

WEI DONG JIANG AND HUA YUN

Department of Information Engineering
Weihai Vocational College
Weihai 264200
Shandong Province, P.R. CHINA.

*E*Mail: jackjwd@163.com

*E*Mail: nyjj2006@163.com



volume 7, issue 3, article 102,
2006.

Received 12 November, 2005;
accepted 07 February, 2006.

Communicated by: P.S. Bullen

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

In this paper, the following inequality:

$$\frac{2}{\pi} + \frac{1}{2\pi^5}(\pi^4 - 16x^4) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}(\pi^4 - 16x^4)$$

is established. An application of this inequality gives an improvement of Yang Le's inequality.

2000 Mathematics Subject Classification: Primary 26A51, 26D07, 26D15.

Key words: Jordan inequality, Yang Le inequality, Upper-lower bound.

Contents

1	Introduction	3
2	Main Result	4
3	Applications	6
	References	

Sharpening of Jordan's Inequality and its Applications

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

Close

Quit

Page 2 of 8

1. Introduction

The following result is known as Jordan's inequality [1]:

Theorem 1.1.

$$(1.1) \quad \frac{\sin x}{x} \geq \frac{2}{\pi}, \quad x \in (0, \pi/2].$$

The inequality (1.1) is sharp with equality if and only if $x = \frac{\pi}{2}$.

Jordan's inequality and its refinements have been considered by a number of other authors (see [2], [3]). In [2] Feng Qi obtained new lower and upper bounds for the function $\frac{\sin x}{x}$; his result reads as follows:

Theorem 1.2. *Let $x \in (0, \pi/2]$, then*

$$(1.2) \quad \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2),$$

with equality if and only if $x = \frac{\pi}{2}$.

In this paper we will consider a new refined form of Jordan's inequality and an application of it on the same problem considered by Zhao [5] – [7]. Our main result is given by the following.



**Sharpening of Jordan's
Inequality and its Applications**

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

Close

Quit

Page 3 of 8

2. Main Result

In order to prove Theorem 2.2 below, we need the following lemma.

Lemma 2.1 ([8]). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two continuous functions which are differentiable on (a, b) , let $g' \neq 0$ on (a, b) , if $\frac{f'}{g'}$ is decreasing on (a, b) , then the functions

$$\frac{f(x) - f(b)}{g(x) - g(b)} \quad \text{and} \quad \frac{f(x) - f(a)}{g(x) - g(a)}$$

are also decreasing on (a, b) .

Theorem 2.2. If $x \in (0, \pi/2]$, then

$$(2.1) \quad \frac{2}{\pi} + \frac{1}{2\pi^5}(\pi^4 - 16x^4) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5}(\pi^4 - 16x^4)$$

with equality if and only if $x = \frac{\pi}{2}$.

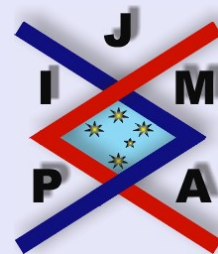
Proof. Let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -16x^4$, $f_3(x) = \sin x - x \cos x$, $f_4(x) = x^5$, and $x \in (0, \pi/2]$, then we have.

$$\begin{aligned} \frac{f_1'(x)}{f_2'(x)} &= \frac{1}{64} \cdot \frac{\sin x - x \cos x}{x^5} = \frac{1}{64} \cdot \frac{f_3(x)}{f_4(x)}. \\ \frac{f_3'(x)}{f_4'(x)} &= \frac{1}{5} \cdot \frac{\sin x}{x^3}. \end{aligned}$$

It is well-known that $\frac{\sin x}{x^3}$ is decreasing on $(0, \frac{\pi}{2})$, so $\frac{f_3'(x)}{f_4'(x)}$ is decreasing on $(0, \frac{\pi}{2})$.

By Lemma 2.1,

$$\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_4(x) - f_4(0)}$$



Sharpening of Jordan's
Inequality and its Applications

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

Close

Quit

Page 4 of 8

is decreasing on $(0, \frac{\pi}{2})$, so $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on $(0, \frac{\pi}{2})$, then

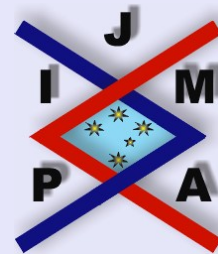
$$h(x) = \frac{f_1(x) - f_1(\frac{\pi}{2})}{f_2(x) - f_2(\frac{\pi}{2})} = \frac{\frac{\sin x}{x} - \frac{\pi}{2}}{\pi^4 - 16x^4}$$

is decreasing on $(0, \frac{\pi}{2})$. By Lemma 2.1.

Furthermore, $\lim_{x \rightarrow 0^+} h(x) = \frac{\pi-2}{\pi^5}$, $\lim_{x \rightarrow \frac{\pi}{2}^-} h(x) = \frac{1}{2\pi^5}$. Thus $\frac{\pi-2}{\pi^5}$ and $\frac{1}{2\pi^5}$ are the best constants in (2.1). So the proof is complete \square

Note: In a similar manner, we can obtain several interesting inequalities similar to (2.2). For example, let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -4x^2$, $f_3(x) = \sin x - x \cos x$, $f_4(x) = x^3$, and $x \in (0, \pi/2]$, then (1.2) is obtained. If we let $f_1(x) = \frac{\sin x}{x}$, $f_2(x) = -8x^3$, $f_3(x) = \sin x - x \cos x$, $f_4(x) = x^4$, then we have

$$\frac{2}{\pi} + \frac{2}{3\pi^4}(\pi^3 - 8x^3) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^4}(\pi^3 - 8x^3).$$



Sharpening of Jordan's Inequality and its Applications

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

Close

Quit

Page 5 of 8

3. Applications

Yang Le's inequality [4] and its generalizations which play an important role in the theory of distribution of values of functions can be stated as follows.

If $A > 0$, $B > 0$, $A + B \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$(3.1) \quad \cos^2 \lambda A + \cos^2 \lambda B - 2 \cos \lambda A \cos \lambda B \cos \lambda \pi \geq \sin^2 \lambda \pi.$$

In [5] – [7] some improvements of Yang Le's inequality are obtained. In a similar way, based on the inequality (2.2) we can give the following.

Theorem 3.1. Let $A_i > 0$ ($i = 1, 2, \dots, n$), $\sum_{i=1}^n A_i \leq \pi$, $n \in \mathbb{N}$ and $n \neq 1$, $0 \leq \lambda \leq 1$, then

$$(3.2) \quad R(\lambda) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq T(\lambda),$$

where

$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda A_i \cos \lambda A_j \cos \lambda \pi,$$

$$R(\lambda) = 4C_n^2 \left(\lambda + \frac{1}{4} \lambda (1 - \lambda^4) \right)^2 \cos^2 \frac{\lambda}{2} \pi,$$

$$T(\lambda) = 4C_n^2 \left(\lambda + \frac{\pi - 2}{2} \lambda (1 - \lambda^4) \right)^2.$$

Proof. Substituting $x = \frac{\lambda}{2} \pi$ in (2.2), we have

$$(3.3) \quad \sin \frac{\lambda}{2} \pi \geq \lambda + \frac{1}{4} \lambda (1 - \lambda^4)$$



Sharpening of Jordan's Inequality and its Applications

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

Close

Quit

Page 6 of 8

and

$$(3.4) \quad \sin \frac{\lambda}{2} \pi \leq \lambda + \frac{\lambda - 2}{2} \lambda (1 - \lambda^4)$$

since

$$(3.5) \quad \sin^2 \lambda \pi = 4 \sin^2 \frac{\lambda}{2} \pi \cos^2 \frac{\lambda}{2} \pi.$$

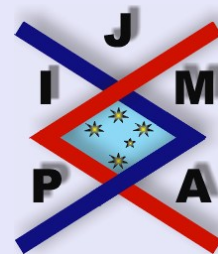
Using the inequality (see [6])

$$(3.6) \quad \sin^2 \lambda \pi \leq H_{ij} \leq 4 \sin^2 \frac{\lambda}{2} \pi$$

and the identity (3.5) it follows that

$$(3.7) \quad 4 \left(\lambda + \frac{1}{4} \lambda (1 - \lambda^4) \right)^2 \cos^2 \frac{\lambda}{2} \pi \leq H_{ij} \leq 4 \left(\lambda + \frac{\pi - 2}{2} \lambda (1 - \lambda^4) \right)^2$$

let $1 \leq i < j \leq n$. Taking the sum for all the inequalities in (3.7), we obtain (3.2), and the proof of Theorem 3.1 is thus complete. \square



**Sharpening of Jordan's
Inequality and its Applications**

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

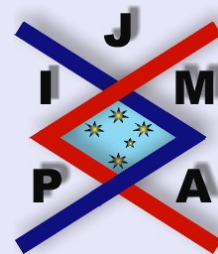
Close

Quit

Page 7 of 8

References

- [1] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, (1970).
- [2] FENG QI, Extensions and sharpenings of Jordan's and Kober's inequality, *Journal of Mathematics for Technology* (in Chinese), **4** (1996), 98–101.
- [3] J.-CH. KUANG, *Applied Inequalities*, 3rd ed., Jinan Shandong Science and Technology Press, 2003.
- [4] L. YANG, Distribution of values and new research, *Beijing Science Press* (in Chinese),(1982).
- [5] C.J. ZHAO AND L. DEBNATH, On generalizations of L. Yang's inequality, *J. Inequal. Pure Appl. Math.*, **4** (3)(2002), Art. 56. [ONLINE <http://jipam.vu.edu.au/article.php?sid=208>]
- [6] C.J. ZHAO, The extension and strength of Yang Le inequality, *Math. Practice Theory* (in Chinese), **4** (2000), 493–497
- [7] C.J. ZHAO, On several new inequalities, *Chinese Quarterly Journal of Mathematics*, **2** (2001), 42–46.
- [8] G.D. ANDERSON, S.-L. QIU, M.K. VAMANAMURTHY AND M. VUORINEN, Generalized elliptic integrals and modular equations, *Pacific J. Math.*, **192** (2000), 1–37.



Sharpening of Jordan's Inequality and its Applications

Wei Dong Jiang and Hua Yun

Title Page

Contents



Go Back

Close

Quit

Page 8 of 8