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Some Classes of Numbers and Derivatives

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Abstract

We prove that three classes of numbers – the non-central Stirling numbers of the first kind, generalized factorial coefficients, and Gould-Hopper numbers – may be defined by the use of derivatives. We derive several properties of these numbers from their definitions. We also prove a result for harmonic numbers. The coefficients of Hermite and Bessel polynomials are a particular case of generalized factorial coefficients, The coefficients of the associated Laguerre polynomials are a particular case of Gould-Hopper numbers. So we obtain some properties of these polynomials. In particular, we derive an orthogonality relation for the coefficients of Hermite and Bessel polynomials.

1 Introduction

The purpose of this paper is to investigate properties of the non-central Stirling numbers of the first kind, the generalized factorial coefficients, and Gould-Hopper numbers by the use of derivatives.

We use the following notation throughout the paper:

- $(a)_n$ denotes the falling factorial of a, that is, $(a)_n = a(a-1)\cdots(a-n+1)$, $(a)_0 = 1$;
- $s(n,k)$ and $\mathbf{s}(n,k)$ denote the signed and the unsigned Stirling numbers of the first kind respectively;
- H_n denotes the harmonic number $\sum_{1\le i\le n} 1/i$;
- $s(n,k,a)$ denotes the non-central Stirling number of the first kind;
- $C(n,k,a)$ denotes the generalized factorial coefficient;

- $C(n, k, b, a)$ denotes the non-central generalized factorial coefficient or Gould-Hopper number.

The notation and the terminology are taken from Charalambides' book [4]. Sloane [6] calls the $s(n, k, a)$ the generalized Stirling numbers. Further,

- $H_n(x)$ denotes the Hermite polynomial;
- $p_n(x)$ denotes the (reverse) Bessel polynomial, and
- $L_n^k(x)$ denotes the associated Laguerre polynomial, where $L_n^0(x) = L_n(x)$ is a Laguerre polynomial.

The paper is organized as follows. The first section is an introduction.

In the second section we prove that the non-central Stirling numbers $s(n, k, a)$ of the first kind naturally appear in the expansion of derivatives of the function $x^{-a} \ln^b x$, where a and b are arbitrary real numbers. We first obtain a recurrence relation for $s(n, k, a)$ and then, using Leibnitz rule, we obtain an explicit formula. We then consider a particular formula for $s(n, 1, a)$ and derive some combinatorial identities. The results are related to a number of sequences from Sloane's *Encyclopedia* [6].

In the third section we first prove that the generalized factorial coefficients appear as coefficients in the expansion of the nth derivative of the function $f(x^a)$, where a is arbitrary real number, and $f \in C^\infty(0, +\infty)$ is arbitrary function. Choosing suitable functions f we derive some properties of generalized factorial coefficients. We are particularly concerned with some properties of coefficients of Hermite and Bessel polynomials. The results of this section are also related to a number of sequences from [6].

In the fourth section we first show that Gould-Hopper numbers are coefficients in the expansion of the nth derivative of the function $x^a f(x^b)$, where a, b are arbitrary real numbers, and $f \in C^\infty(0, +\infty)$ is arbitrary function. The coefficients of associated Laguerre polynomials are particular case of Gould-Hopper numbers. Using similar methods as in the third section we prove a number of properties which describe connections between Gould-Hopper numbers, generalized factorial coefficients, powers, factorials, binomial coefficients, and Stirling numbers. The results are also concerned with some sequences from [6].

Note that these considerations are related with Bell polynomials which naturally appear in derivatives of composition functions [2, 5, 7] .

2 Non-central Stirling numbers of the first kind

We shall first derive a formula for the nth derivative of the function

$$f(x) = x^{-a} \ln^b x, \ (a, b \in \mathbb{R}).$$

Theorem 1. *Let a be a real number, and let n be a nonnegative integer. Then*

$$\frac{d^n}{dx^n} f(x) = x^{-a-n} \sum_{i=0}^{n} p(n, i, a)(b)_i \ln^{b-i} x. \tag{1}$$

where $p(n, i, a)$, $(0 \le i \le n)$ are polynomials of a with integer coefficients.

Proof. Theorem 1 is true for $n = 0$, if we define $p(0, 0, a) = 1$.

If we define

$$p(1, 0, a) = -a,\ p(1, 1, a) = 1$$

then Theorem 1 is also true for $n = 1$.

Assume Theorem 1 is valid for $n \geq 1$.

Taking derivative in (1) we find that

$$\frac{d^{n+1}}{dx^{n+1}} f(x) = x^{-a-n-1} \left[(-a-n) \sum_{i=0}^{n} p(n, i, a)(b)_i \ln^{b-i} x + \sum_{i=0}^{n} p(n, i, a)(b)_{i+1} \ln^{b-i-1} x \right].$$

Replacing $i + 1$ by i in the second sum on the right side yields

$$\frac{d^{n+1}}{dx^{n+1}} f(x) = x^{a-n-1}(-a-n)p(n, 0, a) + s(n, n, a)(b)_{n+1} \ln^{b-n-1} x +$$

$$+ x^{a-n-1} \sum_{i=1}^{n} \left[(-a-n)p(n, i, a) + s(n, i-1, a) \ln^{b-i} x \right] (b)_i.$$

It follows that Theorem 1 is true if we define

$$p(n+1, 0, a) = -(a+n)p(n, 0, a),\ p(n+1, n+1, a) = p(n, n, a),$$

$$p(n+1, i, a) = -(a+n)p(n, i, a) + p(n, i-1, a),\ (i = 1, \ldots, n).$$

□

The preceding equations are the recurrence relations for non-central Stirling numbers of the first kind $s(n, i, a)$, [4, p. 316]. In what follows we shall denote $p(n, i, a)$ by $s(n, i, a)$.

It is easy to see that the following equations hold

$$s(n, 0, a) = (-a)_n,\ (n = 0, 1, 2, \ldots),$$

and

$$s(n, n, a) = 1,\ (n = 0, 1, 2, \ldots).$$

By Leibnitz rule we get

$$\frac{d^n}{dx^n} f(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} x^{-a} \frac{d^{n-k}}{dx^{n-k}} \ln^b x. \tag{2}$$

From the well-known formulas

$$\frac{d^k}{dx^k} x^{-a} = (-a)_k x^{a-k},$$

and

$$\frac{d^{n-k}}{dx^{n-k}} \ln^b x = x^{-n+k} \sum_{i=1}^{n-k} s(n-k, i)(b)_i \ln^{b-i} x,$$

by comparing (1) and (2) we obtain the following:

Proposition 2. *Let a be a real number, and let n, i, $(i \leq n)$ be nonnegative integers. Then*

$$s(n,i,a) = \sum_{k=0}^{n-i} \binom{n}{k} (-a)_k s(n-k,i). \tag{3}$$

Remark 3. Proposition 2 is true in the case $a = 0$ with the convention that $(0)_0 = 1$.

Taking $i = 1$ in (3) we have the following:

Proposition 4. *Let a be a real number, and n be a positive integer. Then*

$$s(n,1,a) = n! \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{\binom{-a}{k}}{n-k}.$$

For $s(n,1,a)$ we have the following recurrence relation:

$$s(1,1,a) = 1,\ s(n,1,a) = (-a-n+1)s(n-1,1,a) + (-a)_{n-1},\ (n \geq 2). \tag{4}$$

We shall now prove that polynomials $r(n,a)$, $(n = 1,2,\ldots)$ defined by

$$r(n,a) = \sum_{k=0}^{n-1} (k+1)s(n,k+1)(-a)^k$$

satisfy (4). For $n = 1$ this is obviously true.

Using the two terms recurrence relation for Stirling numbers of the first kind, for $n > 1$ we have

$$r(n,a) = \sum_{k=0}^{n-1} (k+1)s(n-1,k)(-a)^k - (n-1) \sum_{k=0}^{n-2} (k+1)s(n-1,k+1)(-a)^k.$$

Since $s(n-1,0) = 0$, by replacing $k+1$ instead of k in the first sum on the right side we obtain

$$r(n,a) = (-a-n+1)r(n-1,a) + \sum_{k=0}^{n-2} s(n-1,k+1)(-a)^{k+1}.$$

Furthermore, a well known property of Stirling numbers of the first kind implies

$$\sum_{k=0}^{n-2} s(n-1,k+1)(-a)^{k+1} = (-a)_{n-1},$$

which means that $r(n,a)$ satisfies (4). We have proved the following:

Proposition 5. *Let a be a real number, and let $n \geq 1$ be an integer. Then*

$$n! \sum_{k=0}^{n-1} (-1)^k \frac{\binom{-a}{k}}{n-k} = \sum_{k=0}^{n-1} (k+1)\mathbf{s}(n,k+1)a^k. \tag{5}$$

Remark 6. Proposition 5 is true for $a = 0$ with the convention that $0^0 = 1$.

In the case that a is a negative integer and $n \le -a$, the identity (5) is related to the harmonic numbers.

Proposition 7. *Define $h(n, m)$ such that*

$$h(n, m) = (H_m - H_{m-n})\frac{m!}{(m-n)!}, \ (m = 1, 2, \ldots; n = 1, 2, \ldots, m).$$

Then $h(n, m)$ satisfies (4).

Proof. The proof goes by induction with respect to n. For $n = 1$ we have

$$h(1, m) = \frac{(m)!}{(m-1)!}(H_m - H_{m-1}) = 1.$$

Furthermore, for $n > 1$ we have

$$(m - n + 1)h(n - 1, m) + (m)_{n-1} = \frac{(m)!}{(m-n)!}(H_m - H_{m-n+1}) + (m)_{n-1} =$$

$$= \frac{(m)!}{(m-n)!}(H_m - H_{m-n}),$$

since $\frac{(m)!}{(m-n)!(m-n+1)} = (m)_{n-1}$. It follows that

$$h(n, m) = (m - n + 1)h(n - 1, m) + (m)_{n-1},$$

and the result is proved. □

As an immediate consequence of Proposition 7 we obtain

Proposition 8. *Let m be a positive integer and let n, $(1 \le n \le m)$ be any integers. Then*

$$H_m - H_{m-n} = \frac{(-1)^{n+1}}{\binom{m}{n}} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{m}{k}}{n-k}.$$

Remark 9. The results of this section are concerned with the following sequences in [6]: A001701, A001702, A001705, A001706, A001707, A001708, A001709, A001711, A001712, A001713, A001716, A001717, A001718, A001722, A001723, A001724, A049444, A049458, A049459, A049600, A051338, A051339, A051379, A051523, A051524, A051525, A051545, A051546, A051560, A051561, A051562, A051563, A051564, A051565.

3 Generalized factorial coefficients

The first result in this section is a closed formula for the nth derivative of the function $f(x^a)$, where $f \in C^\infty(0,+\infty)$, and a is a real number. Such one formula may be obtained as a particular case of Faá di Bruno's formula. We obtain here the formula which is easily proved by induction. In addition, we obtain a recurrence relation for coefficients.

Theorem 10. *Let $n > 0$ be an integer, and let a be a real number. Then*

$$\frac{d^n}{dx^n} f(x^a) = \sum_{k=1}^{n} q(n,k,a) x^{ak-n} \frac{d^k}{dx^k} f(t), \tag{6}$$

where $t = x^a$, and $q(n,k,a)$ is a polynomials of a with integer coefficients. The degree of $q(n,k,a)$ is n, and it does not depend on f.

Proof. The result is true for $n = 1$ if we take $q(1,1,a) = a$. Assume that the result is true for $n \geq 1$. Taking derivative in (6) we obtain

$$\frac{d^{n+1}}{dx^{n+1}} f(x^a) = \sum_{k=1}^{n} (ka-n) q(n,k,a) x^{ka-n-1} \frac{d^k}{dx^k} f(t) +$$

$$+ a \sum_{k=1}^{n} q(n,k,a) x^{ka-n+a-1} \frac{d^{k+1}}{dx^{k+1}} f(t) =$$

$$= \sum_{k=2}^{n} \left[(ka-n) q(n,k,a) + a q(n,k-1,a)\right] x^{ka-n-1} \frac{d^k}{dx^k} f(t) +$$

$$+ (a-n) q(n,1,a) x^{a-n-1} \frac{d}{dx} f(t) + a q(n,n,a) x^{(n+1)(a-1)} \frac{d^{n+1}}{dx^{n+1}} f(t).$$

Define

$$q(n,0,a) = 0, \ q(n,k,a) = 0, \ (k > n),$$

and

$$q(n+1,k,a) = (ka-n) q(n,k,a) + a q(n,k-1,a), \ (k = 1,\ldots,n+1), \tag{7}$$

to obtain

$$x^{n+1} \frac{d^{n+1}}{dx^{n+1}} f(x^a) = \sum_{k=1}^{n+1} q(n+1,k,a) t^k \frac{d^k}{dx^k} f(t),$$

and the result is proved. □

If, additionally, we define $q(0,0,a) = 1$ then the formula (6) may be written in the form

$$\frac{d^n}{dx^n} f(x^a) = \sum_{k=0}^{n} q(n,k,a) \frac{d^k}{dx^k} f(t) x^{ak-n}, \ (n = 0,1,\ldots). \tag{8}$$

The equations (7) shows that the polynomials $q(n,k,a)$ are in fact the generalized factorial coefficients $C(n,k,a)$ ([4, p. 309]).

Proposition 11. *Generalized factorial coefficients satisfy the following equations:*

$$C(n,1,a)=(a)_n,\ C(n,n,a)=a^n,\ C(n,k,1)=0,\ (k<n)(n=1,2,\ldots).$$

Proof. For the first equation it is enough to take $f(t)\equiv t$ in (8).

The second equation follows immediately from (7).

If $a=1$ then (8) takes the form

$$x^n\frac{d^n}{dx^n}f(x)=\sum_{k=0}^{n}C(n,k,1)x^k\frac{d^k}{dx^k}f(t).$$

and since f is arbitrary function the third equation is also true. □

The generalized factorial coefficients are related with Hermite and Bessel polynomials. Taking $f(t)=e^{bt}$ in (8) we obtain

$$\frac{d^n}{dx^n}e^{bx^a}=e^{bx^a}\sum_{k=1}^{n}C(n,k,a)b^kx^{ak-n}. \tag{9}$$

Proposition 12. *Let n and m be positive integers. Then*

$$\frac{d^n}{dx^n}e^{-x^m}=e^{-x^m}\sum_{k=\lceil\frac{n}{m}\rceil}^{n}(-1)^kC(n,k,m)x^{mk-n}. \tag{10}$$

Proof. Take $b=-1$ in (9), hence

$$\frac{d^n}{dx^n}e^{-x^m}=e^{-x^m}\sum_{k=0}^{n}C(n,k,m)x^{mk-n},\ (n\geq 0).$$

It is clear that taking derivatives on the left-hand side of this equation can not produce negative powers of x. This means that $C(n,k,m)=0$ if $km-n<0$, and Proposition 12 is proved. □

Remark 13. The equation (10) defines generalized Hermite polynomials, [3].

Proposition 14. *If $H_n(x),\ n=1,\ldots$ are Hermite polynomials then*

$$H_n(x)=\sum_{k=\lceil\frac{n}{2}\rceil}^{n}(-1)^{n+k}C(n,k,2)x^{2k-n}. \tag{11}$$

It is easy to check that functions $f(n,k),\ (n=1,\ldots;\ k=1,\ldots,n)$ defined by

$$f(n,k)=\frac{(-1)^{n-k}(2n-k-1)!}{2^{2n-k}(n-k)!(k-1)!}$$

fulfill the recurrence relation (7) for $a=\frac{1}{2}$. We thus obtain the following:

Proposition 15. *Bessel polynomials $p_n(x)$ satisfy the following equation:*

$$p_n(x) = 2^n \sum_{k=1}^{n} (-1)^{n-k} C\left(n, k, \frac{1}{2}\right) x^k. \tag{12}$$

The next result shows that generalized factorial coefficients are coefficients in the expansion of falling factorials of b in terms of falling factorials of a, where a and b are arbitrary real numbers.

Proposition 16. *Let n be a nonnegative integer, and let a, b be arbitrary real numbers. Then*

$$(b)_n = \sum_{k=0}^{n} C\left(n, k, \frac{b}{a}\right) (a)_k. \tag{13}$$

Proof. Replacing a by $\frac{b}{a}$ in (8) we have

$$x^n \frac{d^n}{dx^n} f(x^{\frac{b}{a}}) = \sum_{k=1}^{n} C\left(n, k, \frac{b}{a}\right) t^k \frac{d^k}{dx^k} f(t),$$

where $t = x^{\frac{b}{a}}$.

Choosing $f(t) = t^a$ implies $f(x^{\frac{b}{a}}) = x^b$, hence

$$x^n \frac{d^n}{dx^n} x^b = \sum_{k=1}^{n} C\left(n, k, \frac{b}{a}\right) t^k \frac{d^k}{dx^k} t^a,$$

that is,

$$(b)_n x^b = \sum_{k=1}^{n} C\left(n, k, \frac{b}{a}\right) (a)_k t^a.$$

Since $x^b = t^a$ the result follows. □

Remark 17. The equation (13) serves as the definition of generalized factorial coefficients in [4, Definition 8.2].

Choosing $b = -a$ implies $(-a)_n = (-1)^n a(a+1)\cdots(a+n-1)$. We thus obtain the expression in which rising factorials are given in terms of falling factorials.

Proposition 18. *Let a be a real number. Then*

$$a(a+1)\cdots(a+n-1) = (-1)^n \sum_{k=1}^{n} C(n, k, -1)(a)_k.$$

Remark 19. The preceding equation means that $C(n, k, -1)$ are Lah numbers.

From the equation (13) we shall derive some properties of coefficients of Hermite and Bessel polynomials. Denote by $b(n,k)$ the coefficient by x^k in the expansion of $P_n(x)$ in (12). Then

$$C\left(n,k,\frac{1}{2}\right) = (-1)^{n-k}2^{-n}b(n,k),\ (n=1,2,\dots,\ k=1,2,\dots,n).$$

Next, denote by $h(n,k)$ the coefficient by x^k in the expansion of $H_n(x)$ in (11). It follows that

$$C(n,k,2) = (-1)^{n+k}h(n,2k-n),$$

where $h(n,2k-n)=0$ if $2k-n<0$. We have thus proved the following:

Proposition 20. *Let a be a real number, and let n be a positive integer. Then the following equations hold*

$$(2a)_n = \sum_{k=1}^{n}(-1)^{n+k}h(n,2k-n)(a)_k,$$

and

$$(a)_n = \sum_{k=1}^{n}(-1)^{n-k}2^{-n}b(n,k)(2a)_k.$$

The following proposition gives a known property of generalized factorial coefficients, [4, Theorem 8.18].

Proposition 21. *Let $n \geq k$ be integers, and let $a,\ b$ be real numbers. Then*

$$C(n,k,a_1a_2) = \sum_{j=k}^{n}C(n,j,a_2)C(j,k,a_1). \tag{14}$$

Proof. Take $f_1(t)=t^{a_1}$ and $f_2(t)=t^{a_2}$, hence

$$f(x^{a_1a_2}) = (f\circ f_1)(x^{a_2}).$$

Firstly, it follows from (6) that

$$x^n\frac{d^n}{dx^n}f(x^{a_1a_2}) = \sum_{k=1}^{n}C(n,k,a_1a_2)x^{a_1a_2k}\frac{d^k}{dx^k}f(t),\ (t=x^{a_1a_2}). \tag{15}$$

On the other hand, (6) also implies

$$x^n\frac{d^n}{dx^n}(f\circ f_1)(x^{a_2}) = \sum_{j=1}^{n}C(n,j,a_2)x^{a_2j}\frac{d^j}{dx^j}(f\circ f_1)(u),\ (u=x^{a_2}).$$

Applying (6) once more yields

$$x^n\frac{d^n}{dx^n}(f\circ f_1)(x^{a_2}) = \sum_{j=1}^{n}\sum_{k=1}^{j}C(n,j,a_2)C(j,k,a_1)x^{a_2j}u^{-j}v^k\frac{d^k}{dx^k}f(v),\ (v=u^{a_1}).$$

Changing the order of summation and taking into account that $v = u^{a_2} = x^{a_1 a_2} = t$ we obtain

$$x^n \frac{d^n}{dx^n}(f \circ f_1)(x^{a_2}) = \sum_{k=1}^{n} \left(\sum_{j=k}^{n} C(n,j,a_2) C(j,k,a_1) \right) x^{a_1 a_2 k} \frac{d^k}{dx^k} f(t).$$

Comparing (15) and the preceding equation shows that Proposition 21 is true. □

From Proposition 21 we derive an orthogonality relation between coefficients of Hermite and Bessel polynomials.

Proposition 22. *If $h(n,k)$ and $b(n,k)$ are the coefficients of Hermite and Bessel polynomials respectively, then*

$$\sum_{k=1}^{n} b(n,k) h(n, 2k-n) = 0.$$

Proof. Since $C(n,k,1) = 0$ for $k < n$ the result follows from (11) and (12). □

Remark 23. The results of this section are related to the following sequences in [6]: A000369, A001497, A001801, A004747, A008297, A013988, A035342, A035469, A049029, A049385, A059343, A092082, A105278, A111596, A122850, A132056, A132062, A136656.

4 Gould-Hopper numbers

In the first result of this section we prove that Gould-Hopper numbers are coefficients in the expansion of the nth derivative of $x^a f(x^b)$, where a, b are arbitrary real numbers, and $f \in C^\infty(0, +\infty)$ is arbitrary function.

Theorem 24. *Let n be a positive integer, and let a, b be real numbers. Then*

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n} \sum_{k=0}^{n} p(n,k,b,a) x^{bk} \frac{d^k}{dx^k} f(t), \tag{16}$$

where $t = x^b$, and $p(n,k,b,a)$ are polynomials of a and b with integer coefficients, which do not depend on f.

Proof. Using Leibnitz rule and (6) we easily obtain

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n} \left[(a)_n f(x^b) + \sum_{j=1}^{n} \sum_{k=1}^{j} \binom{n}{j} C(j,k,b) \frac{d^k}{dx^k} f(t) (a)_{n-j} x^{bk} \right].$$

Changing the order of summation implies

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n} \left[(a)_n f(x^b) + \sum_{k=1}^{n} \left[\sum_{j=k}^{n} \binom{n}{j} C(j,k,b)(a)_{n-j} \right] \frac{d^k}{dx^k} f(t) x^{bk} \right].$$

Theorem 24 is true if we define

$$p(n,0,b,a)=(a)_n,\ p(n,k,b,a)=\sum_{j=k}^{n}\binom{n}{j}C(j,k,b)(a)_{n-j},\ (k=1,\ldots,n).$$

□

Remark 25. According to [4, p. 318] we see that $p(n,k,b,a)$ are Gould-Hopper numbers or non-central generalized factorial coefficients and will be dented by $C(n,k;b,a)$.

Gould-Hopper numbers generalize coefficients of associated Laguerre polynomials $L_n^k(x)$, [1, p. 726].

Namely, $L_n^k(x)$ are defined to be

$$L_n^k(x)=\frac{e^x x^{-k}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+k}),\ L_n^0(x)=L_n(x).$$

Take $a=n+k, b=1, f(x)=e^{-x}$ in (16) to obtain

Proposition 26. *Let $L_n^k(x)$, $(n=0,1,\ldots,\ k=0,1,\ldots)$ be associated Laguerre polynomials. Then*

$$L_n^k(x)=\frac{1}{n!}\sum_{i=0}^{n}(-1)^i C(n,i,1,n+k)x^i.$$

Proposition 27. *Let n be a nonnegative integer, and let a,b,c be nonzero real numbers. Then*

$$(a+bc)_n=\sum_{k=0}^{n}C(n,k;b,a)(c)_k.$$

Proof. Take $f(t)=t^c$ in (16) to obtain $x^a f(x^b)=x^{a+bc}$, and the resul follows. □

Remark 28. The equation from the preceding proposition serves as the definition of Gould-Hopper numbers in [4, p. 317].

The following result shows that Gould-Hopper numbers, with a suitable chosen sign, are coefficients in the expression of falling factorial of a in terms rising factorial of b.

Proposition 29. *Let n be a positive integer, and let a,b be nonzero real numbers. Then*

$$(a)_n=\sum_{k=1}^{n}(-1)^{k-1}C\big(n,k;\frac{a}{b},a\big)\cdot b\cdot(b+1)\cdots(b+k-1).$$

Proof. Take $f(t)=t^{-\frac{a}{c}}$, where $c\neq 0$. Then $x^a f(x^c)=1$, hence $\frac{d^n}{dx^n}(x^a f(x^c))=0,\ (n>0)$. Applying (16) we obtain

$$\sum_{k=0}^{n}C\left(n,k,\frac{a}{b},a\right)(-b)_k=0,$$

where $b=\frac{a}{c}$, and the result holds. □

The next result is an explicit formula for $C(n,k;b,a)$ in terms of generalized factorial coefficients.

Proposition 30. *Let $m \le n$ be nonnegative integers, and let a, b be nonzero real numbers. Then*

$$C(n,m;b,a) = \sum_{k=m}^{n} C\left(k, m, \frac{b}{a}\right)\Big[C(n,k,a) + (k+1)C(n,k+1,a)\Big].$$

Proof. Let us choose $f_1(t) = t,\ f_2(t) = f(t^{\frac{b}{a}})$, where f is arbitrary function. Then

$$f_1(x^a)f_2(x^a) = x^a f(x^b).$$

Using (8) and Leibnitz rule we obtain

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{-n}\sum_{j=0}^{n}\sum_{k=0}^{j} C(n,k,a)\binom{m}{j}\frac{d^j}{dt^j}t\frac{d^{k-j}}{dt^{k-j}}\big[f(t^{\frac{b}{a}})\big]x^{ak},$$

where $t = x^a$. On the right side of this equation only terms obtained for $j = 0$ and $j = 1$ remain. It follows that

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{-n}\sum_{k=0}^{n} C(n,k,a)\frac{d^k}{dt^k}\big[f(t^{\frac{b}{a}})\big]x^{(k+1)a} + \sum_{k=1}^{n} kC(n,k,a)\frac{d^{k-1}}{dt^{k-1}}\big[f(t^{\frac{b}{a}})\big]x^{ak} =$$

$$= \sum_{k=0}^{n}[C(n,k,a) + (k+1)C(n,k+1,a)]\frac{d^k}{dt^k}\big[f(t^{\frac{b}{a}})\big]x^{(k+1)a}.$$

According to (8) we have

$$\frac{d^n}{dx^n}[x^a f(x^b)] = x^{a-n}\sum_{k=0}^{n}\sum_{m=0}^{k} C\left(k, m, \frac{b}{a}\right)\big[C(n,k,a) + (k+1)C(n,k+1,a)\big]f^{(m)}(u)x^{bm},$$

where $u = t^{\frac{b}{a}} = x^b$.

Interchanging the order of summation gives

$$\big[x^a f(x^b)\big]^{(n)} = x^{a-n}\sum_{m=0}^{n}\left[\sum_{k=m}^{n} C\left(k, m, \frac{b}{a}\right)\big[C(n,k,a) + (k+1)C(n,k+1,a)\big]\right]f^{(m)}(t)x^{bm}.$$

Comparing this equation with (16) implies

$$C(n,m,b,a) = \sum_{k=m}^{n} C\left(k, m, \frac{b}{a}\right)\big[C(n,k,a) + (k+1)C(n,k+1,a)\big],\ (m = 0, 1, \dots, n)],$$

and Proposition 30 is proved. □

We finish with a result connecting Gould-Hopper numbers, Stirling numbers of the first kind, powers, binomial coefficients, and falling factorials.

Proposition 31. *Let $j \le n$ be nonnegative integers and let a, b be nonzero real numbers. Then*

$$\sum_{k=j}^{n} C(n,k,b,a)s(k,j) = b^j \sum_{k=j}^{n} \binom{n}{k} (a)_{n-k} s(k,j).$$

Proof. Take $f(t) = \ln^c t$, where c is a real number such that $(c)_i \neq 0,\ (i = 1, 2, \ldots)$. It follows that $x^a f(x^b) = x^a b^c \ln^c x$. From (1) and (3) we conclude that

$$b^c \frac{d^n}{dx^n}(x^a \ln^c x) = b^c (a)_n x^{a-n} \ln x + b x^{a-n} \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{n}{k} (a)_{n-k} s(k,j)(c)_j \ln^{c-j} x.$$

Using (16) yields

$$b^c \left[x^a \ln^c x\right]^{(n)} = b^c (a)_n x^{a-n} \ln^c x + x^{a-n} b^c \sum_{k=1}^{n} \sum_{j=1}^{k} C(n,k,b,a) s(k,j) b^{-j} (c)_j \ln^{c-j} x.$$

Changing the order of summation in both sums leads to the following equation:

$$\sum_{j=1}^{n} \left[\sum_{k=j}^{n} \binom{n}{k} (a)_{n-k} s(k,j) \right] (c)_j \ln^{c-j} x =$$

$$= \sum_{j=1}^{n} \left[\sum_{k=j}^{n} C(n,k,b,a) s(k,j) b^{-j} \right] (c)_j \ln^{c-j} x.$$

Comparing terms by the same $\ln^{c-j} x$, and then dividing by $(c)_j \neq 0$ proves the result. □

Remark 32. The results of this section are concerned with the following sequences in [6]: A000522, A021009, A035342, A035469, A049029, A049385, A072019, A072020, A084358, A092082, A094587, A105278, A111596, A132013, A132014, A132056, A132159, A132681, A132710, A132792, A136215, A136656.

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