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INEQUALITIES FOR 3-LOG-CONVEX FUNCTIONS

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ABSTRACT. This note gives a simple method for obtaining inequalities for ratios involving 3-log-convex functions. As an example, an inequality for Wallis's ratio of Gautchi-Kershaw type is obtained. Inequalities for generalized means are also considered.

Key words and phrases: Inequalities, Logarithmic derivative, Convexity, Gamma function, Digamma function, Extended means.

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1. INTRODUCTION

This paper studies inequalities for positive real valued 3-log-convex (and 3-log-concave) functions. As has become customary (see for instance [23] and [31]), we refer to a function f as 3-log-convex on the interval (a,b) if f is positive and 3-times differentiable on (a,b) and $[\ln(f(t))]''' \geq 0$ for $t \in (a,b)$ (f is referred to as 3-log-concave if instead $[\ln(f(t))]''' \leq 0$). In particular, suppose that g is a positive differentiable function defined on the interval (a,b), and let h be the logarithmic derivative of g, i.e.

$$h(x) = \frac{g'(x)}{g(x)} \tag{1.1}$$

for $x \in (a,b)$.

We will prove the following.

Theorem 1.1. *Suppose that for $a < x < b$, $g(x) > 0$, $h = g'/g$ is twice differentiable and $h''(x) > 0$. Set $R(x) = g(a+b-x)/g(x)$. Then*

$$R(b-)e^{2h\left(\frac{a+b}{2}\right)(b-x)} \leq R(x) \leq R(a+)e^{2h\left(\frac{a+b}{2}\right)(a-x)} \tag{1.2}$$

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365-07

and

$$R(a+)e^{(h(a+)+h(b-))(a-x)} \leq R(x) \leq R(b-)e^{(h(a+)+h(b-))(b-x)}. \tag{1.3}$$

for $a < x < b$, where it is assumed that all four of the one-sided limits, $h(a+)$, $R(a+)$, $h(b-)$ and $R(b-)$ exist and are finite.

In addition, if instead $g(x) > 0$ and $h''(x) < 0$ for $a < x < b$ then the inequalities in (1.2) and (1.3) are reversed.

To see where one might apply Theorem 1.1, consider the function g defined via $g(x) = \Gamma(A+x)$, where $A > 0$ and Γ is the well-known Euler's gamma function. We have $g'(x)/g(x) = \Psi(A+x)$, where Ψ is the digamma function (cf. [2, 3]). It is well-known (see for instance [9]) that Ψ is concave on $(0,\infty)$. Hence Theorem 1.1 is applicable. In Section 3, below, we will prove the following.

Theorem 1.2. *Suppose that $0 < s < 2$ and $v > 0$, then*

$$\left(v+\frac{s}{2}\right)e^{-\Psi\left(v+\frac{s+1}{2}\right)s} \leq \frac{\Gamma(v+1)}{\Gamma(v+s)} \leq \frac{1}{v+\frac{s}{2}}e^{2\Psi\left(v+\frac{s+1}{2}\right)\left(1-\frac{s}{2}\right)} \tag{1.4}$$

and

$$\begin{aligned} \frac{1}{v+\frac{s}{2}}e^{\left(2\Psi\left(v+\frac{s}{2}\right)+\frac{1}{v+\frac{s}{2}}\right)\left(1-\frac{s}{2}\right)} &\leq \frac{\Gamma(v+1)}{\Gamma(v+s)} \\ &\leq \left(v+\frac{s}{2}\right)e^{-\left(2\Psi\left(v+\frac{s}{2}\right)+\frac{1}{v+\frac{s}{2}}\right)\frac{s}{2}}. \end{aligned} \tag{1.5}$$

Note that the inequalities in (1.4) and (1.5) hold in the range $0 < s < 2$ which is somewhat uncustomary for results of this type for the ratio $\frac{\Gamma(v+1)}{\Gamma(v+s)}$ which tend to hold for $0 < s < 1$ (although reversed inequalities hold for $1 < s < 2$) (see [10, 13]). Some comparisons are provided in Section 3.

We remark that recently many functions have been shown to be logarithmically completely monotone (see for instance [4, 7, 8, 19, 20, 24, 25, 27]). Such functions have, in particular, convex (or concave) logarithmic derivatives and hence Theorem 1.1 is applicable in these cases.

The remainder of the paper proceeds as follows. In Section 2, we provide a simple proof of Theorem 1.1. Section 3 is devoted to applications including a proof of Theorem 1.2 and an inequality for generalized means.

2. PROOF OF THEOREM 1.1

In this short section we provide a proof of Theorem 1.1.

Proof of Theorem 1.1. First, suppose $h''(x) > 0$ for $x \in (a,b)$, and for $W \in \mathbb{R}$, define f_W via

$$f_W(x) = R(x)e^{Wx}.$$

Then, we have

$$\log(f_W(x)) = \log(g(a+b-x)) - \log(g(x)) + Wx$$

and

$$\frac{d}{dx}\log(f_W(x)) = W - (h(x) + h(a+b-x)) = W - V(x), \tag{2.1}$$

where $V(x) = h(x) + h(a+b-x)$.

Now, for $x \in \left(a, \frac{a+b}{2}\right)$, $x < a+b-x$ and hence since $h''(x) > 0$, $h'(x) < h'(a+b-x)$ and thus

$$V'(x) = h'(x) - h'(a+b-x) < 0. \tag{2.2}$$

Similarly, for $x \in \left(\frac{a+b}{2}, b\right)$, $x > a+b-x$ and hence

$$V'(x) = h'(x) - h'(a+b-x) > 0. \tag{2.3}$$

Combining (2.2) and (2.3) gives that for $x \in (a,b)$,

$$V\left(\frac{a+b}{2}\right) \le V(x) \le V(a+) = V(b-).$$

Employing (2.1) we then have that f_W is nondecreasing on (a,b) for $W = V(a+)$ and nonincreasing on (a,b) for $W = V\left(\frac{a+b}{2}\right)$. The inequalities in (1.2) and (1.3) then follow. The case $h''(x) < 0$ follows similarly, and the result is proven. □

3. Applications

3.1. Inequalities of Gautschi-Kershaw type. Inequalities for the ratio $\Gamma(v+1)/\Gamma(v+s)$ have been studied extensively by many authors; for results and useful references, see [1, 6, 10, 12, 14, 15, 18, 20, 21, 22, 25, 27, 30, 32].

To see how Theorem 1.2 follows from Theorem 1.1, set $(a,b) = (0,1)$ and $g(x) = \Gamma(A+x)$. Then, note that $h(x) = \Psi(A+x)$, $h(1/2) = \Psi\left(A + \frac{1}{2}\right)$,

$$h(0+) + h(1-) = \Psi(A) + \Psi(A+1) = 2\Psi(A) + \frac{1}{A},$$

$$R(1-) = \lim_{x \to 1^-} \frac{\Gamma(A+1-x)}{\Gamma(A+x)} = \frac{1}{A},$$

and

$$R(0+) = \lim_{x \to 0^+} \frac{\Gamma(A+1-x)}{\Gamma(A+x)} = A.$$

Employing (1.2) and (1.3), since $h''(x) < 0$, we have

$$Ae^{-2\Psi(A+1/2)x} \le \frac{\Gamma(A+1-x)}{\Gamma(A+x)} \le \frac{1}{A} e^{2\Psi(A+1/2)(1-x)} \tag{3.1}$$

and

$$\frac{1}{A} e^{(2\Psi(A)+\frac{1}{A})(1-x)} \le \frac{\Gamma(A+1-x)}{\Gamma(A+x)} \le Ae^{-2\left(\Psi(A)+\frac{1}{A}\right)x} \tag{3.2}$$

for $0 < x < 1$. Theorem 1.2 then follows upon substituting $A = v + s/2$ and $x = s/2$.

From Kershaw [14], we have that for $0 < s < 1$,

$$e^{(1-s)\Psi\left(v+\sqrt{s}\right)} \le \frac{\Gamma(v+1)}{\Gamma(v+s)} \le e^{(1-s)\Psi\left(v+\frac{s+1}{2}\right)} \tag{3.3}$$

and

$$\left(v + \frac{s}{2}\right)^{1-s} \le \frac{\Gamma(v+1)}{\Gamma(v+s)} \le \left(v - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s}. \tag{3.4}$$

In [10, 13], it was proven that the inequalities in (3.3) and (3.4) are reversed for $1 < s < 2$.

Computations suggest that the upper bound in (1.5) is an improvement on both upper bounds in (3.3) and (3.4) for small s and that the lower bound in (1.5) is an improvement on the lower bounds implied by (3.3) and (3.4) for s near 2. Let L_1, U_1, L_2, U_2 denote the lower and upper bounds in (3.3) and (3.4), respectively and $L_1^*, U_1^*, L_2^*, U_2^*$ denote the lower and upper bounds

Table 3.1: Numerical comparisons

(v,s)	$(1,1/4)$	$(1,7/4)$
$\Gamma(v+1)/\Gamma(v+s)$	**1.103262651**	**0.6217515729**
L_1	1.027745410	0.6317370766
L_2	1.092356486	0.6240926184
U_1	1.116801087	0.6188110780
U_2	1.151620182	0.6144792307
L_1^*	1.084327768	0.6118384856
L_2^*	0.980328638	**0.6204985722**
U_1^*	1.150246913	0.6258631306
U_2^*	**1.109373110**	0.6498406288

in (1.4) and (1.5), respectively. Comparison data is given in Table 3.1 for $v = 1$ and $s \in \{1/4, 7/4\}$. We have in particular that for $(v,s) = (1,1/4)$

$$L_2^* < L_1 < L_1^* < L_2 < \frac{\Gamma(v+1)}{\Gamma(v+s)} < U_2^* < U_1 < U_2 < U_1^*,$$

while for $(v,s) = (1,7/4)$

$$L_1^* < U_2 < U_1 < L_2^* < \frac{\Gamma(v+1)}{\Gamma(v+s)} < L_2 < U_1^* < L_1 < U_2^*.$$

In the first case, the best of the four upper bounds is given by U_2^* (the right hand side of (1.5)) while in the second case the best lower bound is given by L_2^* (the left hand side of (1.5)).

Recently, there have been some improvements obtained on the inequalities in (3.3). In particular, results in [21] and [29] (see also [22, 30, 32]) give that for $0 < s < 1$,

$$e^{(1-s)\Psi(L(v+1,v+s))} \le \frac{\Gamma(v+1)}{\Gamma(v+s)} \le e^{(1-s)\Psi(I(v+1,v+s))}, \tag{3.5}$$

where $L(a,b) = (b-a)/(\ln b - \ln a)$ and $I(a,b) = e^{-1}(b^b/a^a)^{1/(b-a)}$ are the logarithmic and exponential means, respectively. Again considering $v = 1$, it can be noted that for small $s > 0$, the lower bound in (1.4), L_1^*, is an improvement on that in (3.5) and the upper bound in (1.5), U_2^*, is an improvement on that in (3.5). In fact, denoting the lower and upper bounds in (3.5) by L_3 and U_3, respectively, we have $\Gamma(2)/\Gamma(1) = 1$ and $\lim_{s\to 0^+} L_2^* = 1 = \lim_{s\to 0^+} U_1^*$, while $\lim_{s\to 0} L_3 < 1 < \lim_{s\to 0^+} U_3$. It is interesting to note that for $(v,s) = (1,1/4)$, computations similar to those above display that U_3 provides a modest improvement on U_2^* ($U_3 = 1.106505726$), but for $(v,s) = (1,s)$ with s near zero we have

$$\frac{\Gamma(2)}{\Gamma(1+s)} < U_2^* < U_2 < U_3 < U_1 < U_1^*.$$

As noted in [21, 29], U_3 is a refinement of U_1 and L_3 is a refinement of L_1.

Values for $(v,s) = (1, 0.02)$ and $(v,s) = (1, 0.10)$ are given in Table 3.2.

Many functions related to the Γ function have recently been shown to be logarithmically completely monotone. As mentioned earlier, strong bounds may be attained in these cases as well, via Theorem 1.1.

Table 3.2: Numerical comparisons

(v,s)	$(1,0.02)$	$(0,0.10)$
$\Gamma(v+1)/\Gamma(v+s)$	$\mathbf{1.011281653}$	$\mathbf{1.051137006}$
L_1	0.6986450960	0.8729765884
L_2	1.009799023	1.044889510
U_1	1.045903237	1.076807140
U_2	1.019219191	1.082081647
L_1^*	1.009075328	1.041402026
L_2^*	0.8690926716	0.9139917416
U_1^*	1.084075243	1.113415941
U_2^*	$\mathbf{1.011330762}$	$\mathbf{1.052276188}$
L_3	0.9941107436	1.038103958
U_3	1.020141278	1.057551215

4. INEQUALITIES FOR FUNCTIONS OF THE FORM $(v^x - u^x)/x$

In [17, 34, 33], functions of the form

$$f(x) = f_{u,v}(x) = \int_u^v s^{x-1} ds = \frac{v^x - u^x}{x} \tag{4.1}$$

for $v > u > 0$ and $x \neq 0$ were studied. Among other results, it was shown in [33] that f is completely monotonic on $(-\infty, +\infty)$ for $0 < u < v < 1$. As of the time of submission, we are unaware of any proof that f possesses a concave logarithmic derivative, for $v > u > 0$ and $0 < x < 1$, hence we will prove that here and apply Theorem 1.1 in order to obtain some new inequalities for the ratios of the form $f(\gamma - x)/f(x)$.[1]

In [33] It was shown that

$$\frac{f(x+\gamma)}{f(x)} \geq \left(\frac{u+v}{2}\right)^\gamma$$

for $\gamma \geq 1$, $x \geq 0$ and $0 < u < v$, and

$$\frac{f(x+\gamma)}{f(x)} \geq (uv)^{\gamma/2}.$$

Here we will prove the following via Theorem 1.1.

Theorem 4.1. *Suppose* $0 < u < v$ *and* $0 < x < 1$. *Then*

$$\begin{aligned} \frac{v-u}{\ln(v)-\ln(u)} e^{-2x\left(\frac{\sqrt{v}\ln(v)-\sqrt{u}\ln(u)}{v-u}-2\right)} &\leq \frac{f_{u,v}(1-x)}{f_{u,v}(x)} \\ &\leq \frac{\ln(v)-\ln(u)}{v-u} e^{2(1-x)\left(\frac{\sqrt{v}\ln(v)-\sqrt{u}\ln(u)}{v-u}-2\right)} \end{aligned} \tag{4.2}$$

[1]Following submission of the original manuscript for this paper, F. Qi and B.-N. Guo [28] announced some results which extend our Lemma 4.3, below.

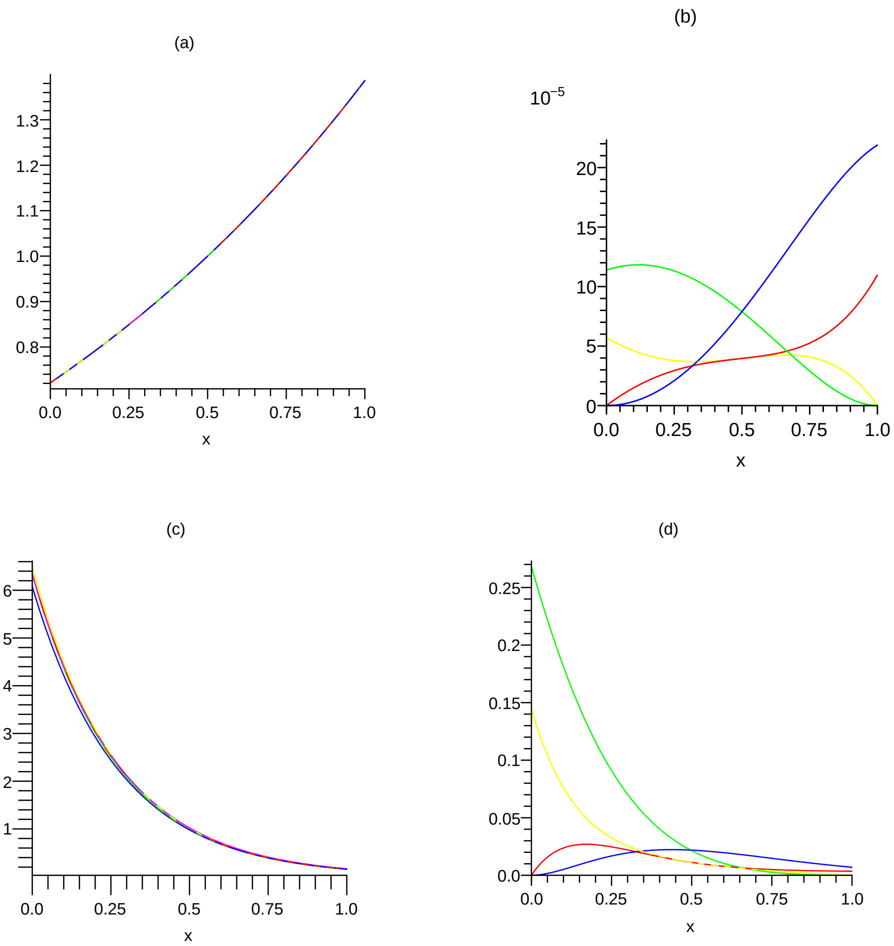

Figure 4.1: Plots of $R(x) = f_{u,v}(1-x)/f_{u,v}(x)$ along with the bounds given in Theorem 4.1 for $x \in (0,1)$ and $(u,v) = (.5,1)$ (Figure (a)) and $(u,v) = (1,20)$ (Figure (c)). The absolute errors are plotted in (b) and (d), respectively.

and

(4.3)
$$\frac{\ln(v)-\ln(u)}{v-u} e^{(1-x)\left(\frac{(3v-u)\ln(v)-(3u-v)\ln(u)}{2(v-u)}-1\right)} \leq \frac{f_{u,v}(1-x)}{f_{u,v}(x)} \leq \frac{v-u}{\ln(v)-\ln(u)} e^{-x\left(\frac{(3v-u)\ln(v)-(3u-v)\ln(u)}{2(v-u)}-1\right)}.$$

Plots comparing the quantities in (4.2) and (4.3), for $(u,v) = (0.5,1)$ and $(u,v) = (1,20)$ are given in Figure 4.

We first prove the following two simple lemmas.

Lemma 4.2. *Define p via*

$$p(y) = (1-y)\left(\frac{1+y}{1-y}e^{-2y} - 1\right).$$

Then $p(y) > 0$ for $y > 0$, $p(y) < 0$ for $y < 0$, and $p(0) = 0$.

Proof. We have

$$p'(y) = 1 - (1+2y)e^{-2y}, \quad \text{and} \quad p''(y) = 4ye^{-2y}.$$

The result follows upon noting that $p'(y) \geq p'(0) = 0$, and hence that $p(y)$ is monotone non-decreasing for $y \in \mathbb{R}$; the only root is $y = 0$. □

Lemma 4.3. *The function $f_{u,v}$ defined as in (4.1) has a concave logarithmic derivative (with respect to x) for $0 < u < v$ and $0 < x < 1$.*

Proof. First note that by dividing through by v^x, it suffices to show the result for $v = 1$ and $u = t < 1$. We then have

$$h(x) = \frac{f'(x)}{f(x)} = -\frac{t^x \ln t}{1-t^x} - \frac{1}{x}, \tag{4.4}$$

$$h'(x) = -\frac{(\ln t)^2 t^x}{(1-t^x)^2} + \frac{1}{x^2}, \tag{4.5}$$

and

$$h''(x) = \frac{(\ln t)^3 t^x (1+t^x)}{(1-t^x)^3} - \frac{2}{x^3}.$$

Note that

$$\begin{aligned} \frac{\partial h''(x)}{\partial t} &= \frac{(\ln t)^2 t^x (3t^{2x} - 3 - xt^{2x}\ln t - x\ln t - 4xt^x \ln t)}{(1-t^x)^4} \\ &= \frac{(\ln t)^2 t^x q(x)}{(1-t^x)^4}, \end{aligned} \tag{4.6}$$

where

$$q'(x) = (5t^{2x} - 1 - 4xt^x \ln t - 4t^x - 2xt^{2x}\ln t)\ln t,$$

and

$$\begin{aligned} q''(x) &= 4(\ln t)^2 t^x \left((2 - x\ln t)t^x - (2 + x\ln t)\right) \\ &= 8(\ln t)^2 t^x \left(1 - \frac{x|\ln t|}{2}\right)\left(\frac{1 + \frac{x|\ln t|}{2}}{1 - \frac{x|\ln t|}{2}} t^x - 1\right). \end{aligned}$$

Employing Lemma (4.2), with $y = x|\ln t|/2$ gives that $q'(x)$ is increasing for $0 < x < 1$ and hence $q'(x) \geq q'(0) = 0$ and finally $q(x) \geq q(0) = 0$.

Returning to (4.6), $h''(x)$ is monotone increasing with respect to t in $(0,1)$.

The concavity of h follows upon noting that for $0 < x < 1$, $\lim_{t\to 1} h''(x) = 0$. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. Note that for $g = f_{u,v}$ and $(a,b) = (0,1)$, in the notation of Theorem 1.1, we have

$$\begin{aligned} R(1-) &= \frac{\ln(v) - \ln(u)}{v-u} = \frac{1}{R(0+)}, \\ h(0+) &= \frac{\ln(v) + \ln(u)}{2}, \\ h(1-) &= \frac{v\ln(v) - u\ln(u)}{v-u} - 1 \end{aligned}$$

and

$$h\left(\frac{a+b}{2}\right) = \frac{\sqrt{v}\ln(v) - \sqrt{u}\ln(u)}{v-u} - 2.$$

The result then follows immediately upon applying Lemma 4.3 and Theorem 1.1. □

Remark 1. The need for bounds of the sort in (4.2) and (4.3) arose recently in the consideration of the behavior of convolution ratios under local approximation (see [5]).

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