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## A MULTIPLICATIVE EMBEDDING INEQUALITY IN ORLICZ-SOBOLEV SPACES

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Abstract

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## Abstract

We prove an Orlicz type version of the multiplicative embedding inequality for Sobolev spaces.

*2000 Mathematics Subject Classification:* 46E35, 26D15, 46E30.

*Key words:* Orlicz spaces, Sobolev embedding theorem, Orlicz-Sobolev spaces.

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# 1. Introduction and Preliminary Results

Let  $\Omega$  be a non-empty bounded open set in  $\mathbb{R}$ ,  $n > 1$  and let  $1 \leq p < n$ . The most important result of Sobolev space theory is the well-known *Sobolev imbedding theorem* (see e.g. [1]), which - in the case of functions vanishing on the boundary - gives an estimate of the norm in the Lebesgue space  $L^q(\Omega)$ ,  $q = np/(n - p)$  of a function  $u$  in the Sobolev space  $W_0^{1,p}(\Omega)$ , in terms of its  $W_0^{1,p}(\Omega)$ -norm. Such an estimate, due to Gagliardo and Nirenberg ([6], [12]) can be stated in the following multiplicative form (see e.g. [4], [10]).

**Theorem 1.1.** *Let  $\Omega$  be a non-empty bounded open set in  $\mathbb{R}$ ,  $n > 1$  and let  $1 \leq p < n$ . Let  $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$  for some  $r \geq 1$ . If  $q$  lies in the closed interval bounded by the numbers  $r$  and  $np/(n - p)$ , then the following inequality holds*

$$(1.1) \quad \|u\|_q \leq c \| \|Du\|_p^\theta \|u\|_r^{1-\theta},$$

where

$$\theta = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{n} - \frac{1}{p} + \frac{1}{r}} \in [0, 1]$$

and

$$c = c(n, p, \theta) = \left[ \frac{p(n-1)}{n-p} \right]^\theta.$$

The constant  $c = c(n, p, \theta)$  is not optimal (see [16], [7] for details).

The goal of this paper is to provide an Orlicz version of inequality (1.1), in which the role of the parameter  $\theta$  is played by a certain concave function. Our



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approach uses a generalized Hölder inequality proved in [8] (see Lemma 1.2 below).

We summarize some basic facts of Orlicz space theory; we refer the reader to Krasnosel'skiĭ and Rutickiĭ [9], Maligranda [11], or Rao and Ren [14] for further details.

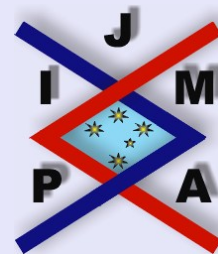
A function  $A : [0, \infty) \rightarrow [0, \infty)$  is an  $N$ -function if it is continuous, convex and strictly increasing, and if  $A(0) = 0$ ,  $A(t)/t \rightarrow 0$  as  $t \rightarrow 0$ ,  $A(t)/t \rightarrow +\infty$  as  $t \rightarrow +\infty$ .

If  $A, B$  are  $N$ -functions (in the following we will adopt the next symbol for the inverse function of  $N$ -functions, too), we write  $A(t) \approx B(t)$  if there are constants  $c_1, c_2 > 0$  such that  $c_1 A(t) \leq B(t) \leq c_2 A(t)$  for all  $t > 0$ . Also, we say that  $B$  dominates  $A$ , and denote this by  $A \preceq B$ , if there exists  $c > 0$  such that for all  $t > 0$ ,  $A(t) \leq B(ct)$ . If this is true for all  $t \geq t_0 > 0$ , we say that  $A \preceq B$  near infinity.

An  $N$ -function  $A$  is said to be doubling if there exists a positive constant  $c$  such that  $A(2t) \leq cA(t)$  for all  $t > 0$ ;  $A$  is called submultiplicative if  $A(st) \leq cA(s)A(t)$  for all  $s, t > 0$ . Clearly  $A(t) = t^r$ ,  $r \geq 1$ , is submultiplicative. A straightforward computation shows that  $A(t) = t^a [\log(e + t)]^b$ ,  $a \geq 1$ ,  $b > 0$ , is also submultiplicative.

Given an  $N$ -function  $A$ , the Orlicz space  $L_A(\Omega)$  is the Banach space of Lebesgue measurable functions  $f$  such that  $A(|f|/\lambda)$  is (Lebesgue) integrable on  $A$  for some  $\lambda > 0$ . It is equipped with the Luxemburg norm  $\|f\|_A = \inf \left\{ \lambda > 0 : \int_{\Omega} A \left( \frac{|f|}{\lambda} \right) dx \leq 1 \right\}$ .

If  $A \preceq B$  near infinity then there exists a constant  $c$ , depending on  $A$  and  $B$ ,



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such that for all functions  $f$ ,

$$(1.2) \quad \|f\|_A \leq c\|f\|_B.$$

This follows from the standard embedding theorem which shows that  $L_B(\Omega) \subset L_A(\Omega)$ .

Given an  $N$ -function  $A$ , the complementary  $N$ -function  $\tilde{A}$  is defined by

$$\tilde{A}(t) = \sup_{s>0} \{st - A(s)\}, \quad t \geq 0.$$

The  $N$ -functions  $A$  and  $\tilde{A}$  satisfy the following inequality (see e.g. [1, (7) p. 230]):

$$(1.3) \quad t \leq A^{-1}(t)\tilde{A}^{-1}(t) \leq 2t.$$

The Hölder's inequality in Orlicz spaces reads as

$$\int_{\Omega} |fg| dx \leq 2\|f\|_A\|g\|_{\tilde{A}}.$$

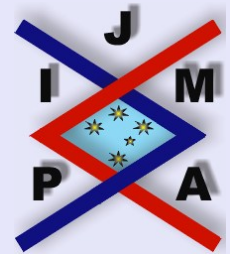
We will need the following generalization of Hölder's inequality to Orlicz spaces due to Hogan, Li, McIntosh, Zhang [8] (see also [3] and references therein).

**Lemma 1.2.** *If  $A$ ,  $B$  and  $C$  are  $N$ -functions such that for all  $t > 0$ ,*

$$B^{-1}(t)C^{-1}(t) \leq A^{-1}(t),$$

then

$$\|fg\|_A \leq 2\|f\|_B\|g\|_C.$$




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If  $A$  is an  $N$ -function, let us denote by  $W^{1,A}(\Omega)$  the space of all functions in  $L^A(\Omega)$  such that the distributional partial derivatives belong to  $L^A(\Omega)$ , and by  $W_0^{1,A}(\Omega)$  the closure of the  $C_0^\infty(\Omega)$  functions in this space. Such spaces are well-known in the literature as *Orlicz-Sobolev* spaces (see e.g. [1]) and share various properties of the classical Sobolev spaces. References for main properties and applications are for instance [5] and [15].

If  $u \in W_0^{1,A}(\Omega)$  and

$$\int_1^\infty \frac{\tilde{A}(s)}{s^{n'+1}} ds = +\infty, \quad n' = n/(n-1)$$

then the continuous embedding inequality

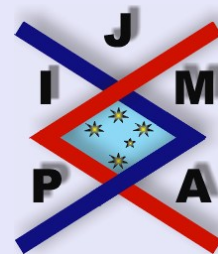
$$(1.4) \quad \|u\|_{A^*} \leq c \|Du\|_A$$

holds, where  $A^*$  is the so-called *Sobolev conjugate* of  $A$ , defined in [1], and  $c$  is a positive constant depending only on  $A$  and  $n$ . In the following it will be not essential, for our purposes, to know the exact expression of  $A^*$ . However, we stress here that one could consider the *best* function  $A^*$  such that inequality (1.4) holds (see [2], [13] for details).

In the sequel we will need the following definition.

**Definition 1.1.** Given an  $N$ -function  $A$ , define the function  $h_A$  by

$$h_A(s) = \sup_{t>0} \frac{A(st)}{A(t)}, \quad 0 \leq s < \infty.$$



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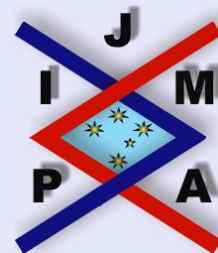
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**Remark 1.** The function  $h_A$  could be infinite if  $s > 1$ , but if  $A$  is doubling then it is finite for all  $0 < s < \infty$  (see Maligranda [11, Theorem 11.7]). If  $A$  is submultiplicative then  $h_A \approx A$ . More generally, given any  $A$ , for all  $s, t \geq 0$ ,  $A(st) \leq h_A(s)A(t)$ .

The property of the function  $h_A$  which will play a role in the following is that it can be inverted, in fact the following lemma holds.

**Lemma 1.3.** If  $A$  is a doubling  $N$ -function then  $h_A$  is nonnegative, submultiplicative, strictly increasing in  $[0, \infty)$  and  $h_A(1) = 1$ .

For the (easy) proof see [3, Lemma 3.1] or [11, p. 84].



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## 2. The Main Result

We will begin by proving two auxiliary results. The first one concerns two functions that we call  $K = K(t)$  and  $H = H(t)$ : they are a way to “measure”, in the final multiplicative inequality, how far the right hand side is with respect to the norms of  $u$  and of  $|Du|$ . In the standard case it is  $K(t) = t^\theta$ ,  $0 \leq \theta \leq 1$  and  $H(t) = t^{1-\theta}$ .

**Lemma 2.1.** *Let  $K \in \mathcal{C}([0, +\infty[) \cap \mathcal{C}^2(]0, +\infty[)$  be:*

- a positive, constant function,

or

-  $K(t) = \alpha t$  for some  $\alpha > 0$ ,

or

- the inverse function of an  $N$ -function which is doubling together with its complementary  $N$ -function.

Then the function  $H : [0, +\infty[ \rightarrow [0, +\infty[$  defined by

$$H(t) = \begin{cases} \frac{t}{K(t)} & \text{if } t > 0 \\ \lim_{t \rightarrow 0} \frac{t}{K(t)} & \text{if } t = 0 \end{cases}$$

belongs to  $\mathcal{C}([0, +\infty[) \cap \mathcal{C}^2(]0, +\infty[)$ , and is:

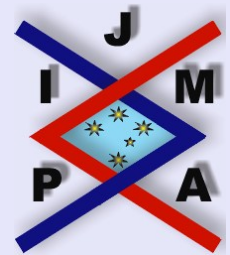
- a positive, constant function,

or

-  $H(t) = \beta t$  for some  $\beta > 0$ ,

or

- is equivalent to the inverse function of an  $N$ -function which is doubling together with its complementary  $N$ -function.



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*Proof.* In the first two possibilities for  $K$  the statement is easy to prove. If  $K$  is the inverse of a doubling  $N$ -function  $A$ , it is sufficient to observe that from inequality (1.3) it is  $H \approx A^{-1}$ .  $\square$

**Lemma 2.2.** *Let  $\Phi$  be an  $N$ -function, and let  $F$  be a doubling  $N$ -function such that  $\Phi \circ F^{-1}$  is an  $N$ -function. The following inequality holds for every  $u \in L^\Phi(\Omega)$ :*

$$(2.1) \quad \|u\|_\Phi \leq \xi_{F^{-1}}(\|F \circ |u|\|_{\Phi \circ F^{-1}}),$$

where  $\xi_{F^{-1}}$  is the increasing function defined by

$$(2.2) \quad \xi_{F^{-1}}(\mu) = \frac{1}{h_F^{-1}\left(\frac{1}{\mu}\right)} \quad \forall \mu > 0.$$

*Proof.* By definition of  $h_F$  (see Definition 1.1; note that by the assumption that  $F$  is doubling,  $h_F$  is everywhere finite, see Remark 1) we have

$$F(s)h_F(t) \geq F(st) \quad \forall s, t > 0$$

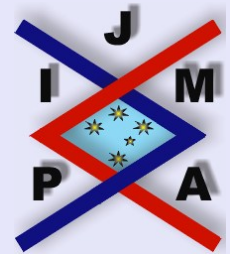
and therefore

$$sh_F(t) \geq F(F^{-1}(s)t) \quad \forall s, t > 0,$$

$$(2.3) \quad F^{-1}(sh_F(t)) \geq F^{-1}(s)t \quad \forall s, t > 0.$$

Setting

$$\mu = \mu(\lambda) = \frac{1}{h_F\left(\frac{1}{\lambda}\right)}$$



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it is

$$\lambda = \frac{1}{h_F^{-1}\left(\frac{1}{\mu}\right)} := \xi_{F^{-1}}(\mu),$$

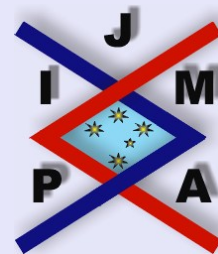
therefore from inequality (2.3), for  $t = \frac{1}{\lambda}$  and  $s = F(|u|)$ , taking into account that  $\xi_{F^{-1}}$  is increasing, we have

$$\begin{aligned} \|u\|_{\Phi} &= \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(\frac{F^{-1}(F(|u|))}{\lambda}\right) dx \leq 1 \right\} \\ &\leq \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(F^{-1}\left(F(|u|)h_F\left(\frac{1}{\lambda}\right)\right)\right) dx \leq 1 \right\} \\ &= \inf \left\{ \xi_{F^{-1}}(\mu) > 0 : \int_{\Omega} \Phi\left(F^{-1}\left(\frac{F(|u|)}{\mu}\right)\right) dx \leq 1 \right\} \\ &= \xi_{F^{-1}}\left(\inf \left\{ \mu > 0 : \int_{\Omega} \Phi\left(F^{-1}\left(\frac{F(|u|)}{\mu}\right)\right) dx \leq 1 \right\}\right) \\ &= \xi_{F^{-1}}(\|F \circ |u|\|_{\Phi \circ F^{-1}}) \end{aligned}$$

□

We can prove now the main theorem of the paper. The symbol  $\xi_K$  which appears in the statement is the function considered in Lemma 2.2, defined in equation (2.2). However, since this symbol is used for any function  $K$  considered in Lemma 2.1, we agree to denote

$$\xi_K(\mu) := 1 \quad \forall \mu \geq 0 \quad \text{if} \quad K \text{ is constant}$$



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and

$$\xi_K(\mu) := \mu \quad \forall \mu \geq 0 \quad \text{if} \quad K(t) = \alpha t \text{ for some } \alpha > 0.$$

The same conventions will be adopted for the symbol  $\xi_H$ . Note that from Lemma 2.1 we know that  $H$  is *equivalent* to the inverse of a doubling  $N$ -function, let us call it  $B^{-1}$ . We will agree to denote still by  $\xi_H$  the function that we should denote by  $\xi_{B^{-1}}$ . This convention does not create ambiguities because if  $B \approx C$  then  $h_B \approx h_C$  and  $\xi_{B^{-1}} \approx \xi_{C^{-1}}$ , therefore  $\xi_H$  is well defined up to a multiplicative positive constant.

**Theorem 2.3.** *Let  $\Omega$  be a non-empty bounded open set in  $\mathbb{R}$ ,  $n > 1$  and let  $P$  be an  $N$ -function satisfying*

$$\int_1^\infty \frac{\tilde{P}(s)}{s^{n'+1}} ds = +\infty, \quad n' = n/(n-1).$$

*Let  $u \in W_0^{1,P}(\Omega) \cap L^R(\Omega)$  for some  $N$ -function  $R$ . If  $Q$  is an  $N$ -function such that*

$$(2.4) \quad K((P^*)^{-1}(s)) \cdot H(R^{-1}(s)) \leq Q^{-1}(s) \quad \forall s > 0$$

*then the following inequality holds*

$$(2.5) \quad \|u\|_Q \leq \xi_K(c \|Du\|_P) \xi_H(\|u\|_R),$$

*where  $K$  and  $H$  are functions as in Lemma 2.1 and  $c$  is a constant depending only on  $n, P, K$ .*



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*Proof.* Let  $K$  and  $H$  be functions as in Lemma 2.1. If  $K$  is a positive, constant function or  $K(t) = \alpha t$  for some  $\alpha > 0$ , then the statement reduces respectively to a direct consequence of inequality (1.2) (with  $A$  and  $B$  replaced respectively by  $Q$  and  $R$ ) or to inequality (1.4) (with  $A$  replaced by  $P$ ). We may therefore assume in the following that  $K$  is the inverse function of an  $N$ -function which is doubling together with its complementary  $N$ -function. Let

$$\Phi_1 = P^* \circ K^{-1} \quad \Phi_2 = R \circ H^{-1}.$$

It is easy to verify that  $\Phi_1$  and  $\Phi_2$  are  $N$ -functions. By assumption (2.4) and Lemma 1.2 we have

$$(2.6) \quad \|u\|_Q = \|K(u)H(u)\|_Q \leq \|K(u)\|_{\Phi_1} \|H(u)\|_{\Phi_2}.$$

By inequality (2.1),

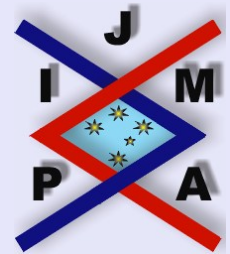
$$(2.7) \quad \|K(u)\|_{\Phi_1} \leq \xi_K(\|u\|_{\Phi_1 \circ K}) = \xi_K(\|u\|_{P^*}) \leq \xi_K(c\|Du\|_P),$$

where  $c$  is a positive constant depending on  $n$  and  $P$  only. On the other hand,

$$(2.8) \quad \|H(u)\|_{\Phi_2} \leq \xi_H(\|u\|_{\Phi_2 \circ H}) = \xi_H(\|u\|_R).$$

From inequalities (2.6), (2.7), (2.8), we get the inequality (2.5) and the theorem is therefore proved.  $\square$

We remark that the natural choice of powers for  $P, Q, R, K, H$  reduce Theorem 2.3 to Theorem 1.1 (in Theorem 2.3 also the case  $p = n$  is allowed); on the other hand, if inequality (2.5) allows growths of  $\xi_K$  different power types, in general it is not true that  $\xi_K(t)\xi_H(t) = t$ , and this is the “price” to pay for the major “freedom” given to the growth  $K$ .




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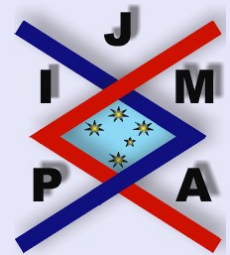
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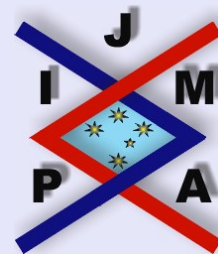
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