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ON MINKOWSKI AND HARDY INTEGRAL INEQUALITIES

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ABSTRACT. The reverse Minkowski's integral inequality:

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leq c\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}},\ \ p>1,$$

where c is a positive constant, and the following Hardy's inequality:

$$\int_0^\infty \left(\frac{F_1(x)F_2(x)\cdots F_i(x)}{x^i}\right)^{\frac{p}{i}} dx \leq \left(\frac{p}{ip-i}\right)^p \int_0^\infty (f_1(x)+f_2(x)+\cdots+f_i(x))^p\,dx, \quad p>1,$$

where

$$F_k(x) = \int_a^x f_k(t)dt, \quad \text{where } k=1,\dots,i$$

are proved.

Key words and phrases: Minkowski's inequality, Hardy's inequality.

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1. THE REVERSE MINKOWSKI INTEGRAL INEQUALITY

In [1, 3, 4], the well- known Minkowski integral inequality is given as follows:

Theorem 1.1. *Let* $p \geq 1$, $0 < \int_a^b f^p(x)dx < \infty$ *and* $0 < \int_a^b g^p(x)dx < \infty$. *Then*

$$\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}} \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}}. \tag{1.1}$$

In this section we establish the following reverse Minkowski integral inequality

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Theorem 1.2. *Let f and g be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \qquad \forall x \in [a,b]. \tag{1.2}$$

Then

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leq c\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}}, \tag{1.3}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \leq M$, $f \leq M(f+g) - Mf$. Therefore

$$(M+1)^p f^p \leq M^p (f+g)^p \tag{1.4}$$

and so,

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leq \frac{M}{M+1}\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.5}$$

On the other hand, since $mg \leq f$. Hence

$$g \leq \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x). \tag{1.6}$$

Therefore,

$$\left(\frac{1}{m}+1\right)^p g^p(x) \leq \left(\frac{1}{m}\right)^p (f(x)+g(x))^p, \tag{1.7}$$

and so,

$$\left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leq \frac{1}{m+1}\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}}. \tag{1.8}$$

Now add the inequalities (1.5)and (1.8) to get the desired inequality (1.1).
Thus, (1.1) is proved. □

2. Hardy Integral Inequality Involving Many Functions

Hardy's inequality [2, 5] reads:

Theorem 2.1. *Let f be a nonnegative integrable function. Define $F(x) = \int_a^x f(t)dt$. Then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty (f(x))^p\,dx, \qquad p > 1. \tag{2.1}$$

Our purpose in this section is to prove the Hardy inequality for several functions.

Theorem 2.2. *Let $f_1, f_2, \ldots, f_i$ be nonnegative integrable functions. Define $F_k(x) = \int_a^x f_k(t)dt$, where $k = 1, \ldots, i$. Then*

$$\int_0^\infty \left(\frac{F_1(x)F_2(x)\cdots F_i(x)}{x^i}\right)^{\frac{p}{i}} dx \leq \left(\frac{p}{ip-i}\right)^p \int_0^\infty (f_1(x)+f_2(x)+\cdots+f_i(x))^p\,dx. \tag{2.2}$$

Proof. By using Jensen's inequality [6, 7]

$$(F_1(x)F_2(x)\cdots F_i(x))^{\frac{1}{i}} \le \frac{\sum_{k=1}^{i} F_k(x)}{i}, \tag{2.3}$$

and so,

$$(F_1(x)F_2(x)\cdots F_i(x))^{\frac{p}{i}} \le \frac{\left(\sum_{k=1}^{i} F_k(x)\right)^p}{i^p}. \tag{2.4}$$

Divide both sides of (2.4) by x^p and integrate resulting the inequality to get

$$\int_0^\infty \left(\frac{F_1(x)F_2(x)\cdots F_i(x)}{x^i}\right)^{\frac{p}{i}} dx \le \frac{1}{i^p}\int_0^\infty \left(\frac{F_1(x)+F_2(x)+\cdots+F_i(x)}{x}\right)^p dx. \tag{2.5}$$

Applying inequality (2.1) to the right hand side of (2.5) we get (2.2). □

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