Journal of Inequalities in Pure and Applied Mathematics

# SHARPENING OF JORDAN'S INEQUALITY AND ITS APPLICATIONS

WEI DONG JIANG AND HUA YUN

Department of Information Engineering
Weihai Vocational College
Weihai 264200
Shandong Province, P.R. CHINA.

*EMail*: jackjwd@163.com
*EMail*: nyjj2006@163.com


ISSN (electronic): 1443-5756
338-05

volume 7, issue 3, article 102, 2006.

*Received 12 November, 2005; accepted 07 February, 2006.*

*Communicated by: P.S. Bullen*

## Abstract

In this paper,the following inequality:

$$\frac{2}{\pi}+\frac{1}{2\pi^5}(\pi^4-16x^4)\leq\frac{\sin x}{x}\leq\frac{2}{\pi}+\frac{\pi-2}{\pi^5}(\pi^4-16x^4)$$

is established. An application of this inequality gives an improvement of Yang Le's inequality.

*2000 Mathematics Subject Classification:* Primary 26A51, 26D07, 26D15.
*Key words:* Jordan inequality, Yang Le inequality, Upper-lower bound.

## Contents

# 1. Introduction

The following result is known as Jordan's inequality [1]:

**Theorem 1.1.**

$$\frac{\sin x}{x} \geq \frac{2}{\pi}, \qquad x \in (0, \pi/2]. \tag{1.1}$$

The inequality (1.1) is sharp with equality if and only if $x = \frac{\pi}{2}$.

Jordan's inequality and its refinements have been considered by a number of other authors (see [2], [3]). In [2] Feng Qi obtained new lower and upper bounds for the function $\frac{\sin x}{x}$; his result reads as follows:

**Theorem 1.2.** *Let $x \in (0, \pi/2]$, then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \tag{1.2}$$

*with equality if and only if $x = \frac{\pi}{2}$.*

In this paper we will consider a new refined form of Jordan's inequality and an application of it on the same problem considered by Zhao [5] – [7]. Our main result is given by the following.

## 2. Main Result

In order to prove Theorem 2.2 below, we need the following lemma.

**Lemma 2.1 ([8]).** *Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on $(a, b)$, let $g' \neq 0$ on $(a, b)$,if $\frac{f'}{g'}$ is decreasing on $(a, b)$, then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)} \quad \textit{and} \quad \frac{f(x) - f(a)}{g(x) - g(a)}$$

*are also decreasing on $(a, b)$.*

**Theorem 2.2.** *If $x \in (0, \pi/2]$, then*

$$\frac{2}{\pi} + \frac{1}{2\pi^5}(\pi^4 - 16x^4) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}(\pi^4 - 16x^4) \tag{2.1}$$

*with equality if and only if $x = \frac{\pi}{2}$.*

*Proof.* Let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -16x^4$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^5$, and $x \in (0, \pi/2]$, then we have.

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{64} \cdot \frac{\sin x - x\cos x}{x^5} = \frac{1}{64} \cdot \frac{f_3(x)}{f_4(x)}.$$

$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{5} \cdot \frac{\sin x}{x^3}.$$

It is well-known that $\frac{\sin x}{x^3}$ is decreasing on $(0, \frac{\pi}{2})$, so $\frac{f_3'(x)}{f_4'(x)}$ is decreasing on $(0, \frac{\pi}{2})$. By Lemma 2.1,

$$\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_4(x) - f_4(0)}$$

is decreasing on $(0, \frac{\pi}{2})$, so $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on $(0, \frac{\pi}{2})$, then

$$h(x) = \frac{f_1(x) - f_1(\frac{\pi}{2})}{f_2(x) - f_2(\frac{\pi}{2})} = \frac{\frac{\sin x}{x} - \frac{\pi}{2}}{\pi^4 - 16x^4}$$

is decreasing on $(0, \frac{\pi}{2})$. By Lemma 2.1.

Furthermore, $\lim_{x\to 0+} h(x) = \frac{\pi-2}{\pi^5}$, $\lim_{x\to\frac{\pi}{2}-} h(x) = \frac{1}{2\pi^5}$. Thus $\frac{\pi-2}{\pi^5}$ and $\frac{1}{2\pi^5}$ are the best constants in (2.1). So the proof is complete □

**Note:** In a similar manner, we can obtain several interesting inequalities similar to (2.2). For example, let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -4x^2$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^3$, and $x \in (0, \pi/2]$, then (1.2) is obtained. If we let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -8x^3$, $f_3(x) = \sin x - x\cos x$, $f_4(x) = x^4$, then we have

$$\frac{2}{\pi} + \frac{2}{3\pi^4}(\pi^3 - 8x^3) \le \frac{\sin x}{x} \le \frac{2}{\pi} + \frac{\pi-2}{\pi^4}(\pi^3 - 8x^3).$$

## 3. Applications

Yang Le's inequality [4] and its generalizations which play an important role in the theory of distribution of values of functions can be stated as follows.

If $A > 0$, $B > 0$, $A + B \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A + \cos^2 \lambda B - 2\cos \lambda A \cos \lambda B \cos \lambda\pi \geq \sin^2 \lambda\pi. \tag{3.1}$$

In [5] – [7] some improvements of Yang Le's inequality are obtained. In a similar way, based on the inequality (2.2) we can give the following.

**Theorem 3.1.** *Let* $A_i > 0$ $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} A_i \leq \pi$, $n \in \mathbb{N}$ *and* $n \neq 1$, $0 \leq \lambda \leq 1$, *then*

$$R(\lambda) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq T(\lambda), \tag{3.2}$$

*where*

$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos \lambda A_i \cos \lambda A_j \cos \lambda\pi,$$

$$R(\lambda) = 4C_n^2 \left(\lambda + \frac{1}{4}\lambda(1 - \lambda^4)\right)^2 \cos^2 \frac{\lambda}{2}\pi,$$

$$T(\lambda) = 4C_n^2 \left(\lambda + \frac{\pi - 2}{2}\lambda(1 - \lambda^4)\right)^2.$$

*Proof.* Substituting $x = \frac{\lambda}{2}\pi$ in (2.2),we have

$$\sin \frac{\lambda}{2}\pi \geq \lambda + \frac{1}{4}\lambda(1 - \lambda^4) \tag{3.3}$$

and

$$\sin\frac{\lambda}{2}\pi \le \lambda + \frac{\lambda-2}{2}\lambda(1-\lambda^4) \tag{3.4}$$

since

$$\sin^2\lambda\pi = 4\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi. \tag{3.5}$$

Using the inequality (see [6])

$$\sin^2\lambda\pi \le H_{ij} \le 4\sin^2\frac{\lambda}{2}\pi \tag{3.6}$$

and the identity (3.5) it follows that

$$4\left(\lambda+\frac{1}{4}\lambda\left(1-\lambda^4\right)\right)^2\cos^2\frac{\lambda}{2}\pi \le H_{ij} \le 4\left(\lambda+\frac{\pi-2}{2}\lambda(1-\lambda^4)\right)^2 \tag{3.7}$$

let $1 \le i < j \le n$.Taking the sum for all the inequalities in (3.7), we obtain (3.2), and the proof of Theorem 3.1 is thus complete. □

# References

[1] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, (1970).

[2] FENG QI, Extensions and sharpenings of Jordan's and Kober's inequality, *Journal of Mathematics for Technology* (in Chinese), **4** (1996), 98–101.

[3] J.-CH. KUANG, *Applied Inequalities*, 3rd ed., Jinan Shandong Science and Technology Press, 2003.

[4] L. YANG, Distribution of values and new research, *Beijing Science Press* (in Chinese),(1982).

[5] C.J. ZHAO AND L. DEBNATH, On generalizations of L.Yang's inequality, *J. Inequal. Pure Appl. Math.*, **4** (3)(2002), Art. 56. [ONLINE http://jipam.vu.edu.au/article.php?sid=208]

[6] C.J. ZHAO, The extension and strength of Yang Le inequality, *Math. Practice Theory* (in Chinese), **4** (2000), 493–497

[7] C.J. ZHAO, On several new inequalities, *Chinese Quarterly Journal of Mathematics*, **2** (2001), 42–46.

[8] G.D. ANDERSON, S.-L. QIU, M.K. VAMANAMURTHY AND M. VUORINEN, Generalized elliptic integrals and modular equations, *Pacific J. Math.*, **192** (2000), 1–37.