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Hessenberg Matrices and Integer Sequences

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Dedicated to the memory of Professor Veselin Perić

Abstract

We consider a particular case of upper Hessenberg matrices, in which all subdiagonal elements are -1. We investigate three type of matrices related to polynomials, generalized Fibonacci numbers, and special compositions of natural numbers. We give the combinatorial meaning of the coefficients of the characteristic polynomials of these matrices.

1 Introduction

We investigate a particular case of upper Hessenberg matrices, in which all subdiagonal elements are -1. Several mathematical objects may be represented as determinants of such matrices. We consider three type of matrices related to polynomials, generalized Fibonacci numbers, and a special kind of composition of natural numbers. Our objective is to find the combinatorial meaning of the coefficients of the characteristic polynomials.

The coefficients of the characteristic polynomials of matrices of the first type, that is, the sums of principal minors, are related to some binomial identities.

The characteristic polynomials of matrices of the second kind give, as a particular case, the so-called convolved Fibonacci numbers defined by Riordan [4].

Coefficients of the characteristic polynomial of matrices of the third kind are connected with a special kind of composition of natural numbers, in which there are two different types of ones. These were introduced by Deutsch [2], and studied by Grimaldi [3].

We prove several formulas which generate a number of sequences in Slonae's *Encyclopedia* [5].

The following result about upper Hessenberg matrices, which may be easily proved by induction, will be used in this paper.

Theorem 1. *Let the matrix* P_n *be defined by*

$$P_n = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ 0 & 0 & 0 & \cdots & -1 & p_{n,n} \end{bmatrix}, \tag{1}$$

and let the sequence $a_1, a_2, \ldots$ *be defined by*

$$a_{n+1} = \sum_{i=1}^{n} p_{i,n} a_i, \ (n = 1, 2, \ldots). \tag{2}$$

Then

$$a_{n+1} = a_1 \det P_n, \ (n = 1, 2, \ldots).$$

The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ Also, we use the well-known fact that the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix.

2 Polynomials

We start our investigations with Hessenbeg matrices whose determinants are polynomials.

According to Theorem 1 we have

Proposition 2. *Let the matrix* P_{n+1} *be defined by:*

$$P_{n+1} = \begin{bmatrix} 1 & p_1 & p_2 & \cdots & p_{n-1} & p_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & -1 & x \end{bmatrix}.$$

Then

$$\det P_{n+1} = x^n + p_1 x^{n-1} + \cdots + p_n.$$

We let $S_{n+1-k}(x)$ denote the sum of all principal minors of order $n+1-k$ of P_{n+1}, where $k = 0, \ldots, n$, and $S_0 = 1$.

Proposition 3. *The following formulas are valid*

$$S_{n+1-k}(x) = \binom{n}{k-1} x^{n-k+1} + x^{n-k} \sum_{i=0}^{n-k} p_i \binom{n-i}{k} x^{-i},\ (k = 0, 1, 2, \ldots, n). \tag{3}$$

Proof. For $k = 0$ equation (3) means that $S_{n+1} = \det P_{n+1}$, which is clear.

If we delete the first row and the first column of P_{n+1} we obtain a lower triangular matrix of order n with x's on the main diagonal. Hence, all principal minors of order $n-k$ obtained by deleting the first row and the first column of P_{n+1} and another $k-1$ rows and columns with the same indices are equal to x^{n-k+1}. There are $\binom{n}{k-1}$ such minors. This gives the first term in equation (3).

We shall next calculate the minor M_{n+1-k} obtained by deleting the rows and columns with indices $2 \leq m_1 < \cdots < m_k \leq n+1$. By deleting the mth row and the mth column of P_{n+1}, for $m > 1$, we obtain an upper triangular block matrix where the upper block is P_{m-1} and thus its determinant is equal $\sum_{i=0}^{m-2} p_i x^{m-2-i}$. The lower block is a lower triangular matrix of order $n+1-m$ with x's on the main diagonal. It follows that $M_{n+1-k}(x) = \sum_{i=0}^{m-2} p_i x^{m-2-i} x^{n-k-m+2}$. For a fixed $m > 1$ there are $\binom{n+1-m}{k-1}$ such minors.

We thus obtain

$$S_{n+1-k}(x) = \binom{n}{k-1} x^{n-k+1} + \sum_{m=2}^{n-k+2} \sum_{i=0}^{m-2} p_i \binom{n-m+1}{k-1} x^{n-k-i},\ (k = 1, 2, \ldots, n).$$

Changing the order of summation yields

$$S_{n+1-k}(x) = \binom{n}{k-1} x^{n-k+1} + \sum_{i=0}^{n-k} p_i x^{n-k-i} \sum_{m=i+2}^{n-k+2} \binom{n-m+1}{k-1},\ (k = 1, 2, \ldots, n).$$

The corollary now holds by the following recurrence relation for binomial coefficients:

$$\sum_{i=0}^{n-k} \binom{n-i}{k} = \binom{n+1}{k+1}.$$

□

Corollary 4. *If* $x = 1,\ p_i = 1,\ (i = 0, 1, \ldots, n)$ *then*

$$S_{n+1} = n+1,\ S_{n+1-k} = \binom{n}{k-1} + \binom{n+1}{k+1},\ (k = 1, 2, \ldots, n).$$

Several sequences from Sloane's OIES [5] are generated by the preceding formulas. We state some of them: A000124, A000217, A001105, A001845, A004006, A005744, A005893, A006522, A017281, A027927, A056220, A057979, A080855, A080856, A080857, A105163, A121555, A168050.

Corollary 5. *If* $x \neq 1,\ p_i = 1,\ (i = 1, \ldots, n)$ *in (3) then*

$$S_{n+1}(x) = \frac{x^{n+1}-1}{x-1},\ S_{n+1-k}(x) = \binom{n}{k-1} x^{n-k+1} + \sum_{j=0}^{n-k} \binom{k+j}{k} x^j,\ (k = 0, 1, \ldots, n).$$

According to the well-known binomial identity

$$\binom{k+j}{j} = (-1)^j \binom{-k-1}{j},$$

we see that the second term

$$\sum_{j=0}^{n-k} \binom{k+j}{k} x^j,$$

in the preceding equation is the partial sum of the expansion of the function $\frac{1}{(1-x)^{k+1}}$ into powers of x.

The characteristic matrix $Q_{n+1}(t)$ of P_{n+1} has the form

$$Q_{n+1}(t) = -\begin{bmatrix} 1-t & p_1 & p_2 & \cdots & p_{n-1} & p_n \\ -1 & x-t & 0 & \cdots & 0 & 0 \\ 0 & -1 & x-t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x-t & 0 \\ 0 & 0 & 0 & \cdots & -1 & x-t \end{bmatrix}.$$

This is also a Hessenberg matrix. If $f_{n+1}(t)$ is the characteristic polynomial of P_{n+1} then

$$f_{n+1}(t) = \det(Q_{n+1}(t)) = (-1)^{n+1} \left[\sum_{k=1}^{n} p_k (x-t)^{n-k} + (1-t)(x-t)^n \right].$$

We thus obtain

Proposition 6. *The following equation holds*

$$\sum_{k=0}^{n+1} (-1)^{n-k+1} S_{n-k+1}(x) t^k = (-1)^{n+1} \left[\sum_{k=1}^{n} p_k (x-t)^{n-k} + (1-t)(x-t)^n \right]. \tag{4}$$

As a consequence we shall prove a curious binomial identity.

Proposition 7. *Let m, n $(0 \le m \le n)$ be arbitrary integers. Then*

$$\sum_{j=m}^{n} \sum_{k=0}^{n-j} (-1)^k \binom{k+j}{j} \binom{j}{m} = 1.$$

Proof. Take $p_i = 1,\ (i = 0, \ldots, n)$, $t = 1$, and $x+1$ instead of x in (4). The left side becomes

$$L = 1 + \sum_{k=0}^{n} (-1)^{n-k+1} \left[\binom{n}{k-1} (x+1)^{n-k+1} + \sum_{j=0}^{n-k} (-1)^{n-k+1} \binom{k+j}{j} (x+1)^j \right].$$

It is easily seen that

$$\sum_{k=0}^{n} (-1)^{n-k+1} \binom{n}{k-1} (x+1)^{n-k+1} = (-x)^n - 1.$$

Therefore,

$$L = (-x)^n + \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{m=0}^{j} (-1)^{n-k+1} \binom{k+j}{j} \binom{j}{m} x^m.$$

Changing the order of summation we have

$$L = (-x)^n + \sum_{m=0}^{n} \left[\sum_{j=m}^{n} \sum_{k=0}^{n-j} (-1)^{n-k+1} \binom{k+j}{j} \binom{j}{m} \right] x^m.$$

The right side of (4) has the form $R = (-1)^{n+1}(x^{n-1} + x^{n-2} + \cdots + 1)$.
In this way we obtain

$$\sum_{m=0}^{n} \left[\sum_{j=m}^{n} \sum_{k=0}^{n-j} (-1)^{n-k+1} \binom{k+j}{j} \binom{j}{m} \right] x^m = (-1)^{n+1}(x^n + x^{n-1} + \cdots + 1).$$

The proposition follows by comparing coefficients of the same powers of x in this equation.
□

3 Generalized Fibonacci Numbers

Consider the sequence recursively given by $l_1 = 1, l_2 = x, l_{n+1} = yl_{n-1} + xl_n,\ (n > 2)$, and an upper Hessenberg matrix L_n of order n defined by:

$$L_n = \begin{bmatrix} x & y & 0 & \cdots & 0 & 0 \\ -1 & x & y & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ 0 & 0 & 0 & \cdots & -1 & x \end{bmatrix}.$$

From Theorem 1 we obtain

$$\det L_n = l_{n+1},\ (n = 1, 2, \ldots).$$

For $l_1 = l_2 = x = y = 1$ we have $l_{n+1} = f_{n+1}$, where f_{n+1} is the $n+1$th Fibonacci number. The numbers l_n are usually called generalized Fibonacci numbers.

Proposition 8. *Let $S_{n-k},\ (k = 0, 1, \ldots, n-1)$ be the sum of all principal minors of L_n of order $n - k$. Then*

$$S_{n-k} = \sum_{j_1+j_2+\cdots+j_{k+1}=n+1} l_{j_1} l_{j_2} \cdots l_{j_{k+1}}, \tag{5}$$

where the sum is taken over $j_t \geq 1,\ (t = 1, 2, \ldots, k+1)$.

Proof. Principal minors of L_n are some convolutions of l_n. For example, the minor obtained by deleting the ith row and the ith column of L_n is obviously equal to $l_i \cdot l_{n-i+1}$. Therefore, the principal minor $M(i_1, i_2, \ldots, i_k)$ of L_n obtained by deleting the rows and columns with indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ is

$$M(i_1, i_2, \ldots, i_k) = l_{i_1} \cdot l_{i_2 - i_1} \cdots l_{i_k - i_{k-1}} \cdot l_{n-i_k+1}.$$

Denoting $i_1 = j_1, i_2 - i_1 = j_2, \ldots, i_k - i_{k-1} = j_k, n - i_k + 1 = j_{k+1}$ yields

$$M(i_1, i_2, \ldots, i_k) = l_{j_1} \cdot l_{j_2} \cdots l_{j_k} \cdot l_{j_{k+1}},$$

where $j_t \ge 1,\ (t = 1, 2, \ldots, k+1)$, and $j_1 + \cdots + j_{k+1} = n + 1$.

Summing over all $1 \le i_1 < i_2 < \cdots < i_k \le n$ we obtain (5).

□

From Proposition 8 we easily derive the following known result for compositions of a natural number.

Proposition 9. *The number $c(n+1, k+1)$ of compositions of $n+1$ into $k+1$ parts is*

$$c(n+1, k+1) = \binom{n}{k}.$$

Proof. Take specifically $l_1 = x = 1,\ y = 0$. Then L_n is a lower triangular matrix with 1's on the main diagonal. Therefore, all of its principal minors equal 1, and there are $\binom{n}{k}$ minors of order $n - k$. It follows that the sum (5) equals $\binom{n}{k}$, and that all its summands are equal 1. On the other hand, this sum is taken over all compositions of $n+1$ into $k+1$ parts and the proposition follows. □

In Riordan's book [4] the sum on the right side of (5) for $x = y = 1$ is called a convolved Fibonacci number and is denoted by $f_{n-k+1}^{(k+1)}$. Hence,

Corollary 10. *The convolved Fibonacci number $f_{n-k+1}^{(k+1)}$ is equal, up to sign, to the coefficient of t^k in the characteristic polynomial $p_n(t)$ of L_n, in the case $x = y = 1$.*

The following result is an explicit formula for convolved Fibonacci numbers.

Corollary 11. *Let $f_{n-k+1}^{(k+1)}$ denote the convolved Fibonacci number. Then*

$$f_{n-k+1}^{(k+1)} = \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n-i}{i} \binom{n-2i}{k}.$$

Proof. The characteristic matrix of L_n has the form:

$$\begin{bmatrix} t-1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & t-1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & t-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t-1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & t-1 \end{bmatrix}.$$

It follows that $p_n(t) = f_{n+1}(t-1)$, where $f_{n+1}(t)$ is a Fibonacci polynomial. The corollary follows from the preceding corollary and the following explicit formula for Fibonacci polynomials:

$$f_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}.$$

□

Since $f_{n+1}(1) = f_{n+1}$ we also have the following identity for Fibonacci numbers:

Corollary 12. *For Fibonacci numbers f_n, $(n = 0, 1, \ldots)$ we have*

$$f_{n+1} = (-1)^n \sum_{k=0}^{n} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} (-2)^k \binom{n-i}{i} \binom{n-2i}{k}.$$

Remark 13. Taking particular values for x and y in the matrix L_n we may obtain different kinds of generalized Fibonacci numbers. For example, taking $x = y = 1$, and taking 3 instead of 1 in the upper left corner of L_n we obtain a matrix whose determinant is a Lucas number. In this case Proposition 8 shows that the coefficients of the characteristic polynomial are convolutions between Fibonacci and Lucas numbers.

Many of the sequences in Sloane's *Encyclopedia* are generated by the sums from Proposition 8. Here are some of them: A000045, A000129, A001076, A001628, A001629, A001872, A006190, A006503, A006504, A006645, A023607, A052918, A054457, A073380, A073381. A077985, A152881.

4 Particular Compositions of Natural Numbers

We shall now consider a type of Hessenberg matrices whose determinants are Fibonacci numbers with odd indices. This will lead us to the compositions of a natural number with two different types of ones, introduced by E. Deutsch in [2].

Proposition 14. *Let G_n be the matrix of order n defined by:*

$$G_n = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 2 & 1 & \cdots & 1 & 1 \\ 0 & -1 & 2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

Then

$$\det G_n = f_{2n+1}.$$

Proof. According to Theorem 1 we have

$$a_1 = 1, a_2 = 2, \ldots, a_{n+1} = a_1 + a_2 + \cdots + a_{n-1} + 2a_n, \ (n > 2).$$

It follows from Identity 2 in [1] that $a_{n+1} = f_{2n+1}$. □

Proposition 15. *Let $S_{n-k}, \ (0 \le k \le n)$ be the sum of all minors of order $n-k$ of G_n. Then*

$$S_{n-k} = \sum_{j_1+j_2+\cdots+j_{k+1}=n-k} f_{2j_1+1} f_{2j_2+1} \cdots f_{2j_{k+1}+1},$$

where the sum is taken over $j_t \ge 0, \ (t = 1, 2, \ldots, k+1)$.

Proof. Let $M(i_1, i_2, \ldots, i_k)$ be the minor of order $n-k$ obtained by deleting the rows and columns with indices $1 \le i_1 < i_2 < \cdots < i_k \le n$. Then

$$M = \det(G_{i_1-1}) \cdot \det(G_{i_2-i_1-1}) \cdots \det(G_{i_k-i_{k-1}-1}) \cdot \det(G_{n-i_k}).$$

Applying Proposition 14 we obtain

$$M = f_{2(i_1-1)+1} f_{2(i_2-i_1-1)+1} \cdots f_{2(i_k-i_{k-1}-1)+1} f_{2(n-i_k)+1}.$$

It follows that

$$S_{n-k} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{2(i_1-1)+1} f_{2(i_2-i_1-1)+1} \cdots f_{2(i_k-i_{k-1}-1)+1} f_{2(n-i_k)+1}.$$

If we set

$$i_1 - 1 = j_1, i_2 - i_1 - 1 = j_2, \ldots, i_k - i_{k-1} - 1 = j_k, n - i_k = j_{k+1},$$

we have

$$S_{n-k} = \sum_{j_1+j_2+\cdots+j_{k+1}=n-k} f_{2j_1+1} f_{2j_2+1} \cdots f_{2j_{k+1}+1},$$

where the sum is taken over all $j_t \ge 0, \ (t = 1, 2, \ldots, k+1)$. □

For compositions of n in which there are two types of ones Grimaldi, in his paper [3], shows that its number is f_{2n+1}.

We let $c_{n,k}$ denote the number of such compositions in which exactly k parts equal 0. We finish the paper by showing that the number S_{n-k}, from the preceding proposition, is in fact equal to $c_{n-k,k}$.

Proposition 16. *Let n be a positive integer, and let k be a nonnegative integer. Then*

$$c_{n,k} = \sum_{j_1+j_2+\cdots+j_{k+1}=n} f_{2j_1+1} f_{2j_2+1} \cdots f_{2j_{k+1}+1}, \tag{6}$$

where the sum is taken over $j_t \ge 0, \ (t = 1, 2, \ldots, k+1)$.

Proof. We use induction with respect of k. The theorem is true for $k = 0$, according to the preceding proposition.

Assume the theorem is true for $k-1$. The greatest value of j_{k+1} is n, and is obtained for $j_1 = j_2 = \cdots = j_k = 0$. We thus may write (6) in the form:

$$c_{n,k} = \sum_{j=0}^{n} f_{2j+1} \sum_{j_1+j_2+\cdots+j_k=n-j} f_{2j_1+1} f_{2j_2+1} \cdots f_{2j_k+1}.$$

By the induction hypothesis we have

$$c_{n,k} = \sum_{j=0}^{n} f_{2j+1} \cdot c_{n-j,k-1}. \tag{7}$$

Let $(i_1, i_2, \ldots)$ be a composition of n with two different types of ones and exactly k zeroes, and let first index, where 0 is a summand, be i_p. Then $(i_1, i_2, \ldots, i_{p-1})$ is a composition of $j = i_1 + \cdots + i_{p-1}$ with no zeroes, and $(i_{p+1}, \ldots)$ is a composition of $n-j$ with exactly $k-1$ zeroes. The number of such compositions is $f_{2j+1} \cdot c(n-j, k-1)$, which is a summand on the right side of (7). In this way the sum on the right side of (7) counts all of the required compositions.

□

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References

[1] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count,* MAA, 2003.

[2] E. Deutsch, Advanced exercise H-641, *Fibonacci Quart.* **44** (2006), 188.

[3] R. P. Grimaldi, Compositions and the alternate Fibonacci numbers, *Congr. Numer.* **186** (2007), 81–96.

[4] J. Riordan, *Combinatorial Identities,* Wiley, 1968.

[5] N. J. A. Sloane, *Online Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

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