$$\frac{x_1}{kx_1 + x_{k+1}} + \frac{x_2}{kx_2 + x_{k+2}} + \dots + \frac{x_n}{kx_n + x_k} \leq \frac{n}{k + 1}.$$

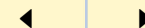
In the particular case  $k = 1$ , we get

$$\frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \dots + \frac{x_n}{x_n + x_1} \geq \frac{n}{2}$$



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for  $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ , and

$$\frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_n}{x_n + x_1} \leq \frac{n}{2}$$

for  $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ .

**Remark 26.** By Corollary 1.2 and Corollary 1.4, we can see that the following more general statement holds:

Let  $n \geq 2$  and  $1 \leq k \leq n - 1$  be natural numbers, and let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $\sqrt[n]{a_1 a_2 \cdots a_n} = r$ .

(a) If  $r \geq 1$ , and at least  $n - k$  of  $a_1, a_2, \dots, a_n$  are smaller than or equal to  $r$ , then

$$\frac{1}{1 + k a_1} + \frac{1}{1 + k a_2} + \cdots + \frac{1}{1 + k a_n} \geq \frac{n}{1 + k r};$$

(b) If  $r \leq 1$ , and at least  $n - k$  of  $a_1, a_2, \dots, a_n$  are greater than or equal to  $r$ , then

$$\frac{1}{a_1 + k} + \frac{1}{a_2 + k} + \cdots + \frac{1}{a_n + k} \leq \frac{n}{r + k}.$$

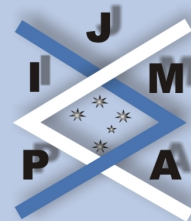
**Proposition 2.5.** Let  $a_1, a_2, \dots, a_n$  be positive numbers such that  $a_1 a_2 \cdots a_n = 1$ .

(a) If  $a_1 \leq \cdots \leq a_{n-1} \leq 1 \leq a_n$ , then

$$\frac{1}{\sqrt{1 + 3a_1}} + \frac{1}{\sqrt{1 + 3a_2}} + \cdots + \frac{1}{\sqrt{1 + 3a_n}} \geq \frac{n}{2};$$

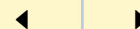
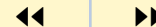
(b) If  $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$ , then

$$\frac{1}{\sqrt{1 + 2a_1}} + \frac{1}{\sqrt{1 + 2a_2}} + \cdots + \frac{1}{\sqrt{1 + 2a_n}} \leq \frac{n}{\sqrt{3}}.$$



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*Proof.* (a) We will apply Corollary 1.2 (case  $k = 1$  and  $r = 1$ ) to the function  $g(x) = \frac{1}{\sqrt{1+3x}}$ . The function  $f(u) = g(e^u) = \frac{1}{\sqrt{1+3e^u}}$  has the second derivative

$$f''(u) = \frac{1}{2}e^u(3e^u - 2)(1 + 3e^u)^{-\frac{5}{2}}.$$

Since  $f'' > 0$  for  $u \geq 0$ ,  $f$  is convex for  $u \geq 0$ . Therefore, to finish the proof, we have to show that  $g(x) + g(y) \geq 2g(1)$  for any  $x, y > 0$  with  $xy = 1$ . This inequality is equivalent to

$$\frac{1}{\sqrt{1+3x}} + \sqrt{\frac{x}{x+3}} \geq 1.$$

Using the substitution  $\frac{1}{\sqrt{1+3x}} = t$ ,  $0 < t < 1$ , transforms the inequality into

$$\sqrt{\frac{1-t^2}{8t^2+1}} \geq 1-t.$$

By squaring, we get  $t(1-t)(2t-1)^2 \geq 0$ , which is clearly true. Equality occurs for  $a_1 = a_2 = \dots = a_n = 1$ .

(b) We will apply Corollary 1.4 (case  $k = 1$  and  $r = 1$ ) to the function  $g(x) = \frac{1}{\sqrt{1+2x}}$ . The function  $f(u) = g(e^u) = \frac{1}{\sqrt{1+2e^u}}$  is concave for  $u \leq 0$ , since

$$f'' = e^u(e^u - 1)(1 + 2e^u)^{-\frac{5}{2}} \leq 0.$$

Thus, it suffices to show that  $g(x) + g(y) \leq 2g(1)$  for any  $x, y > 0$  with  $xy = 1$ . This inequality follows from the Cauchy-Schwarz inequality, as follows

$$\sqrt{\frac{3}{1+2x}} + \sqrt{\frac{3}{1+2y}} \leq \sqrt{\left(\frac{3}{1+2x} + 1\right) \left(1 + \frac{3}{1+2y}\right)} = 2.$$

Equality occurs for  $a_1 = a_2 = \dots = a_n = 1$ . □



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**Remark 27.** Using the substitution  $a_1 = \frac{x_2}{x_1}$ ,  $a_2 = \frac{x_3}{x_2}$ ,  $\dots$ ,  $a_n = \frac{x_1}{x_n}$ , we get the following statement:

Let  $x_1, x_2, \dots, x_n$  be positive real numbers.

(a) If  $x_1 \geq x_2 \geq \dots \geq x_n$ , then

$$\sqrt{\frac{x_1}{x_1 + 3x_2}} + \sqrt{\frac{x_2}{x_2 + 3x_3}} + \dots + \sqrt{\frac{x_n}{x_n + 3x_1}} \geq \frac{n}{2};$$

(b) If  $x_1 \leq x_2 \leq \dots \leq x_n$ , then

$$\sqrt{\frac{3x_1}{x_1 + 2x_2}} + \sqrt{\frac{3x_2}{x_2 + 2x_3}} + \dots + \sqrt{\frac{3x_n}{x_n + 2x_1}} \leq n.$$

**Remark 28.** By Corollary 1.2 and Corollary 1.4, the following more general statement holds:

Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $\sqrt[n]{a_1 a_2 \dots a_n} = r$ .

(a) If  $r \geq 1$  and  $a_1 \leq \dots \leq a_{n-1} \leq r \leq a_n$ , then

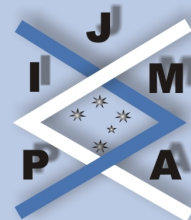
$$\frac{1}{\sqrt{1 + 3a_1}} + \frac{1}{\sqrt{1 + 3a_2}} + \dots + \frac{1}{\sqrt{1 + 3a_n}} \geq \frac{n}{\sqrt{1 + 3r}};$$

(b) If  $r \leq 1$  and  $a_1 \leq r \leq a_2 \leq \dots \leq a_n$ , then

$$\frac{1}{\sqrt{1 + 2a_1}} + \frac{1}{\sqrt{1 + 2a_2}} + \dots + \frac{1}{\sqrt{1 + 2a_n}} \leq \frac{n}{\sqrt{1 + 2r}}.$$

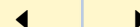
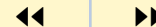
**Proposition 2.6.** Let  $a_1, a_2, \dots, a_n$  be positive numbers such that  $a_1 a_2 \dots a_n = 1$ .





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(a) If  $a_1 \leq \dots \leq a_{n-1} \leq 1 \leq a_n$ , then the following inequality holds for  $0 \leq p \leq p_0$ , where  $p_0 \cong 1.5214$  is the positive root of the equation  $p^3 - p - 2 = 0$ :

$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \dots + \frac{1}{(p+a_n)^2} \geq \frac{n}{(p+1)^2};$$

(b) If  $a_1 \leq 1 \leq a_2 \leq \dots \leq a_n$ , then the following inequality holds for  $p \geq 1 + \sqrt{2}$ :

$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \dots + \frac{1}{(p+a_n)^2} \leq \frac{n}{(p+1)^2}.$$

*Proof.* (a) We will apply Corollary 1.2 (case  $k = 1$  and  $r = 1$ ) to the function  $g(x) = \frac{1}{(p+x)^2}$ . Notice that the function  $f(u) = g(e^u) = \frac{1}{(p+e^u)^2}$  is convex for  $u \geq 0$ , because

$$f''(u) = \frac{2e^u(2e^u - p)}{(p + e^u)^4} > 0.$$

Consequently, we have to show that  $g(x) + g(y) \geq 2g(1)$  for any  $x, y > 0$  with  $xy = 1$ ; that is

$$\frac{1}{(p+x)^2} + \frac{1}{(p+y)^2} \geq \frac{2}{(p+1)^2}.$$

Using the substitution  $x + y = 2t$ ,  $t \geq 1$ , the inequality transforms into

$$\frac{2t^2 + 2pt + p^2 - 1}{(2pt + p^2 + 1)^2} \geq \frac{1}{(p+1)^2},$$

or, equivalently,

$$(t-1)[(1+2p-p^2)t + (1-p)(p^2+1)] \geq 0.$$



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It is true, because  $1 + 2p - p^2 > p(2 - p) > 0$  and

$$(1 + 2p - p^2)t + (1 - p)(p^2 + 1) \geq (1 + 2p - p^2) + (1 - p)(p^2 + 1) \\ = 2 + p - p^3 \geq 0$$

for  $0 \leq p \leq p_0$ . Equality holds for  $a_1 = a_2 = \dots = a_n = 1$ .

(b) We will apply Corollary 1.4 (case  $k = 1$  and  $r = 1$ ) to the function  $g(x) = \frac{1}{(p+x)^2}$ .

The function  $f(u) = g(e^u) = \frac{1}{(p+e^u)^2}$  is concave for  $u \leq 0$ , since

$$f''(u) = \frac{2e^u(2e^u - p)}{(p + e^u)^4} < 0.$$

By Corollary 1.4, it suffices to show that  $g(x) + g(y) \leq 2g(1)$  for any  $x, y > 0$  with  $xy = 1$ ; that is

$$\frac{1}{(p+x)^2} + \frac{1}{(p+y)^2} \leq \frac{2}{(p+1)^2}.$$

Using the notation  $x + y = 2t$ ,  $t \geq 1$ , the inequality becomes

$$(t-1)[(p^2 - 2p - 1)t + (p-1)(p^2 + 1)] \geq 0.$$

It is true, since  $p^2 - 2p - 1 \geq 0$  for  $p \geq 1 + \sqrt{2}$ . Equality holds for  $a_1 = a_2 = \dots = a_n = 1$ .  $\square$

*Remark 29.* Using the substitution  $a_1 = \frac{x_2}{x_1}$ ,  $a_2 = \frac{x_3}{x_2}$ ,  $\dots$ ,  $a_n = \frac{x_1}{x_n}$ , we get the following statement:

Let  $x_1, x_2, \dots, x_n$  be positive real numbers.

(a) If  $0 \leq p \leq p_0 \cong 1.5214$  and  $x_1 \geq x_2 \geq \dots \geq x_n$ , then

$$\left(\frac{x_1}{px_1 + x_2}\right)^2 + \left(\frac{x_2}{px_2 + x_3}\right)^2 + \dots + \left(\frac{x_n}{px_n + x_1}\right)^2 \geq \frac{n}{(p+1)^2};$$



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(b) If  $p \geq 1 + \sqrt{2}$  and  $x_1 \leq x_2 \leq \dots \leq x_n$ , then

$$\left(\frac{x_1}{px_1 + x_2}\right)^2 + \left(\frac{x_2}{px_2 + x_3}\right)^2 + \dots + \left(\frac{x_n}{px_n + x_1}\right)^2 \leq \frac{n}{(p+1)^2}.$$

*Remark 30.* By Corollary 1.2 and Corollary 1.4, the following more general statement holds:

Let  $a_1, a_2, \dots, a_n$  be positive real numbers such that  $\sqrt[n]{a_1 a_2 \dots a_n} = r$ .

(a) If  $r \geq 1$  and  $a_1 \leq \dots \leq a_{n-1} \leq r \leq a_n$ , then the following inequality holds for  $0 \leq p \leq p_0$ , where  $p_0 \cong 1,5214$  is the positive root of the equation  $p^3 - p - 2 = 0$ :

$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \dots + \frac{1}{(p+a_n)^2} \geq \frac{n}{(p+r)^2};$$

(b) If  $r \leq 1$  and  $a_1 \leq r \leq a_2 \leq \dots \leq a_n$ , then the following inequality holds for  $p \geq 1 + \sqrt{2}$ :

$$\frac{1}{(p+a_1)^2} + \frac{1}{(p+a_2)^2} + \dots + \frac{1}{(p+a_n)^2} \leq \frac{n}{(p+r)^2}.$$

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Vasile Cîrtoaje

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