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Counting Keith Numbers

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Abstract

A Keith number is a positive integer N with the decimal representation $a_1a_2\cdots a_n$ such that $n \geq 2$ and N appears in the sequence $(K_m)_{m\geq 1}$ given by the recurrence $K_1 = a_1, \ldots, K_n = a_n$ and $K_m = K_{m-1} + K_{m-2} + \cdots + K_{m-n}$ for $m > n$. We prove that there are only finitely many Keith numbers using only one decimal digit (i.e., $a_1 = a_2 = \cdots = a_n$), and that the set of Keith numbers is of asymptotic density zero.

1 Introduction

With the number 197, let $(K_m)_{m\geq 1}$ be the sequence whose first three terms $K_1 = 1,\ K_2 = 9$ and $K_3 = 7$ are the digits of 197 and that satisfies the recurrence $K_m = K_{m-1}+K_{m-2}+K_{m-3}$

for all $m > 3$. Its initial terms are

$$1,\ 9,\ 7,\ 17,\ 33,\ 57,\ 107,\ 197,\ 361,\ 665, \ldots$$

Note that 197 itself is a member of this sequence. This phenomenon was first noticed by Mike Keith and such numbers are now called *Keith numbers*. More precisely, a number N with decimal representation $a_1 a_2 \cdots a_n$ is a Keith number if $n \geq 2$ and N appears in the sequence $K^N = (K_m^N)_{m \geq 1}$ whose n initial terms are the digits of N read from left to right and satisfying $K_m^N = K_{m-1}^N + K_{m-2}^N + \cdots + K_{m-n}^N$ for all $m > n$. These numbers appear in Keith's papers [3, 4] and they are the subject of entry $A007629$ in Neil Sloane's Encyclopedia of Integer Sequences [11] (see also [7, 8, 9]).

Let $\mathcal{K}$ be the set of all Keith numbers. It is not known if $\mathcal{K}$ is infinite or not. The sequence $\mathcal{K}$ begins

$$14,\ 19,\ 28,\ 47,\ 61,\ 75,\ 197,\ 742,\ 1104,\ 1537,\ 2208, 2580,\ 3684,\ 4788, \ldots$$

M. Keith and D. Lichtblau found all 94 Keith numbers smaller than 10^{29} [4]. D. Lichtblau found the first *pandigital* Keith number (containing each of the digits 0 to 9 at least once): 27847652577905793413.

Recall that a rep-digit is a positive integer N of the form $a(10^n - 1)/9$ for some $a \in \{1, \ldots, 9\}$ and $n \geq 1$; i.e., a number which is a string of the same digit a when written in base 10. Our first result shows that there are only finitely many Keith numbers which are rep-digits.

Theorem 1.1. *There are only finitely many Keith numbers that are rep-digits and their set can be effectively determined.*

We point out that some authors refer to the Keith numbers as *replicating Fibonacci digits* in analogy with the Fibonacci sequence $(F_n)_{n \geq 1}$ given by $F_1 = 1,\ F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. F. Luca showed [5] that the largest rep-digit Fibonacci number is 55.

The proof of Theorem 1.1 uses Baker-type estimates for linear forms in logarithms. It will be clear from the proof that it applies to all *base b Keith numbers* for any fixed integer $b \geq 3$, where these numbers are defined analogously starting with their base b expansion (see the remark after the proof of Theorem 1.1).

For a positive integer x we write $\mathcal{K}(x) = \mathcal{K} \cap [1, x]$. As we mentioned before, $\mathcal{K}(10^{29}) = 94$. A heuristic argument [4] suggests that $\#\mathcal{K}(x) \gg \log x$, and, in particular, that $\mathcal{K}$ should be infinite. Going in the opposite way, we show that $\mathcal{K}$ is of asymptotic density zero.

Theorem 1.2. *The estimate*

$$\#\mathcal{K}(x) \ll \frac{x}{\sqrt{\log x}}$$

holds for all positive integers $x \geq 2$.

The above estimate is very weak. It does not even imply that that sum of the reciprocals of the members of $\mathcal{K}$ is convergent. We leave to the reader the task of finding a better upper bound on $\#\mathcal{K}(x)$. Typographical changes (see the remark after the proof of Theorem 1.2) show that Theorem 1.2 also is valid for the set of base b Keith numbers if $b \ge 4$. Perhaps it can be extended also to the case $b = 3$. For $b = 2$, Kenneth Fan has an unpublished manuscript (mentioned by Keith [4]) showing how to construct all Keith numbers and that, in particular, there are infinitely many of them. For example, any power of 2 is a binary Keith number.

Throughout this paper, we use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with their usual meaning. Recall that for functions A and B the inequalities $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent to the fact that there exists a positive constant c such that the inequality $|A| \le cB$ holds. The constants in the inequalities implied by these symbols may occasionally depend on other parameters. For a real number x we use $\log x$ for the natural logarithm of x. For a set $\mathcal{A}$, we use $\#\mathcal{A}$ and $|\mathcal{A}|$ to denote its cardinality.

2 Preliminary Results

For an integer $N > 0$, recall the definition of the sequence $K^N = (K_m^N)_{m\ge1}$ given in the Introduction. In K^N we allow N to be any string of the digits $0, 1, \dots, 9$, so N may have initial zeros. So, for example, $K^{020} = (0, 2, 0, 2, 4, 6, 12, 22, \dots)$. For $n \ge 1$ we define the sequence L^n as $L^n = K^M$ where $M = 11 \cdots 1$ with n digits 1. In particular, $L^1 = (1, 1, 1, \dots)$ and $L^2 = (1, 1, 2, 3, 5, 8, \dots)$, the Fibonacci numbers. In the following lemma, which will be used in the proofs of both Theorems 1 and 2, we establish some properties of the sequences K^N and L^n.

Lemma 2.1. *Let N be a string of the digits $0, 1, \dots, 9$ with length $n \ge 1$. If N does not start with 0, we understand it also as the decimal representation of a positive integer.*

(a) *If N has at least $k \ge 1$ nonzero entries, then $K_m^N \ge L_{k+m-n}^k$ holds for every $m \ge n+1$.*

(b) *If N has at least one nonzero entry, then $K_m^N \ge L_{m-n}^n$ holds for every $m \ge n+1$. We have $K_m^N \le 9L_m^n$ for every $m \ge 1$.*

(c) *If $n \ge 3$ and $N = K_m^N$ for some $m \ge 1$ (so N is a Keith number), then $2n < m < 7n$.*

(d) *For fixed $n \ge 2$ and growing $m \ge n+1$,*

$$L_m^n = 2^{m-n-1}(n-1)(1 + O(m/2^n)) + 1$$

where the constant in O is absolute.

Proof. (a). By the recurrences defining K^N and L^k, the inequality clearly holds for the first k indices $m = n+1, n+2, \dots, n+k$. For $m > n+k$ it holds by induction.

(b). We have $K_m^N \ge 1 = L_{m-n}^n$ for $m = n+1, n+2, \dots, 2n$ and the inequality holds. For $m > 2n$ it holds by induction. The second inequality follows easily by induction.

(c). The lower bound $m > 2n$ follows from the fact that K^N is nondecreasing and that

$$K^N_{2n} \le 9L^n_{2n} = 9 \cdot 2^{n-1}(n-1) + 9 < 10^{n-1} \le N$$

for $n \ge 3$. To obtain the upper bound, note that for $m \ge n$ we have by induction that $L^n_m \ge L^2_{m-n+2} \ge \phi^{m-n}$ where $\phi = 1.61803\cdots$ is the golden ratio. Thus, by part (b),

$$10^n > N = K^N_m \ge L^n_{m-n} \ge \phi^{m-2n}$$

and $m < (2 + \log 10/\log\phi)n < 7n$.

(d). We write L^n_m in the form $L^n_m = (2^{m-n-1} - d(m))(n-1) + 1$ and prove by induction on m that for $m \ge n+1$,

$$0 \le d(m) < m2^{m-2n}.$$

This will prove the claim.

It is easy to see by the recurrence that $L^n_{n+1}, L^n_{n+2}, \ldots, L^n_{2n+1}$ are equal, respectively, to $2^0(n-1)+1, 2^1(n-1)+1, \ldots, 2^n(n-1)+1$. So $d(m) = 0$ for $n+1 \le m \le 2n+1$ and the claim holds. For $m \ge 2n+1$,

$$\begin{aligned} L^n_m &= L^n_{m-1} + L^n_{m-2} + \cdots + L^n_{m-n} \\ &= \sum_{k=1}^{n} \Big((2^{m-n-1-k} - d(m-k))(n-1) + 1 \Big) \\ &= \Big(2^{m-n-1} - 2^{m-2n-1} + 1 - \sum_{k=1}^{n} d(m-k) \Big)(n-1) + 1 \end{aligned}$$

and the induction hypothesis gives

$$\begin{aligned} 0 \le d(m) &= 2^{m-2n-1} - 1 + \sum_{k=1}^{n} d(m-k) \\ &< 2^{m-2n-1} + (m-1)\sum_{k=1}^{n} 2^{m-2n-k} \\ &< m2^{m-2n}. \end{aligned}$$

□

In part (d), if m is roughly of size 2^n or larger then the error term swallows the main term and the asymptotic estimate is useless. Indeed, the actual asymptotic behavior of L^n_m when $m \to \infty$ is $c\alpha^m$ where $c > 0$ is a constant and $\alpha < 2$ is the only positive root of the polynomial $x^n - x^{n-1} - \cdots - x - 1$. But for m small relative to 2^n, say $m = O(n)$ (ensured for Keith numbers by part (c)), this "incorrect" asymptotic estimate of L^n_m is very precise and useful, as we shall demonstrate in the proofs of Theorems 1.1 and 1.2.

In the proof of Theorem 1.1 we will apply also a lower bound for a linear form in logarithms. The following bound can be deduced from a result due to Matveev [6, Corollary 2.3].

Lemma 2.2. *Let $A_1, \dots, A_k$, $A_i > 1$, and $n_1, \dots, n_k$ be integers, and let $N = \max\{|n_1|, \dots, |n_k|, 2\}$. There exist positive absolute constants c_1 and c_2 (which are effective), such that if*

$$\Lambda = n_1 \log A_1 + n_2 \log A_2 + \cdots + n_k \log A_k \neq 0,$$

then

$$\log |\Lambda| > -c_1 c_2^k (\log A_1) \cdots (\log A_k) \log N.$$

For the proof of Theorem 2 we will need an upper bound on sizes of antichains (sets of mutually incomparable elements) in the poset (partially ordered set)

$$P(k,n) = (\{1, 2, \dots, k\}^n, \leq_p)$$

where $\leq_p$ is the product ordering

$$a = (a_1, a_2, \dots, a_n) \leq_p b = (b_1, b_2, \dots, b_n) \iff a_i \leq b_i \text{ for } i = 1, 2, \dots, n.$$

We have $|P(k,n)| = k^n$ and for $k = 2$ the poset $P(2,n)$ is the Boolean poset of subsets of an n-element set ordered by inclusion. The classical theorem of Sperner [1, 2] asserts that the maximum size of an antichain in $P(2,n)$ equals the middle binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$. In the next lemma we obtain an upper bound for any $k \geq 2$.

Lemma 2.3. *If $k \geq 2, n \geq 1$ and $X \subset P(k,n)$ is an antichain to $\leq_p$, then*

$$|X| < \frac{(k/2) \cdot k^n}{n^{1/2}}.$$

Proof. We proceed by induction on k. For $k = 2$ this bound holds by Sperner's theorem because

$$\binom{n}{\lfloor n/2 \rfloor} < \frac{2^n}{n^{1/2}}$$

for every $n \geq 1$. Let $k \geq 3$ and $X \subset P(k,n)$ be an antichain. For A running through the subsets of $[n] = \{1, 2, \dots, n\}$, we partition X in the sets X_A where X_A consists of the $u \in X$ satisfying $u_i = k \iff i \in A$. If we delete from all $u \in X_A$ all appearances of k, we obtain (after appropriate relabelling of coordinates) a set of $|X_A|$ distinct $(n - |A|)$-tuples from $P(k-1, n-|A|)$ that must be an antichain to $\leq_p$. Thus, by induction, for $|A| < n$ we have

$$|X_A| < \frac{((k-1)/2) \cdot (k-1)^{n-|A|}}{(n-|A|)^{1/2}}$$

and $|X_{[n]}| \leq 1$. Summing over all As and using the inequality $\sqrt{n/m} \leq (n+1)/(m+1)$

(which holds for $1 \leq m \leq n$) and standard properties of binomial coefficients, we get

$$\begin{aligned}
|X| &= \sum_{A \subset [n]} |X_A| \\
&< 1 + \sum_{i=0}^{n-1} \binom{n}{i} \frac{((k-1)/2) \cdot (k-1)^{n-i}}{(n-i)^{1/2}} \\
&= \frac{1}{\sqrt{n}} \left(\sqrt{n} + \frac{1}{2} \sum_{i=0}^{n-1} \binom{n}{i} \sqrt{n/(n-i)} \cdot (k-1)^{n-i+1} \right) \\
&\leq \frac{1}{\sqrt{n}} \left(\sqrt{n} + \frac{1}{2} \sum_{i=0}^{n-1} \binom{n+1}{n-i+1} (k-1)^{n-i+1} \right) \\
&< \frac{k^{n+1}}{2\sqrt{n}}.
\end{aligned}$$

□

We conclude this section with three remarks as to the last lemma.

1. Various generalizations and strengthenings of Sperner's theorem were intensively studied, see, e.g., the book of Engel and Gronau [2]. Therefore, we do not expect much originality in our bound.

2. It is clear that for $k = 2$ the exponent $1/2$ of n in the bound of Lemma 2.3 cannot be increased. The same is true for any $k \geq 3$. We briefly sketch a construction of a large antichain when $k = 3$; for $k > 3$ similar constructions can be given. For $k = 3$ and $n = 3m \geq 3$ consider the set $X \subset P(3, n)$ consisting of all u which have i 1s, $n - 2i$ 2s and i 3s, where $i = 1, 2, \ldots, m = n/3$. It follows that X is an antichain and that

$$|X| = \sum_{i=1}^{m} \binom{n}{i, i, n-2i} = \sum_{i=1}^{m} \frac{n!}{(i!)^2 (n-2i)!}.$$

By the usual estimates of factorials, if $m - \sqrt{n} < i \leq m$ then

$$\binom{n}{i, i, n-2i} \gg \binom{n}{m, m, m} \gg \frac{3^n}{n}.$$

Hence X is an antichain in $P(3, n)$ with size

$$|X| \gg \sqrt{n} \cdot \frac{3^n}{n} = \frac{3^n}{\sqrt{n}}.$$

3. For composite k we can decrease the factor $k/2$ in the bound of Lemma 2.3. Suppose that $k = lm$ where $l \geq m \geq 2$ are integers and let $X \subset P(k, n)$ be an antichain. We associate with every $u \in X$ the pair of n-tuples $(v^u, w^u) \in P(m, n) \times P(l, n)$ defined by $v_i^u = u_i - m\lceil u_i/m \rceil + m$ and $w_i^u = \lceil u_i/m \rceil$, $1 \leq i \leq n$. Note that the pair (v^u, w^u) uniquely determines u and that if $w^u = w^{u'}$ then v^u and $v^{u'}$ are incomparable by $\leq_p$. Thus, by

Lemma 2.3, for fixed $w \in P(l,n)$ there are less than $(m/2)m^n/\sqrt{n}$ elements $u \in X$ with $w^u = w$. The number of ws is at most $|P(l,n)| = l^n$. Hence

$$|X| < \frac{(m/2)\cdot m^n}{n^{1/2}} \cdot l^n = \frac{(m/2)\cdot k^n}{n^{1/2}}.$$

In particular, if k is a power of 2 then $|X| < k^n/\sqrt{n}$ for every antichain $X \subset P(k,n)$.

3 The proof of Theorem 1.1

Let $N = a(10^n - 1)/9 = aa\cdots a$, $1 \le a \le 9$, be a rep-digit. Since $K^N = aL^n$, N is a Keith number if and only if the rep-unit $M = (10^n-1)/9 = 11\cdots 1$ is a Keith number. Suppose that M is a Keith number: for some m we have

$$M = \frac{10^n - 1}{9} = L_m^n = 2^{m-n-1}(n-1)\left(1 + O\left(\frac{m}{2^n}\right)\right),$$

where the asymptotic relation was proved in part (d) of Lemma 2.1. We rewrite this relation as

$$\frac{2^{2n+1-m}5^n}{9(n-1)} - 1 = \frac{1}{9(n-1)2^{m-n-1}} + O\left(\frac{m}{2^n}\right).$$

Since $2n < m < 7n$ by part (c) of Lemma 2.1, we get

$$\frac{2^{2n+1-m}5^n}{9(n-1)} - 1 = O\left(\frac{n}{2^n}\right).$$

Because $5^n > 9(n-1)$ for every $n \ge 1$, the left side is always non-zero (the power of 5 cannot be canceled). Writing it in the form $e^\Lambda - 1$ and using that $e^\Lambda - 1 = O(\Lambda)$ (as $\Lambda \to 0$), we get

$$0 \ne \Lambda = (2n+1-m)\log 2 + n\log 5 - \log(9(n-1)) \ll \frac{n}{2^n}.$$

Taking logarithms and applying Lemma 2.2, we finally obtain

$$-d(\log n)^2 < \log|\Lambda| < c(\log n - n\log 2)$$

where $c, d > 0$ are effectively computable constants. This implies that n is effectively bounded and completes the proof of Theorem 1.1. □

Remark. The same argument shows that for every integer $b \ge 3$ there are only effectively finitely many base b rep-digits, i.e., positive integers of the form $a(b^n-1)/(b-1)$ with $a \in \{1,\dots,b-1\}$, which are base b Keith numbers. Indeed, we argue as for $b = 10$ and derive the equation

$$\frac{b^n}{(b-1)(n-1)2^{m-n-1}} - 1 = O(n/2^n).$$

In order to apply Lemma 2.2, we need to justify that the left side is not zero. If b is not a power of 2, it has an odd prime divisor p, and p^n cannot be cancelled, for big enough n, by $(b-1)(n-1)$. If $b \ge 3$ is a power of 2, then $b-1$ is odd and has an odd prime divisor, which cannot be cancelled by the rest of the expression.

4 The proof of Theorem 1.2

For an integer $N > 0$, we denote by n the number of its digits: $10^{n-1} \le N < 10^n$. We shall prove that there are $\ll 10^n/\sqrt{n}$ Keith numbers with n digits; it is easy to see that this implies Theorem 2. There are only few numbers with n digits and $\ge n/2$ zero digits: their number is bounded by

$$\sum_{i \ge n/2} \binom{n}{i} 9^{n-i} \le 2^n 9^{n/2} = 6^n < (10^n)^{0.8}.$$

Hence it suffices to count only the Keith numbers with n digits, of which at least half are nonzero.

Let N be a Keith number with $n \ge 3$ digits, at least half of them nonzero. So, $N = K_m^N$ for some index $m \ge 1$. By part (c) of Lemma 2.1, $2n < m < 7n$ and we may use the asymptotic estimate in part (d). Setting $k = \lfloor n/2 \rfloor$ and using the inequality in part (a) of Lemma 2.1, we get

$$10^n > N = K_m^N \ge L_{k+m-n}^k.$$

Part (d) of Lemma 2.1 gives that for big n,

$$L_{k+m-n}^k > \frac{2^{m-n-1}(k-1)}{2} > \frac{2^{m-n}n}{12}.$$

On the other hand, the second inequality in part (b) of Lemma 2.1 and part (d) give, for big n,

$$10^{n-1} \le N = K_m^N \le 9L_m^n < 9 \cdot 2^{m-n}n.$$

Combining the previous inequalities, we get

$$\frac{10^n}{90} < 2^{m-n}n < 12 \cdot 10^n.$$

This implies that, for $n > n_0$, the index m attains at most 12 distinct values and

$$m = (1 + \log 10/\log 2 + o(1))n = (\kappa + o(1))n.$$

Now we partition the set S of considered Keith numbers (with n digits, at least half of them nonzero) in blocks of numbers N having the same value of the index m and the same string of the first (most significant) $k = \lfloor n/2 \rfloor$ digits. So, we have at most $12 \cdot 10^k$ blocks. We show in a moment that the numbers in one block B, when regarded as $(n-k)$-tuples from $P(10, n-k)$, form an antichain to $\le_p$. Assuming this, Lemma 2.3 implies that $|B| < 10^{n-k+1}/2\sqrt{n-k}$. Summing over all blocks, we get

$$|S| < 12 \cdot 10^k \cdot \frac{10^{n-k+1}}{2\sqrt{n-k}} \ll \frac{10^n}{\sqrt{n}},$$

which proves Theorem 2.

To show that B is an antichain, we suppose for the contradiction that N_1 and N_2 are two Keith numbers from B with $N_1 <_p N_2$. Let $M = N_2 - N_1$ and $M^* = 00 \cdots 0M \in P(10, n)$ (we complete M to a string of length n by adding initial zeros). It follows that M has at most $n - k$ digits and $M < 10^{n-k}$. On the other hand, by the linearity of recurrence and by $N_1 <_p N_2$, we have

$$M = N_2 - N_1 = K_m^{N_2} - K_m^{N_1} = K_m^{M^*}.$$

Since M^* has some nonzero entry, the first inequality in part (b) of Lemma 2.1 and part (d) give, for big n,

$$K_m^{M^*} \geq L_{m-n}^n > 2^{m-2n-2}n.$$

Thus

$$10^{n-k} = 10^{n-\lfloor n/2 \rfloor} > M > 2^{m-2n-2}n.$$

Using the above asymptotic estimate of m in terms of n, we arrive at the inequality

$$\begin{aligned} \exp((\tfrac{1}{2}\log 10 + o(1))n) &> \exp((\kappa \log 2 - 2\log 2 + o(1))n) \\ &= \exp((\log 5 + o(1))n) \end{aligned}$$

that is contradictory for big n because $10^{1/2} < 5 = 10/2$. This finishes the proof of Theorem 2. □

Remark. The above proof generalizes, with small modifications, to all bases $b \geq 4$. We replace base 10 by b, modify the proof accordingly, and have to satisfy two conditions. First, in the beginning of the proof we delete from the numbers with n base b digits those with $> \alpha n$ zero digits, for some constant $0 < \alpha < 1$. In order that we delete negligibly many, compared to b^n, numbers, we must have $2 \cdot (b-1)^{1-\alpha} < b$. Second, for the final contradiction we need that $b^\alpha < b/2$. For $b \geq 5$, both conditions are satisfied with $\alpha = 1/2$, as in case $b = 10$. For $b = 4$ they are satisfied with $\alpha = 0.49$, say. However, for $b = 3$ they cannot be satisfied by any α. Thus, the case $b = 3$ seems to require more substantial changes.

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