journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 1, Article 3, 16 pp.

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VACCINATION STRATEGIES OF POPULATION GROUPS WITH DISTINCT PERCEIVED PROBABILITIES OF INFECTION

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Received 08 December, 2007; accepted 07 August, 2008

Communicated by S.S. Dragomir

ABSTRACT. Previous modelling studies have formalized the "Tragedy of the Commons" that can occur under a voluntary vaccination policy, when there is a significant payoff not to vaccinate under conditions where high vaccine coverage affords indirect protection to nonvaccinators through herd immunity effects. Most of these previous studies have considered only a homogeneous population. However, in real populations, vaccine uptake can vary enormously across different social groups, often leading to localized outbreaks. In this paper, we consider a population under a voluntary vaccination policy consisting of distinct social groups. Unlike previous work on vaccination game theory in heterogeneous populations, these social groups differ both in the perceived vaccine risk as well as the perceived probability of becoming infected. Using game theory, projected dynamical systems theory, and variational inequality theory, we characterize the Nash equilibria of the system and analyze the game dynamics. The approach allows us to predict, in principle, the vaccine coverage in various social groups with distinct perceived vaccine and infection risks, where individuals are attempting to minimize health risks. We find that, under a wide range of parameter values, the vaccine coverage in a multi-group population can be higher than the vaccine coverage in the corresponding homogeneous population with the same average perceived relative risk of vaccination. This paper generalizes previous work by Cojocaru et al [10] on applications of PDS and VI in vaccine game theory.

Key words and phrases: Vaccination strategies games; Population dynamics; Dynamical systems and games.

2000 *Mathematics Subject Classification.* 92D25, 91A10, 37N25.

1. INTRODUCTION

Voluntary vaccination policies have sometimes been compared to a Prisoner's Dilemma [5, 6]. When vaccine coverage is very high, unvaccinated individuals are protected through the herd immunity phenomenon, which can create an individual incentive not to vaccinate (particularly if there is a perceived risk associated with the vaccine). Hence, a voluntary vaccination policy can be a victim of its own success. This strategic interaction between individuals, where the

M.-G. Cojocaru and C.T. Bauch are supported by the Natural Sciences and Engineering Council of Canada.

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payoff (health level) to an individual for vaccinating or not vaccinating depends partly upon whether or not other individuals in the population have decided to vaccinate, describes a game and can be analyzed using game theory [5, 6]. Previous game theoretical analyses of voluntary vaccination policies have shown how this Prisoner's Dilemma effect may lead to suboptimal vaccine coverage levels in the population.

Perhaps partly because of this effect, vaccine "scares" have existed since the first vaccines were invented, and have occurred for smallpox, pertussis, Hepatitis B, polio, and measles-mumps-rubella vaccines, among others [19, 14, 1, 27, 28, 8, 22]. In addition to the rapid declines in vaccine coverage that occur during vaccine "scares", vaccine coverage can be persistently low for a number of reasons having to do with supply (for instance, insufficient resources to deliver vaccines) or demand (for instance, lack of knowledge of vaccination programmes, perception that diseases are not sufficiently dangerous to vaccinate against, religious beliefs against vaccination).

Here, we describe the literature on vaccine/disease risk perception and how it influences vaccine uptake. A number of studies indicate widespread belief that vaccines are dangerous, relative to the diseases they prevent, and that this misperception of vaccine and disease risk can influence uptake [3, 29, 24, 16, 30, 7, 31]. For instance, a study in the Hackney region of London indicated that 34% of parents thought that immunization is more dangerous than getting childhood diseases [31]. Likewise, a population-wide study in Germany showed widespread belief that experiencing certain common pediatric infectious diseases is a natural and beneficial process (23%), and that vaccines are dangerous (25%) [30]. Nonvaccinating behaviour is apparently correlated with such beliefs. For instance, a study in readers of *Mothering* magazine indicated a positive correlation between non-vaccinating behaviour for DTP vaccine and beliefs that (a) vaccines are risky, and (b) diseases are natural [3].

Nonvaccinating behaviour is also related to individual's beliefs as to how likely they are to become infected. The same study in readers of *Mothering* magazine indicated a positive correlation between non-vaccinating behaviour and the belief that vaccination is not needed because other parents have vaccinated and disease is under control [3] (see also Ref. [21]. For influenza, it has been shown in a large number of studies that whether or not an individual decides to vaccinate depends to a significant degree upon their perceived probability of their becoming infected [9], and similar effects have been documented for measles [20].

Unsurprisingly, perceptions of vaccine and disease risk, and vaccine uptake, can also vary across distinct groups in a given population, with differences occurring along socioeconomic and religious divisions [14, 24, 31, 32]. It has been speculated that causative factors in low vaccine uptake in certain social groups include mistrust of authorities and lack of effective communication between communities and health authorities [24, 27, 31, 7, 32].

It is increasingly recognized that accounting for the interaction between human behaviour and disease transmission in epidemiological models is a necessary and valuable goal [17, 25]. The situation of vaccine uptake and risk perception illustrates a case in point, where individual vaccinating decisions influence overall vaccine coverage and hence the force of infection, which in turn influences individual vaccination decisions. Most previous game theoretical analyses of vaccination have assumed a homogeneous population where all individuals have the same perceived risks of complications due to the vaccine, risks of becoming infected, or risks of experiencing significant complications due to infection. Some recent game theoretical work has started to consider the dynamics of vaccination behaviour in a heterogeneous population with distinct social groups [10]. This work concluded that, for the same average perceived risk of the vaccine compared to the risk of having the disease, a 2-group population with a vaccine-averse minority group can, at many parameter values, have a higher overall vaccine coverage level than the corresponding 1-group population. This work assumed that groups vary only in

the perceived risks of complications due to getting vaccinated or having the disease, and not in the perceived probability of becoming infected. For the present study, we generalize this work by (1) allowing the perceived probability of becoming infected to also vary across social groups, (2) exploring an alternative functional form for the perceived probability of becoming infected, and (3) exploring model dynamics when there is a relationship between the relative risk of vaccine/disease, and the perceived probability of infection.

The mathematical approach used in [10] for deriving solutions to the vaccination game is that of finite-dimensional projected dynamical systems (PDS) and variational inequalities (VI). This approach is widely used in operations research, economic theory, finance and network analysis (see for example [26] and the references therein). Most recently, in [11], the problem of time-dependent vaccination games has been considered, through the use of infinite-dimensional PDS and infinite-dimensional VI (called *evolutionary* variational inequalities).

In general, a PDS is a dynamical system whose flow is constrained to evolve on a closed and convex subset, generically denoted by $\mathbb{K}$, of the ambient space. In this paper we consider the ambient space to be the Euclidean space $\mathbb{R}^k$ and we consider the constraint set $\mathbb{K}$ to be a k-dimensional cube in $\mathbb{R}^k$. The results present in the PDS literature (both on Euclidean spaces and on more general Hilbert spaces) are based on nonlinear and convex analysis and differential inclusions (see for example [4, 26]).

Our motivation to use a projected dynamical system is twofold. First, it is known that there exists an intimate relation between Nash games and variational inequality problems [18] and between variational inequality problems and projected dynamical systems (finite- [26] and infinite-dimensional [13]). Thus the critical points of a projected system coincide with the solutions of the underlying game and vice versa. Second, although the projected dynamical system used here is finite-dimensional, the existing literature in finite dimensions does not offer a way of visualizing a projected flow; therefore we use more recent results [12] to compute projected trajectories and their critical points, without using variational inequality algorithms.

Finally, the present paper refines the work in [10] by allowing the perceived probability of becoming infected to vary across population groups. This is achieved by considering an alternative functional form for the perceived probability of becoming infected, and by exploring the dynamics when there is a relationship between the relative risk and the probability of infection.

The paper is organized as follows: Section 2 gives a general overview of how vaccination strategies can be formulated as Nash games. Section 3 shows that the vaccination games we consider have solutions which are stable with respect to global perturbations. Section 4 presents a sample of examples and questions that could be studied using the theoretical context introduced in previous sections. Finally Section 5 contains conclusions and some ideas for future work.

2. VACCINATION GAMES FOR POPULATION GROUPS WITH DISTINCT PERCEIVED PROBABILITIES OF INFECTION

We present here in brief the setup of a vaccination strategies game, using similar notation to that in [6, 10]. We consider a population consisting of a finite number (k) of social groups, where each group may have a different perception of risks associated with vaccination and infection, and therefore may adopt different vaccination strategies. We consider a disease for which there is lifelong natural immunity, and in which individuals are typically infected early in life in the absence of vaccination (this describes the so-called paediatric infectious diseases, such as measles, mumps, rubella, pertussis and chickenpox) [2]. Likewise we consider a vaccine which is administered primarily in the youngest age classes, and in which vaccination coverage

is typically low later in life. In particular, in our case discussions and examples we will refer to parameter values associated with such diseases (see Section 4 below).

We let $i \in \{1, \ldots, k\}$ represent the i-th social group in a population with a finite number of individuals. For the i-th group, we let the perceived probability of significant complications due to vaccination be denoted by r_v^i, the perceived probability of becoming infected given that a proportion p of the population is vaccinated be denoted by π_p^i, and the perceived probability of significant morbidity upon infection be r_{inf}^i. The overall probability of experiencing significant morbidity because of not vaccinating is thus $r_{inf}^i \pi_p^i$. We denote by $r_i := \frac{r_v^i}{r_{inf}^i}$ the relative perceived risk of vaccination versus infection.

Assumption 2.1. *We assume that all individuals within a group share a common assessment of the risks involved with vaccination and infection, r_i, and of the probability of becoming infected, π_p^i, however different groups have different relative risk assessments and distinct perceived probabilities of becoming infected.*

We consider the strategy set for all individuals in group i to be $\{P_i | P_i \in [0, 1_P]\}$, where P_i is the probability that a child in group i is vaccinated. Here $1_P < 1$, but could be chosen very close to 1. This choice of a constraint set is a mathematical necessity (as will be seen in Theorem 3.4), however it does not impact on the interpretation of the results. We therefore wish to find a Nash equilibrium strategy $\underline{P}^* := (P_1^*, P_2^*, \ldots, P_k^*)$, such that when everyone in group i plays P_i^*, no sufficiently small subset of individuals in any group can achieve a higher utility (payoff) by switching to a different strategy $P_i \neq P_i^*$. At P_i^* there should be no incentive to switch strategies, so such strategies should be stable equilibrium solutions of our game. In [10] we derived existence and uniqueness results for solutions of a vaccination game similar to the above using variational inequalities and projected dynamical systems. We will use an analogous approach below.

We let the utility function in a group where the perceived relative risk is r_i, and where the vaccine coverage in the population as a whole is p, be given by

$$u_i(P_i, p) = -r_v^i P_i - r_{inf}^i \pi_p^i (1 - P_i) \text{ subject to } P_i \in [0, 1_P].$$

After rescaling one can rewrite the above as

$$u_i(P_i, p) = -r_i P_i - \pi_p^i (1 - P_i) \text{ subject to } P_i \in [0, 1_P], \text{ where } r_i = \frac{r_v^i}{r_{inf}^i}. \tag{2.1}$$

The players in a given round of the game are the parents of a given cohort of children, who play the game only once (they can decide only once whether or not to vaccinate their child). Future rounds of the game are played by the parents of later cohorts.

In order to find a mathematical expression for π_p^i, one approach is to use equilibrium solutions of a deterministic SIR compartmental model and assume that individuals have perfect knowledge of their probability of eventually becoming infected [6]. However, individuals do not in fact have perfect knowledge of their probability of being infected. In [10] we assumed $\pi_p^i = \pi_p^j$, $\forall i, j \in \{1, \ldots, k\}$ to be a decreasing function of the form $\pi_p^i = \frac{a}{b+p}$, where a and b were constants chosen according to the epidemiology of common paediatric infectious diseases, and p is the proportion vaccinated. This expressed the fact that disease prevalence is a function of how many individuals have been vaccinated. Hence, a higher vaccine coverage p in a population implies a lower perceived probability π_p^i of becoming infected. This simplification made the initial analysis easier.

However, the function π_p^i should represent the perceived probability of infection, not the actual probability of infection, since it is the perceived probability that dictates vaccinating behaviour. There are currently no data that would allow us to know whether one functional

form is more realistic than another. Hence, it is important to explore model predictions under alternative functional forms to see whether the insights of [10] continue to hold up. In this paper we explore the dynamics using the functional form $\pi_p^i := e^{-a^i p}$, where $a^i \in [1, 10]$. Obviously we suppose the value of a^i varies across groups, to capture the fact that different groups may have different perceived probabilities of infection, as well as different perceptions of disease and vaccine risk. By comparison, in [10], the perceived probability of infection was the same across groups. We also note that for highly transmissible childhood diseases such as measles and pertussis, we assume the effect of time lags to be small, since most vaccination and disease transmission occurs in the youngest age classes.

3. OPTIMAL SOLUTIONS AND EQUILIBRIUM VACCINE COVERAGE

In this section we use a Nash game setting to study vaccination behaviour in heterogeneous populations as described in Section 2. However, in order to assert existence of an optimal solution for such a game, we make use of variational inequalities (VI) and projected systems (PDS) theories on the Euclidean space $\mathbb{R}^k$. For ease of presentation, before we proceed to analyze the game, we recall in brief the definitions of VI, PDS, Nash games and their interrelations.

3.1. Nash Games, VI and PDS.

We assume the reader to be familiar with the notions of closed convex sets, tangent cones and monotone mappings in $\mathbb{R}^k$ (for a quick reference see [4]).

We first remind the reader on the definition of a Nash game. We consider a game with m players, each player i having at his/her disposal a strategy vector $x_i = \{x_{i1}, \ldots, x_{in}\}$ selected from a closed, convex set $K_i \subset \mathbb{R}^n$, with a utility (or pay-off) function $u_i : K \to \mathbb{R}$, where $K = K_1 \times K_2 \times \cdots \times K_m \subset \mathbb{R}^{mn}$. The rationality postulate is that each player i selects a strategy vector $x_i \in K_i$ that maximizes his/her utility level $u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$ given the decisions $(x_j)_{j \neq i}$ of the other players. In this framework one then has:

Definition 3.1 (Nash Equilibrium)**.** A Nash equilibrium is a strategy vector $x^* = (x_1^*, \ldots, x_m^*) \in K$ such that

$$u_i(x_i^*, \hat{x}_i^*) \geq u_i(x_i, \hat{x}_i^*), \qquad \forall x_i \in K_i, \forall i, \tag{3.1}$$

where $\hat{x}_i^* = (x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_m^*)$.

Next we recall the definitions of finite-dimensional variational inequality problems and projected dynamical systems.

Definition 3.2. Let $K \subset \mathbb{R}^k$ be a closed, convex, nonempty set and $F : K \to \mathbb{R}^k$ a mapping. A **variational inequality problem** given by F and K is:

$$\text{find } x \in K \text{ so that } \langle F(x), y - x \rangle \geq 0, \qquad \text{for all } y \in K, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^k$, defined by $\langle x, y \rangle = \sum_{i=1}^k x_i y_i$, for any $x, y \in \mathbb{K}$.

Definition 3.3. Let $K \subset \mathbb{R}^k$ be a closed, convex, nonempty set and $F : K \to \mathbb{R}^k$ be a mapping. The initial value problem

$$\frac{dx(\tau)}{d\tau} = P_{T_K(x(\tau))}(-F(x(\tau))), \qquad x(0) = x_0 \in K, \tag{3.3}$$

is called a **projected differential equation**, where $P_K : \mathbb{R}^k \to K$ is given by $||P_K(z) - z|| = \inf_{x \in K} ||x - z||$, and $T_K(x)$ is the tangent cone to K at x.
A **projected dynamical system** is therefore the flow given by an equation of type (3.2).

In general, a VI problem is related to a PDS by the following (see [26, 13] for proofs):

Theorem 3.1. *Any solution of (3.2) is a critical point of the projected equation (3.3) and vice versa.*

The next result shows when such problems admit solutions (see [23, 12] for (3.2) and (3.3) respectively):

Theorem 3.2. *Assume F is Lipschitz continuous on K and monotone. Then problems (3.2) and (3.3) have solutions; moreover, problem (3.3) has a unique solution in the absolutely continuous class of functions defined on $[0, \infty)$ to K.*

Finally, a game of this form can be formulated as a VI as follows (for a proof see [18]).

Theorem 3.3. *Provided the utility functions u_i are of class $\mathcal{C}^1$ and concave (meaning $-u_i$ is convex) with respect to the variables x_i, then $x^* \in K$ is a Nash equilibrium if and only if it satisfies the VI*

$$\langle F(x^*), x - x^* \rangle \geq 0, \qquad \forall x \in K, \tag{3.4}$$

where $F(x) = (-\nabla_{x_1} u_1(x), \ldots, -\nabla_{x_m} u_m(x))$ and where

$$\nabla_{x_i} u_i(x) = \left(\frac{\partial u_i(x)}{\partial x_{i1}}, \ldots, \frac{\partial u_i(x)}{\partial x_{in}} \right).$$

To summarize, in this subsection we showed how we can equivalently reformulate solutions of a generic Nash game as critical points of a projected dynamical system using a variational inequality. In our study we are not making use of the theory of VI for computation purposes, as is traditional in operations research [26]; in fact we compute solutions to our vaccination game by using a projected system. Next, we apply these reformulation techniques to the vaccination game we started to build in Section 2.

3.2. **Vaccination Strategies Game.** We assume the population has a finite number of individuals divided into k distinct groups. The division is made according to Assumption 2.1 in Section 2. We thus consider a game with k players where each player has a 1-dimensional vaccination strategy vector. We denote by P_i, $i \in \{1, 2, \ldots, k\}$ the vaccination strategy corresponding to the i-th group and by ϵ_i the proportion of individuals in group i. In this context we have

$$\epsilon_i \in (0, 1) \quad \text{and} \quad \sum_{i=1}^{k} \epsilon_i = 1.$$

Evidently we are not interested in $\epsilon_i = 0$. For if this is true for some $i \in \{1, 2, \ldots, k\}$, then the problem is reduced to a population with $k - 1$ or less distinct groups. We are also not interested in $\epsilon_i = 1$ for some i, otherwise the problem reduces to the social homogeneous case considered in previous work [6]. We now denote by r_i the relative risk assessment and by $\pi_p^i = e^{-a_i p}$ the perceived probability of infection for the i-th group. We are interested in the cases $r_i \neq r_j$, or $a_i \neq a_j \; \forall i, j \in \{1, 2, \ldots, k\}$, otherwise the problem reduces to the case of a population with $k - 1$ or less distinct groups.

Under these hypotheses the vaccination coverage level of the entire population is assumed to be $p = \sum_{i=1}^{k} \epsilon_i P_i$. Following Section 2, the expected payoff function for a player is given by

$$u_i(P_i, p) = -r_i P_i - \pi_p^i (1 - P_i), \qquad \forall i \in \{1, 2, \ldots, k\}, \tag{3.5}$$

where $\pi_p^i = e^{-a_i \left(\sum_{i=1}^{k} \epsilon_i P_i \right)}$.

Let $\mathbb{K} := \{\underline{P} := (P_1, \ldots, P_k) \mid P_i \in [0, 1_P]\}$ and let the mapping $u : \mathbb{K} \to \mathbb{R}^k$ be given by $u(\underline{P}) = (u_1(P_1, p), \ldots, u_k(P_k, p))$. This game can be formulated (see [18]) as the variational inequality problem

$$\text{find } \underline{P}^* \in \mathbb{K} \text{ s.t. } \sum_{i=1}^{k} \left\langle -\frac{\partial u_i(P_i, p)}{\partial P_i}\bigg|_{P_i^*}, P_i - P_i^* \right\rangle \geq 0, \quad \forall\, \underline{P} = (P_1, \ldots, P_k) \in \mathbb{K},$$

since each u_i is of class $\mathcal{C}^1$ and concave with respect to P_i. This VI is further equivalent to

(3.6) find $\underline{P}^* \in \mathbb{K}$ s.t.

$$\sum_{i=1}^{k} \left\langle r_i - e^{-a_i\left(\sum_{i=1}^{k} \epsilon_i P_i\right)}\left[a_i \epsilon_i (1 - P_i) + 1\right]\Big|_{\underline{P}^*}, P_i - P_i^* \right\rangle \geq 0, \ \forall\, \underline{P} \in \mathbb{K}.$$

In order to study the proposed vaccination dynamics, we let $F : \mathbb{K} \to \mathbb{R}^k$ with $F(\underline{P}) = \left(-\frac{\partial u_1}{\partial P_1}, \ldots, -\frac{\partial u_k}{\partial P_k}\right)$ and we associate to the VI problem (3.6) the projected dynamical system given by

$$\Pi_{\mathbb{K}}(\underline{P}, -F(\underline{P})) = P_{T_K(\underline{P})}(-F(\underline{P})) \text{ with } \underline{P}(0) \in \mathbb{K}. \tag{3.7}$$

According to Theorem 3.1 above, the stationary points of PDS (3.7) coincide with the solutions of the Nash game. To study the question of stability of these game solutions under perturbations we use the notion of monotone mappings. Monotonicity is a generalization of the usual notion of a monotone real function of one variable. In the theory of PDS, monotonicity and its extensions, like strict monotonicity above, play a central role in the sense that they give information about the behaviour of perturbed equilibria. One of these results states that a PDS with a strictly monotone field F can only have a unique equilibrium and that all solutions are monotonically attracted to this point. The attraction can happen for solutions starting in a neighbourhood of the equilibrium, or can extend to all solutions starting anywhere in the set $\mathbb{K}$ [26, 13]. We are now able to prove the central result of the paper.

Theorem 3.4. *The Nash game above has a unique solution. This solution is a global strict monotone attractor for the vaccination strategies dynamics.*

Proof. Step 1. We show first that the field $F : K \to \mathbb{R}^k$ is strictly monotone on $\mathbb{K}$. This is relatively easy to see if we keep in mind that for continuously differentiable functions like F, strict monotonicity is equivalent to (see [26])

$$\underline{z}^T (\nabla F)\underline{z} > 0, \quad \text{for all } \underline{z} \neq 0 \in \mathbb{R}^k \text{ and } \forall \underline{P} \in \mathbb{K}. \tag{3.8}$$

In this case,

$$\nabla F(\underline{P}) = \begin{bmatrix} a_1^2\epsilon_1^2 e^{-a_1 p}(1 - P_1) & a_1^2\epsilon_1\epsilon_2 e^{-a_1 p}(1 - P_1) & \cdots & a_1^2\epsilon_1\epsilon_k e^{-a_1 p}(1 - P_1) \\ \cdots & \cdots & \cdots & \cdots \\ a_k^2\epsilon_k\epsilon_1 e^{-a_k p}(1 - P_k) & a_k^2\epsilon_k\epsilon_2 e^{-a_k p}(1 - P_k) & \cdots & a_k^2\epsilon_k^2 e^{-a_k p}(1 - P_k) \end{bmatrix}$$
$$+ \begin{bmatrix} 2a_1\epsilon_1 e^{-a_1 p} & a_1\epsilon_2 e^{-a_1 p} & \cdots & a_1\epsilon_k e^{-a_1 p} \\ \cdots & \cdots & \cdots & \cdots \\ a_k\epsilon_1 e^{-a_k p} & a_k\epsilon_2 e^{-a_k p} & \cdots & 2a_k\epsilon_k e^{-a_k p} \end{bmatrix},$$

where $p = \sum_{i=1}^{k} \epsilon_i P_i$. Then

$$\begin{aligned}\underline{z}^T(\nabla F)\underline{z} = \Big(&a_1^2\epsilon_1^2 e^{-a_1 p} z_1^2(1-P_1) + a_1^2\epsilon_1\epsilon_2 e^{-a_1 p} z_1 z_2(1-P_1) + \cdots \\ &+ a_1^2\epsilon_1\epsilon_k e^{-a_1 p} z_1 z_k(1-P_1) + \cdots + a_k^2\epsilon_k\epsilon_1 e^{-a_k p} z_k z_1(1-P_k) \\ &+ a_k^2\epsilon_k\epsilon_2 e^{-a_k p} z_k z_2(1-P_k) + \cdots + a_k^2\epsilon_k^2 e^{-a_k p} z_k^2(1-P_k)\Big) \\ &+ \Big(2a_1\epsilon_1 e^{-a_1 p} z_1^2 + a_1\epsilon_2 e^{-a_1 p} z_1 z_2 + \cdots + a_1\epsilon_k e^{-a_1 p} z_1 z_k \\ &\qquad + \cdots + a_k\epsilon_1 e^{-a_k p} z_k z_1 + a_k\epsilon_2 e^{-a_k p} z_k z_2 + \cdots + 2a_k\epsilon_k e^{-a_k p} z_k^2\Big),\end{aligned}$$

where $p = \sum_{i=1}^{k} \epsilon_i P_i$. This is further equal to

$$\begin{aligned}\underline{z}^T(\nabla F)\underline{z} = &\sum_{i=1}^{k} z_i^2\Big[a_i^2\epsilon_i^2 e^{-a_i p}(1-P_i) + 2a_i\epsilon_i e^{-a_i p}\Big] \\ &+ z_1 z_2\Big[a_1^2\epsilon_1\epsilon_2 e^{-a_1 p}(1-P_1) + a_2^2\epsilon_1\epsilon_2 e^{-a_2 p}(1-P_2) + a_1\epsilon_2 e^{-a_1 p} \\ &\quad + a_2\epsilon_1 e^{-a_2 p}\Big] + \cdots + z_1 z_k\Big[a_1^2\epsilon_1\epsilon_k e^{-a_1 p}(1-P_1) \\ &\quad + a_k^2\epsilon_1\epsilon_k e^{-a_k p}(1-P_k) + a_1\epsilon_k e^{-a_1 p} + a_k\epsilon_1 e^{-a_k p}\Big] \\ &\quad + \cdots + z_{k-1} z_k\Big[a_{k-1}^2\epsilon_{k-1}\epsilon_k e^{-a_{k-1} p}(1-P_{k-1}) \\ &\qquad + a_k^2\epsilon_{k-1}\epsilon_k e^{-a_k p}(1-P_k) + a_{k-1}\epsilon_k e^{-a_{k-1} p} + a_k\epsilon_{k-1} e^{-a_k p}\Big].\end{aligned} \tag{3.9}$$

Since $\forall i \in \{1, \ldots, k\}\ P_i \in [0, 1_P]$ we have that $(1 - P_i) > 0$. Since $z \in \mathbb{R}^k$ and $z \neq 0$, then at least one z_i, $i \in \{1, \ldots, k\}$ is not zero. However, we notice that all the coefficients of z_i^2 and of the products $z_i z_j$ in (3.9) are strictly positive. Hence

$$\underline{z}^T(\nabla F)\underline{z} > 0, \quad \forall z \in \mathbb{R}^k \text{ and } \underline{P} \in \mathbb{K},$$

therefore F is strictly monotone on $\mathbb{K}$. Since F is clearly continuous, by [23] game (3.1) has a unique solution.

Step 2. Next, we see that $-F : K \to \mathbb{R}^k$ is a Lipschitz continuous vector field since it is continuously differentiable and so by Theorem 3.2 we have that solutions of (3.7) starting at each initial point exist and are unique. They are also globally attracted towards the game solution. □

The game solution is unique. Moreover, it is a global monotone attractor for the trajectory of a PDS starting at an initial point in K. This latter fact is key in computing the approximate optimal group strategies. In the following section we derive such approximate optimal group strategies and vaccine coverage levels solely using a PDS approach. We then proceed to run comparisons between various game scenarios of interest to population biology.

4. EXAMPLES AND DISCUSSIONS

In our previous paper discussing a less refined game than the one here, namely, where all groups share the same value of $\pi_p = \frac{b}{a+p}$ [10], we have analyzed the impact of the heterogeneity of perceived relative risks r_i over the equilibrium vaccine coverage levels p^* in a population with two groups. The first group, the "majority" ($\epsilon_1 > \epsilon_2$), was considered more vaccine inclined than the second group, the "minority" (i.e., $r_1 < r_2$). This particular scenario has been chosen

for analysis based on observations [15] that generally, a small minority of nonvaccinators can produce a significant drop in the vaccine coverage levels in a population, should an outbreak occur in this minority group. In [10] we showed, using a setting similar to the above, that we can capture theoretically this very fact (p^*, the vaccine coverage value, is dropping in the presence of less vaccine inclined minorities). Moreover, a key point of our previous work was also to show that analyzing the population via heterogeneous groups leads, overall, to higher equilibrium vaccine coverage values than analyzing it as one homogeneous monolith.

In our model, the function π_p represents the *perceived* risk of being infected, not the actual risk. There have been a number of models, game theoretical or otherwise, which have attempted to capture human behaviour and they always rely on such simple phenomenological functions. Unfortunately, the data are not advanced to the point where functions can be accurately parameterized and validated, so authors tend to opt for simple functions with the right qualitative behaviour. This is our approach as well. We did give arguments for the "ballpark" values of a and b of π_p in [10], however we also raised the question of whether or not the results we obtained in [10] depend on the type of function π_p we considered.

In the present paper we essentially show that our analyses in [10] still hold when we vary the expression of the function π_p from $\frac{b}{a+p}$ to e^{-ap}, as well as when we consider heterogeneity of groups via both distinct perceived relative risks r_i and distinct perceived risks of being infected $\pi_p^i = e^{-a_i p}$. Essentially, considering again two groups, a minority and a majority characterized by distinct attitudes toward vaccination, we compute and analyze the equilibrium vaccine coverage values and see that these levels drop[1] in the presence of vaccine averse minorities. We also comment upon the values of the parameters a_i and r_i, $i \in \{1, 2\}$ below.

Before we proceed, we set in all the examples below $1_P := 0.9$; consequently, the constraint set will be set to $\mathbb{K} = [0, 0.9]^k$.

I. Our first discussion concerns a population with 2 groups, where we choose the first group to be the majority group. In our previous notation, we therefore let $0 < \epsilon_2 < \epsilon_1 < 1$. We consider however that one of the groups has a fixed "reference behaviour" with specified values of a and r. Because there is still relatively little empirical data on the relationship between risk perception and vaccinating behaviour, we can only make educated guesses as to the values of a and r. For the reference behaviour, we set $a = 3$, which gives a perceived probability of infection of only 7% at 90% vaccine coverage, and 55% at 20% vaccine coverage. This represents a sensible middle ground which avoids unrealistic extremes where the perceived probability is very high at high coverage levels [21], or very low at low coverage levels. Likewise, we pick a value of $r = 0.01$ for the reference behaviour, representing a situation where there is a significant level of trust in vaccination, and the disease is thought to be 100 times more dangerous than becoming vaccinated (the actual value is much higher for most vaccine-preventable infections, but r is a perceived relative risk, not an actual relative risk). In the other group with "variable behaviour" we will assume that $r = 0.0033a$. Hence, a decreased perceived risk of becoming infected corresponds to an increased perceived risk of the vaccine relative to the disease. In essence, the relation between the relative risk r and π_p comes from an assumption we make on the model, namely, that lower values of the perceived probability of infection in a group correspond in general to larger values of the perceived relative risk: individuals who think having the disease is less dangerous may also think that their risk of becoming infected is lower. Note that the functional form of the perceived probabilities of infection here is $\pi_p^i = e^{-a_i p}$ in both cases.

[1] In general a vaccine coverage level of 80% and above is considered very good for almost eradicating certain pediatric diseases. We find drops in p^* to approx. 50%, indicating an increase in the overall number of infected children.

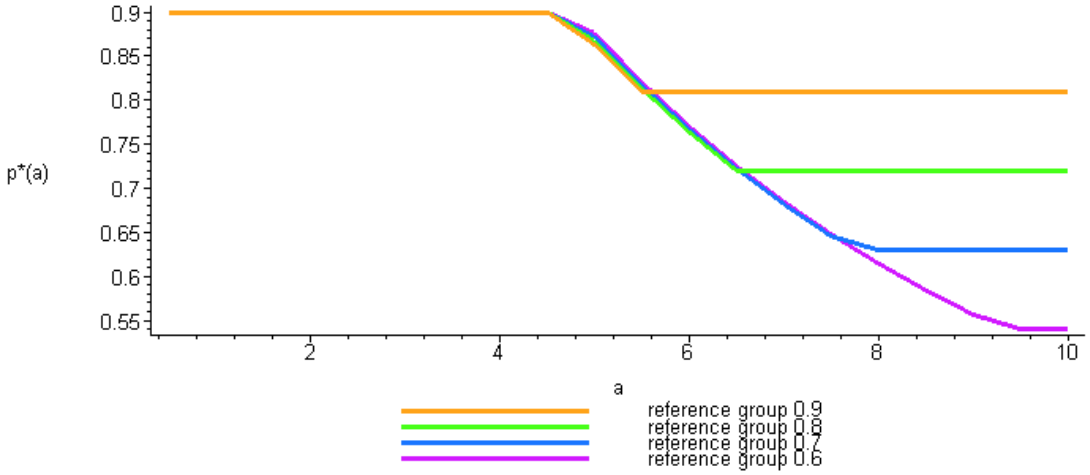

Figure 4.1: Plot of the overall equilibrium vaccine coverage of a 2 group population versus the value of parameter $a := a_2$ in the minority population, for 4 different values of ϵ_1 and ϵ_2. In all of the 4 cases we considered the majority group (of ϵ_1 size) to be the reference group with $a_1 = 3$, $r_1 = 0.01$ and $\pi_p^1 = e^{-3p}$, and the minority (of ϵ_2 size) to have varying risk $r_2 = 0.0033a_2$ and probability of infection $\pi_p^2 = e^{-a_2 p}$.

We divide the analysis into two cases: the first when $a_1 = 3$, $r_1 = 0.01$ (i.e., group 1 is the reference) and the second when $a_2 = 3$, $r_2 = 0.01$. For both of these cases, we assume that in the variable group the parameter a takes values in the interval $[1, 10]$. Figures 4.1 and 4.2 below show the equilibrium vaccine coverage $p^* = p^*(a)$ for the two cases. These figures show that, regardless of the group size and whether or not the minority or majority group are the "reference" group, a drop in overall coverage starts to occur when $a \approx 5$ in the group where risk perception is described by $r = 0.0033a$.

In Figure 4.1 where the majority is the "reference" group with fixed $a_1 = 3$, as the value of a_2 in the minority group increases (corresponding to a lower perceived risk of becoming infected), the vaccination coverage drops. For all values of $\epsilon_1 > \epsilon_2$, the drop occurs at $a_2 \approx 5$. For sufficiently large a, the minority group consists mostly or entirely of nonvaccinators while the majority group behaviour is not changed: hence, for $\epsilon_1 = 0.90$, where 10% of the population is in the minority group, the overall coverage level drops 10% for sufficiently large a. When $\epsilon_1 = 0.60$, the drop is approximately 40%, etc.

In Figure 4.2, where the minority is now the "reference" group ($a_2 = 3$) and the majority group can have various values of a_1, the results are somewhat different to those in Figure 4.1. In this case, the drop again starts to occur at $a_1 \approx 5$. However, the decrease is the same for various values of $\epsilon_1 > \epsilon_2$, in the range $a_1 \in [1, 10]$. For $a_1 > 10$, one would see a pattern of vaccine coverage flattening out similar to that in Figure 4.1, as the majority group turns to an entirely nonvaccinating strategy for sufficiently large a_1, leaving the minority group entirely responsible for vaccination coverage at the level ϵ_2.

II. Our next examples compare the overall vaccine coverage levels in a 1-group population and in a 2-groups population, the latter with a vaccine-averse minority (in our previous notation $0 < \epsilon_2 < \epsilon_1 < 1$). We want to determine whether vaccine coverage is higher or lower in the heterogeneous population compared to the homogeneous population, for the same overall perception of relative risk. This analysis generalizes and solidifies a similar one in [10].

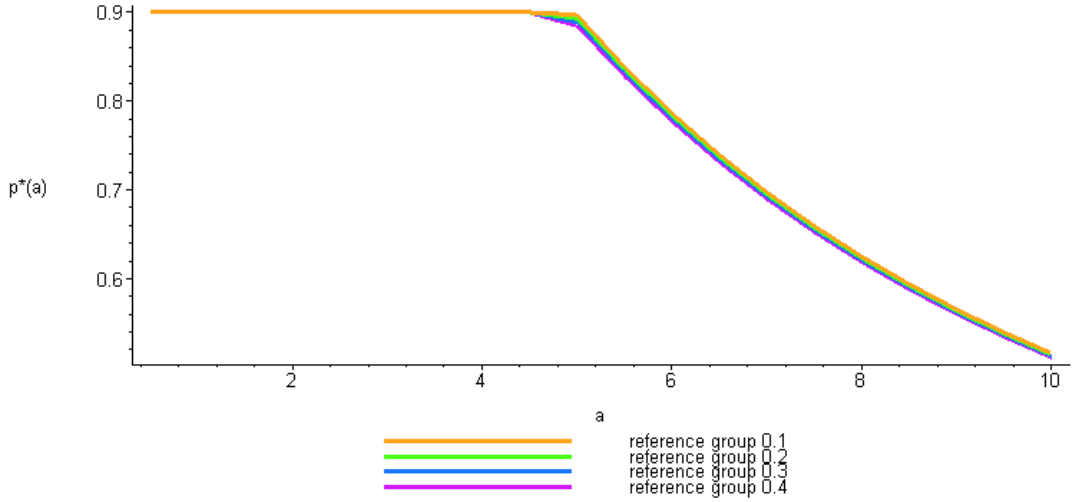

Figure 4.2: Plot of the overall equilibrium vaccine coverage of a 2 group population versus the value of parameter $a := a_1$ in the majority population, for 4 different values of ϵ_1 and ϵ_2. In all of the 4 cases we considered the minority group (of ϵ_2 size) to be the reference group with $a_2 = 3$, $r_2 = 0.01$, $\pi_p^2 = e^{-3p}$, and the majority (of ϵ_1 size) to have varying risk $r_1 = 0.0033a_1$ and probability of infection $\pi_p^1 = e^{-a_1 p}$.

To make this comparison sensible, the perceived relative risk in the 1 group case is related to the perceived relative risks in the 2 groups case via the relation

$$r = r_1\epsilon_1 + r_2\epsilon_2. \tag{4.1}$$

We divide our analysis in two cases: first we suppose that

$$a_1 = a_2 = a, \text{ i.e., } \pi_p^1 = \pi_p^2, \ a \in [1, 10], \ r_1 = 0.0033a, \text{ and } r_2 = \frac{r - \epsilon_1 r_1}{\epsilon_2}.$$

We illustrate this analysis in Figures 4.3 and 4.4.

In the second case we consider

$$a_1 = 4, \ a := a_2 \in [1, 10], \text{ i.e., } \pi_p^1 \neq \pi_p^2, \ r_2 = 0.0033a, \text{ and } r_1 = \frac{r - \epsilon_2 r_2}{\epsilon_1}.$$

We illustrate this case in Figures 4.5 and 4.6.

Case 1. Note that in this case the heterogeneity of the 2 groups is only given by the difference in relative risk perceptions, since $\pi_p^1 = \pi_p^2 = e^{-ap}$, $a \in [1, 10]$. To have that $r_2 > r_1$ for all $a \in [1, 10]$ (so that the minority group perceives a higher relative risk of the vaccine to the disease than the majority group), the relation $r = r_1\epsilon_1 + r_2\epsilon_2$ implies that we consider only the case $r > 0.05$. Figure 4.3 shows a 3-dimensional plot of equilibrium vaccine coverage surfaces $p^* = p^*(a, r)$, $a \in [1, 10]$, $r \in (0.05, 2.05)$ for 4 possible values of ϵ_1 and ϵ_2 (see figure caption).

In order to better highlight the relation between $p^*(a, r)$ in the homogeneous and heterogeneous cases, we compute $p^*(a, r)$ for 4 different fixed values of $r := 0.05 + j/5$, $j \in \{1, \ldots, 4\}$. Figures 4.4 (see also the caption) compare overall vaccine coverage in the homogeneous and heterogeneous cases, as a function of a.

Figures 4.3 and 4.4 show that, in general, the vaccine coverage is higher in the heterogeneous (2 group) populations than in the homogeneous (1 group) populations, except when a or r are sufficiently low (corresponding to high perceived probability of infection and low relative risk

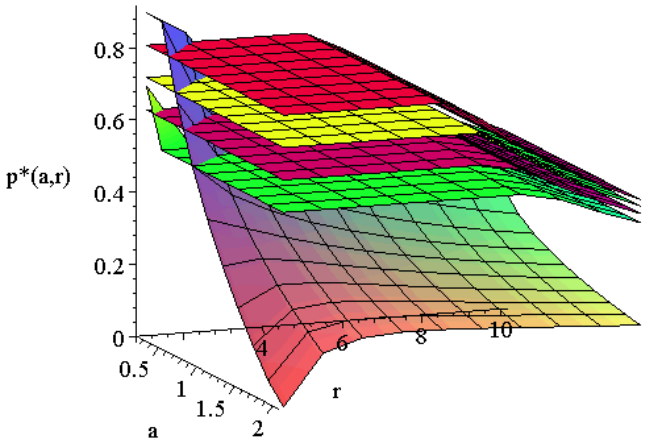

Figure 4.3: Plot of equilibrium vaccine coverage $p^*(a,r)$ of heterogeneous cases versus the homogeneous case. The highest (red) surface represents $p^*(a,r)$ for the heterogeneous case $\epsilon_1 = 0.9$, $\epsilon_2 = 0.1$; the next (yellow) surface represents $p^*(a,r)$ for the heterogeneous case $\epsilon_1 = 0.8$, $\epsilon_2 = 0.2$; the magenta surface represents $p^*(a,r)$ for the heterogeneous case $\epsilon_1 = 0.7$, $\epsilon_2 = 0.3$; the green surface represents $p^*(a,r)$ for the heterogeneous case $\epsilon_1 = 0.6$, $\epsilon_2 = 0.4$; finally the multicolored surface represents $p^*(a,r)$ for the 1 group case with $r = \epsilon_1 r_1 + \epsilon_2 r_2$.

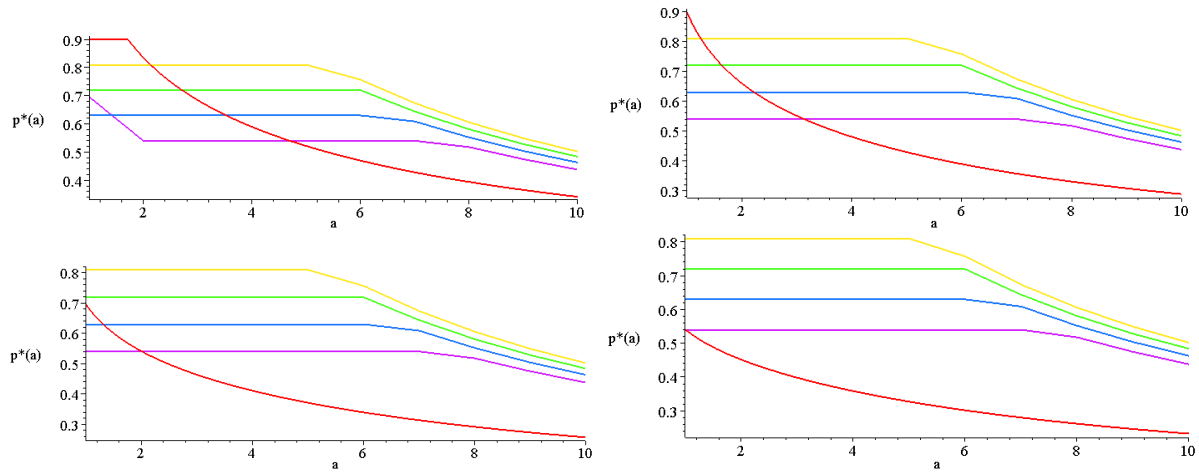

Figure 4.4: Plots of the equilibrium vaccine coverage $p^*(a, r = fixed)$ of the homogeneous case versus the heterogeneous cases. In all figures the curves represent: red curve - 1 group case; yellow curve - $\epsilon_1 = 0.9$, $\epsilon_2 = 0.1$ case; green curve - $\epsilon_1 = 0.8$, $\epsilon_2 = 0.2$ case; blue curve - $\epsilon_1 = 0.7$, $\epsilon_2 = 0.3$ case; purple curve - $\epsilon_1 = 0.6$, $\epsilon_2 = 0.4$ case. The upper left figure represents $p^*(a, r = 0.25)$, the upper right represents $p^*(a, r = 0.45)$, the lower left represents $p^*(a, r = 0.65)$ and the lower right represents $p^*(a, r = 0.85)$.

respectively). This is consistent with what was found in [10]. However, we note that the "real-world" parameter values may fall anywhere on the (a, r) plane. We summarize these results in the table below:

$r := 0.25$ (Figure 4.4	group sizes	a values for which $p^*(a) \geq p^*(a, 1 \text{ group})$
upper left)	$\epsilon_1 = 0.9,\ \epsilon_2 = 0.1$	$a \geq 2.18$
	$\epsilon_1 = 0.8,\ \epsilon_2 = 0.2$	$a \geq 2.7$
	$\epsilon_1 = 0.7,\ \epsilon_2 = 0.3$	$a \geq 3.5$
	$\epsilon_1 = 0.6,\ \epsilon_2 = 0.4$	$a \geq 4.68$
$r := 0.45$ (Figure 4.4	group sizes	a values for which $p^*(a) \geq p^*(a, 1 \text{ group})$
upper right)	$\epsilon_1 = 0.9,\ \epsilon_2 = 0.1$	$a \geq 1.245$
	$\epsilon_1 = 0.8,\ \epsilon_2 = 0.2$	$a \geq 1.63$
	$\epsilon_1 = 0.7,\ \epsilon_2 = 0.3$	$a \geq 2.2$
	$\epsilon_1 = 0.6,\ \epsilon_2 = 0.4$	$a \geq 3.12$
$r := 0.65$ (Figure 4.4	group sizes	a values for which $p^*(a) \geq p^*(a, 1 \text{ group})$
lower left)	$\epsilon_1 = 0.9,\ \epsilon_2 = 0.1$	$a \geq 1$
	$\epsilon_1 = 0.8,\ \epsilon_2 = 0.2$	$a \geq 1$
	$\epsilon_1 = 0.7,\ \epsilon_2 = 0.3$	$a \geq 1.31$
	$\epsilon_1 = 0.6,\ \epsilon_2 = 0.4$	$a \geq 2.01$
$r := 0.85$ (Figure 4.4	group sizes	a values for which $p^*(a) \geq p^*(a, 1 \text{ group})$
lower right)	$\epsilon_1 = 0.9,\ \epsilon_2 = 0.1$	$a \geq 1$
	$\epsilon_1 = 0.8,\ \epsilon_2 = 0.2$	$a \geq 1$
	$\epsilon_1 = 0.7,\ \epsilon_2 = 0.3$	$a \geq 1$
	$\epsilon_1 = 0.6,\ \epsilon_2 = 0.4$	$a \geq 1$

Case 2. Here we generalize our discussion from *Case 1* to heterogeneity of not only risk perceptions, but also heterogeneity of perceived probabilities of infection. We now take $\pi_p^1 = e^{-4_1 p} \neq \pi_p^2 = e^{-a_2 p}$. Figure 4.5 shows a 3-dimensional plot of equilibrium vaccine coverage surfaces $p^* = p^*(a_2, r)$, $a_2 \in [1, 10]$, $r \in (0.05, 2.05)$ for 4 possible values of ϵ_1 and ϵ_2 (see figure caption).

In order to highlight the relation between $p^*(a_1, r)$ in the homogeneous and heterogeneous cases, we compute $p^*(a_1, r)$ for 4 different fixed values of $r := 0.05 + j/5$, $j \in \{1, \ldots, 4\}$. Figures 4.6 (see also the caption) compare overall vaccine coverage in the homogeneous and heterogeneous cases, as a function of a_2.

Figures 4.5 and 4.6 show again that the vaccine coverage is higher in the heterogeneous (2 group) populations than in the homogeneous (1 group) populations, except when a or r are sufficiently low, leading us to conclude that extending the incorporation of heterogeneity showed a consolidation of our earlier conclusion, namely that heterogeneous populations have better overall vaccine coverage than homogeneous ones.

III. Finally, our last example illustrates the case of a heterogeneous population with 4 groups, where the first 2 groups have an exponential perceived probability of infection and the last two groups have a perceived probability of infection

$$\pi_p^j = \frac{b}{c + \sum_{i=1}^{4} \epsilon_i P_i}, \ \ j \in \{3, 4\} \quad \text{and} \quad c = 0.1,\ b = 0.09$$

(as in [10]). The parameter values are given in the table below, together with their respective equilibrium strategies.

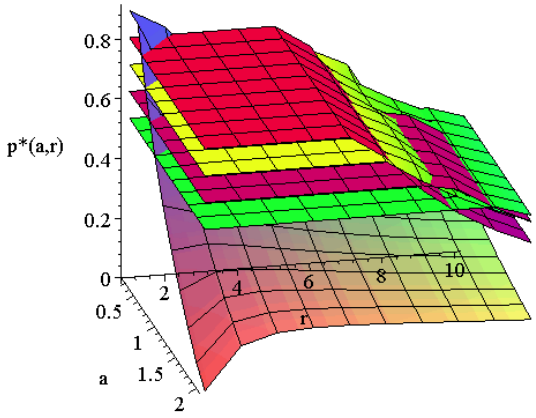

Figure 4.5: Plot of equilibrium vaccine coverage $p^*(a := a_2, r)$ of heterogeneous cases versus the homogeneous case when $a_1 = 4$. The highest (red) surface represents $p^*(a, r)$ for the heterogeneous case $\epsilon_1 = 0.9$, $\epsilon_2 = 0.1$; the next (yellow) surface represents $p^*(a, r)$ for the heterogeneous case $\epsilon_1 = 0.8$, $\epsilon_2 = 0.2$; the magenta surface represents $p^*(a, r)$ for the heterogeneous case $\epsilon_1 = 0.7$, $\epsilon_2 = 0.3$; the green surface represents $p^*(a, r)$ for the heterogeneous case $\epsilon_1 = 0.6$, $\epsilon_2 = 0.4$; finally the multicolored surface represents $p^*(a, r)$ for the 1 group case with $r = \epsilon_1 r_1 + \epsilon_2 r_2$.

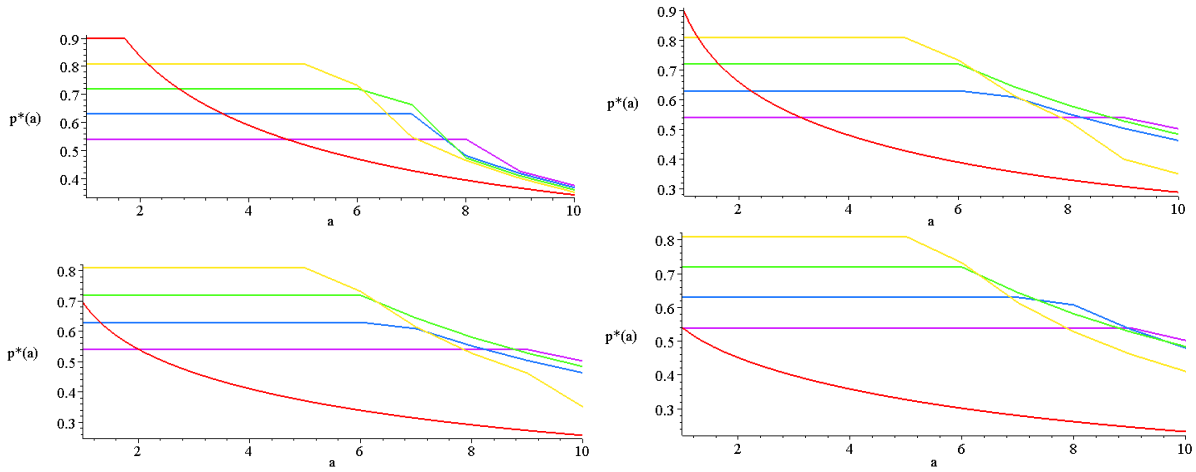

Figure 4.6: Plots of the equilibrium vaccine coverage $p^*(a := a_2, r = fixed)$ of the homogeneous case versus the heterogeneous cases where $a_1 = 4$. In all figures the curves represent: red curve - 1 group case; yellow curve - $\epsilon_1 = 0.9$, $\epsilon_2 = 0.1$ case; green curve - $\epsilon_1 = 0.8$, $\epsilon_2 = 0.2$ case; blue curve - $\epsilon_1 = 0.7$, $\epsilon_2 = 0.3$ case; purple curve - $\epsilon_1 = 0.6$, $\epsilon_2 = 0.4$ case. The upper left figure represents $p^*(a, r = 0.25)$, the upper right represents $p^*(a, r = 0.45)$, the lower left represents $p^*(a, r = 0.65)$ and the lower right represents $p^*(a, r = 0.85)$.

risks	group sizes	π_p	equilibrium strategies
$r_1 = 0.3$	$\epsilon_1 = 0.1$	$\pi_p^1 = \frac{0.09}{0.1+\sum_{i=1}^4 \epsilon_i P_i}$	$P_1^* = 0$
$r_2 = 0.01$	$\epsilon_2 = 0.2$	$\pi_p^2 = \frac{0.09}{0.1+\sum_{i=1}^4 \epsilon_i P_i}$	$P_2^* = 0.9$
$r_3 = 0.0066$	$\epsilon_3 = 0.4$	$\pi_p^3 = e^{-2\left(\sum_{i=1}^4 \epsilon_i P_i\right)}$	$P_3^* = 0.9$
$r_4 = 0.1$	$\epsilon_4 = 0.3$	$\pi_p^4 = e^{-4\left(\sum_{i=1}^4 \epsilon_i P_i\right)}$	$P_4^* = 0.506$

In this case, the overall vaccine coverage at equilibrium is equal to $p^* = 0.6918$.

5. CONCLUSIONS

Our examples confirm and generalize the conclusions of [10], namely that for a wide range of parameter values, the vaccine coverage in a multi-group population can be higher than the vaccine coverage in a homogeneous population where the average perceived relative risk is given by (4.1). This work goes beyond [10] by allowing the perceived probability of infection, π_p, to vary across groups using a different functional form, and supposing a relationship between π_p and r in some cases. This work, together with [11] shows again the versatility and usability of both finite dimensional PDS and VI for various formulations of vaccination strategies games.

Future work may consider the relative risk r_i as a variable of the model that evolves in response to vaccination coverage, rather than treating it as a fixed quantity. Future work should also analyze the effects of heterogeneity on the equilibria of vaccinating strategies when the probability of becoming infected is a function not of vaccine coverage p, but of the actual number of infected individuals at any given time in the population. This would require incorporation of compartmental epidemic models such as the SIR model [2] into the PDS/VI framework.

REFERENCES

[1] M.R. ALBERT, K.G. OSTHEIMER AND J.G. BREMAN, The last smallpox epidemic in Boston and the vaccination controversy 1901–1903, *N. Engl. J. Med.,* **344** (2001), 375–379.

[2] R.M. AND ERSON AND R.M. MAY, *Infectious Diseases of Humans*, Oxford University Press, Oxford (1991).

[3] D.A. ASCH, J. BARON, J.C. HERSHEY, H. KUNREUTHER, J. MESZAROS, I. RITOV AND M. SPRANCA, Omission bias and pertussis vaccination, *Med. Decis. Making,* **14** (1994), 118–123.

[4] J.P. AUBIN AND A. CELLINA, *Differential Inclusions*, Springer-Verlag, Berlin (1984).

[5] C.T. BAUCH, A.P. GALVANI AND D.J.D. EARN, Group interest versus self interest in smallpox vaccination policy, *Proc. Natl. Acad. Sci.,* **100** (2003), 10564–10567.

[6] C.T. BAUCH AND D.J.D. EARN, Vaccination and the theory of games, *Proc. Natl. Acad. Sci.,* **101** (2004), 13391–13394.

[7] P. BELLABY, Communication and miscommunication of risk: understanding UK parents attitudes to combined MMR vaccination, *Br. Med. J.,* **327** (2003), 725–728.

[8] B. BIROSCAK, A. FIORE, N. FASANO, P. FINEIS, M. COLLINS AND G. STOLTMAN, Impact of the thimerosal controversy on hepatitis B vaccine coverage of infants born to women of unknown hepatitis B surface antigen status in Michigan, *Pediatrics,* **111** (2003), e645–e649.

[9] G.B. CHAPMAN AND E.J. COUPS, Predictors of influenza vaccine acceptance among healthy adults, *Preventitive Medicine,* **29** (1999), 249–262.

[10] M.-G. COJOCARU, C.T. BAUCH AND M.D. JOHNSTON, Dynamics of vaccination strategies via projected dynamical systems, *Bulletin of Mathematical Biology,* **69** (2007), 1453–1476.

[11] M.-G. COJOCARU, Dynamic equilibria of group vaccination strategies in a heterogeneous population, *J. Glob. Opt.,* **40**(1-3) (2008), 51–63.

[12] M.-G. COJOCARU AND L.B. JONKER, Projected differential equations in Hilbert spaces, *Proc. Amer. Math. Soc.,* **132**(1) (2004), 183–193.

[13] G. ISAC AND M.-G. COJOCARU, Variational inequalities, complementarity problems and pseudomonotonicity, Dynamical aspects, in: *Proceedings of the International Conference on Nonlinear Operators, Differential Equations and Applications,* Babes-Bolyai University of Cluj-Napoca III (2002), 41–62.

[14] N. DURBACH, They might as well brand us: working class resistance to compulsory vaccination in Victorian England, *Soc. Hist. Med.,* **13**(1) (2000), 45–62.

[15] *Eurosurveillance Weekly Release*, **5**10, (2005). [ONLINE http://www.eurosurveillance.org/ew/2005/050519.asp].

[16] M. EVANS, H. STODDART, L. CONDON, E. FREEMAN, M. GRIZZELL AND R. MULLEN, Parents perspectives on the MMR immunisation: a focus group study, *Br. J. Gen. Pract.*, **51** (2001), 904–910.

[17] B. FISCHHOFF, Assessing and communicating the risks of terrorism. In: Teich, A.H., Nelson, S.D., Lita, S.J. Eds, *Science and Technology in a Vulnerable World. Suppl to the AAAS Science and Technology Policy Yearbook 2003*, Washington DC: AAAS, (2002), 51–64.

[18] D. GABAY AND H. MOULIN, On the uniqueness and stbility of Nash-equilibria in noncooperative games, in *Applied Stochastic Control in Econometrics and Management Science*, North Holland, Amsterdam (1980).

[19] E.J. GANGAROSA, A.M. GALAZKA, C.R. WOLFE, L.M. PHILLIPS, R.E. GANGAROSA, E. MILLER AND R.T. CHEN, Impact of anti-vaccine movements on pertussis control: the untold story, *Lancet*, **351** (1998), 356–361.

[20] K.P. GOLDSTEIN, T.J. PHILIPSON, H. JOO AND R.S. DAUM, The effect of epidemic measles on immunization rates, *JAMA*, **276** (1996), 56–58.

[21] Health Canada, Immunisation and Respiratory Diseases Division Website. http://www.phac-aspc.gc.ca/im/vs-sv/vs-faq_e.html

[22] V.A. JANSEN, N. STOLLENWERK, H.J. JENSEN, M.E. RAMSAY, W.J. EDMUNDS AND C.J. RHODES, Measles outbreaks in a population with declining vaccine uptake, *Science*, **301** (2003), 804.

[23] D. KINDERLEHRER AND G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Academic Press (1980).

[24] N. LASHUAY, T. TJOA, M.L.Z. de NUNCIO, M. FRANKLIN, J. ELDER AND M. JONES, Exposure to immunization media messages among African American parents, *Prev. Med.*, **31** (2000), 522–528.

[25] E. McKENZIE AND F. ROBERTS, Modeling social responses to bioterrorism involving infectious agents, *DIMACS Technical Report*, **2003-30** (2003) Rutgers University, http://dimacs.rutgers.edu/Workshops/Modeling/Report.doc

[26] A. NAGURNEY AND D. ZHANG, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers (1996).

[27] G. POLAND AND R. JACOBSEN, Understanding those who do not understand: a brief review of the anti-vaccine movement, *Vaccine*, **19** (2001), 2440–2445.

[28] S. PLOTKIN, Lessons learned concerning vaccine safety, *Vaccine*, **20** (Suppl. 1) (2002), S16–S19.

[29] R.J. ROBERTS, Q.D. SANDIFER, M.R. EVANS, M.Z. NOLAN-FARELL AND P.M. DAVIS, Reasons for non-uptake of measles, mumps and rubella catch up immunisation in a measles epidemic and side effects of the vaccine, *Br. Med. J.*, **310** (1995), 1629–1639.

[30] H.J. SCHMITT, Factors influencing vaccine uptake in Germany, *Vaccine*, **20** (Suppl. 1) (2002), S2–S4.

[31] M.S. SMAILBEGOVIC, G.J. LAING AND H. BEDFORD, 2003. Why do parents decide against immunization? The effect of health beliefs and health professionals, *Child Care Health Dev.*, **29** (2003), 303–311.

[32] B. WARSHAWSKY, S. WILSON-CLARK, D. SIDER, M. BRAGG, V. DUBEY, et al., Issues of under-immunized populations: the tale of three outbreaks, *Can. J. Infect. Dis. Med. Microbiol.*, **17**(6) (2006), 351.