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A Generalization of the Question of Sierpiński on Geometric Progressions

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Abstract

In this paper we prove that there is no geometric progression that contains four distinct integers of the form $Dm^2 + C, D, m \in \mathbb{N}, C = \pm 1, \pm 2, \pm 4$.

1 Introduction

The integers of the form $T_n = n(n+1)/2, n \in \mathbb{N}$, are called triangular numbers. Sierpiński [7, D23] asked whether or not there exist four (distinct) triangular numbers in geometric progression. Szymiczek [10] conjectured that the answer is negative. The problem of finding three such triangular numbers is readily reduced to finding solutions to a Pell equation (by an old result of Gérardin [6]; see also [9, 4]). This implies that there are infinitely many

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such triples, the smallest of which is (T_1, T_3, T_8). In fact, an easy calculation shows that if $T_n = m^2$ then

$$(T_n,\, T_{n+2m} = m(2n+3m+1),\, T_{3n+4m+1} = (2n+3m+1)^2)$$

forms a geometric progression.

Recently M. Bennett [1] proved that there do not exist four distinct triangular numbers in geometric progression with the ratio being a positive integer. Chen and Fang [3] extended Bennett's result to the rational ratio and proved that there do not exist four distinct triangular numbers in geometric progression. Using the theory of Pell equations and a result of Bilu-Hanrot-Voutier [2] on primitive divisors of Lucas and Lehmer numbers, Yang-He [15] and Yang [14] claimed that there is no geometric progression that contains four distinct triangular numbers. But their proof is under the assumption that the geometric progression has an integral common ratio. Fang [5], using only the Störmer theorem on Pell's equation, showed that there is no geometric progression which contains four distinct triangular numbers.

Note that if $T_n = n(n+1)/2, n \in \mathbb{N}$ is a triangular number, then $8T_n = m^2 - 1$, where $m = 2n+1$. Thus the Sierpiński problem is equivalent to whether or not there exist four distinct integers of the form $m^2 - 1$ in geometric progression. In this paper, we consider the more general question whether or not there exists a geometric progression which contains four distinct integers of the form $Dm^2 + C$ with $D, m \in \mathbb{N}, C = \pm 1, \pm 2, \pm 4$. We use Störmer theory on Pell equations to prove the following results:

Theorem 1. *Let D be a positive integer. Then there is no geometric progression which contains four distinct integers of the form $Dm^2 + C$, $m \in \mathbb{N}$, $C \in \{-4, 4\}$.*

By Theorem 1, we have the following two Corollaries immediately.

Corollary 2. *Let D be a positive integer. Then there is no geometric progression which contains four distinct integers of the form $Dm^2 + C$, $m \in \mathbb{N}$, $C \in \{-1, 1\}$.*

Corollary 3. *Let D be a positive integer. Then there is no geometric progression which contains four distinct integers of the form $Dm^2 + C$, $m \in \mathbb{N}$, $C \in \{-2, 2\}$.*

2 Some Lemmas

To prove the above theorem, we need the following lemmas. Throughout this paper, we assume that k, l are coprime positive integers and kl nonsquare; and let $2 \nmid kl$ when $C = 2$ or 4. We need some results on the solutions of the diophantine equations

$$kx^2 - ly^2 = C,\, C = 1,\, 2,\, 4. \tag{1}$$

We recall that the minimal positive solution of Diophantine equation (1) is the positive integer solution (x, y) of equation (1) such that $x\sqrt{k} + y\sqrt{l}$ is the smallest. One can easily see that this is equivalent to determining a positive integer solution (x, y) of equation (1) such that x and y are the smallest. By abuse of language, we shall also refer to $x\sqrt{k} + y\sqrt{l}$ instead of the pair (x, y) as a solution to (1) and call $x\sqrt{k}+y\sqrt{l}$ the minimal positive solution.

If $x_1\sqrt{k} + y_1\sqrt{l}$ is the minimal positive solution of (1), then we have the following result.

Lemma 4. ([11]) *All positive integer solutions of (1) are given by*

$$\frac{x\sqrt{k}+y\sqrt{l}}{\sqrt{C}}=\left(\frac{x_1\sqrt{k}+y_1\sqrt{l}}{\sqrt{C}}\right)^n,\ n\in\mathbb{N}.$$

Moreover, we have $2\nmid n$ when $k>1$ or $C=2$.

Störmer (see [4, p. 391]) proved a result on divisibility properties of solutions of Pell equations. More new results extending Störmer theory had been obtained over the years. We will list some known results that will be used in the proofs in this paper.

Lemma 5. (Störmer's theorem [4, p. 391]) *Let D be a positive nonsquare integer. Let (x_1, y_1) be a positive integer solution of Pell equation*

$$x^2-Dy^2=C,\ C\in\{-1,1\}.$$

If every prime divisor of y_1 divides D, then $x_1+y_1\sqrt{D}$ is the minimal positive solution.

Considering the Diophantine equation

$$kx^2-ly^2=1,\quad k>1, \tag{2}$$

D. T. Walker [12] obtained a result similar to Störmer's theorem. See also Q. Sun and P. Yuan [11].

Lemma 6. ([12, 11]) *Let (x,y) be a positive integer solution of (2) .*
(i) If every prime divisor of x divides k, then either

$$x\sqrt{k}+y\sqrt{l}=\varepsilon$$

or

$$x\sqrt{k}+y\sqrt{l}=\varepsilon^3,\quad \textit{and}\quad x=3^sx_1,\ 3\nmid x_1,\ 3^s+3=4kx_1^2,$$

where in both cases $\varepsilon=x_1\sqrt{k}+y_1\sqrt{l}$ is the minimal positive solution of (2) , $s\in\mathbb{N}$.
(ii) If every prime divisor of y divides l, then either

$$x\sqrt{k}+y\sqrt{l}=\varepsilon$$

or

$$x\sqrt{k}+y\sqrt{l}=\varepsilon^3,\quad \textit{and}\quad y=3^sy_1,\ 3\nmid y_1,\ 3^s-3=4ly_1^2,\quad s\geq 2.$$

Using the method in [11], the first author proved the following results.

Lemma 7. ([8]) *Let k,l be coprime positive odd integers and kl nonsquare. Suppose that (x,y) is a positive integer solution of the Diophantine equation*

$$kx^2-ly^2=2. \tag{3}$$

(i) If every prime divisor of x divides k , then either

$$x\sqrt{k}+y\sqrt{l}=\varepsilon$$

or

$$\frac{x\sqrt{k}+y\sqrt{l}}{\sqrt{2}}=\left(\frac{\varepsilon}{\sqrt{2}}\right)^3, \quad \textit{and} \quad x=3^s x_1, 3^s+3=2kx_1^2,$$

where in both cases $\varepsilon=x_1\sqrt{k}+y_1\sqrt{l}$ is the minimal positive solution of (3) , $s\in\mathbb{N}$.
(ii) If every prime divisor of y divides l , then either

$$x\sqrt{k}+y\sqrt{l}=\varepsilon$$

or

$$\frac{x\sqrt{k}+y\sqrt{l}}{\sqrt{2}}=\left(\frac{\varepsilon}{\sqrt{2}}\right)^3, \quad \textit{and} \quad y=3^s y_1, 3^s-3=2ly_1^2, \quad s\geq 2.$$

Lemma 8. ([8]) *Let k,l be coprime positive odd integers and kl nonsquare. Suppose that (x,y) is a positive integer solution of the Diophantine equation*

$$kx^2-ly^2=4. \tag{4}$$

(i) If every prime divisor of x divides k , then $x\sqrt{k}+y\sqrt{l}=\varepsilon$ is the minimal positive solution of equation (4) except for the case $(k,l,x,y)=(5,1,5,11)$.
(ii) If every prime divisor of y divides l , then $x\sqrt{k}+y\sqrt{l}=\varepsilon$ is the minimal positive solution of equation (4) .

Lemma 9. *Let $k, l=a_0a^m$ be coprime positive integers and kl nonsquare with $m>1$ an integer. If (x,a^r) is a positive integer solution of the Diophantine equation*

$$kx^2-ly^2=C, \qquad C\in\{-1,1,-2,2,-4,4\}, \tag{5}$$

where r is a non-negative integer. Then $x\sqrt{k}+a^r\sqrt{l}=\varepsilon$ is the minimal positive solution of equation (5).

Proof. We only consider the case of $C=1$ (the proofs of the other cases are similar). If $x\sqrt{k}+a^r\sqrt{l}$ is not the minimal positive solution of equation (5) , then by Lemma 6(ii) and since $u\sqrt{k}+v\sqrt{l}$ is the minimal positive solution, we have $a^r=3^s v$ and $3\nmid v$. Therefore, $3|a$. Since $m\geq 2, 3^s-3=4a_0a^m v^2$ is also divisible by 9. Hence $9|3$, which is a contradiction. This completes the proof of Lemma 9.

□

Remark 10. Lemma 9 is also true for $l=a_0a^m/2^t, C=\pm 1$ with $t\leq m$ is a nonnegative integer and $2|l$.

Lemma 11. ([13]) *Let $x_1\sqrt{k}+y_1\sqrt{l}$ be the minimal positive solution of (1) such that $2\nmid x_1y_1$ when $C=2$ or 4. If $x\sqrt{k}+y\sqrt{l}$ is a positive integer solution of (1), then $y_1|y$. And if $k>1$ or $C=2$, then $x_1|x$.*

Lemma 12. *Let $k, l, a, b, r_2, r_3, r_4$ be positive integers such that* $\gcd(k, lab) = 1, \gcd(a, b) = 1, a > b, r_2 < r_3 < r_4$*. If $2|ab$ but $2 \nmid l$, then following system of Diophantine equations*

$$kx_1^2 - la^{r_4} = C, \tag{6}$$

$$kx_2^2 - la^{r_4-r_2}b^{r_2} = C, \tag{7}$$

$$kx_3^2 - la^{r_4-r_3}b^{r_3} = C, \tag{8}$$

$$kx_4^2 - lb^{r_4} = C, \tag{9}$$

where $C \in \{-4, 4\}$, has no positive integer solutions (x_1, x_2, x_3, x_4) with $x_1 > x_2 > x_3 > x_4$.

Proof. We now suppose that $2|a$ but $2 \nmid b$.

Case 1: $2|r_4$.

If $2|r_2$, then we get $k \equiv l \equiv C/4 \pmod 4$ by considering the equations $k\left(\frac{x_1}{2}\right)^2 - l\left(\frac{a^{r_4/2}}{2}\right)^2 = C/4$ and (9) mod 4. And so by taking mod 4 for $k\left(\frac{x_2}{2}\right)^2 - l\left(\frac{a^{(r_4-r_2)/2}}{2}\right)^2 = C/4$, we have $\left(\frac{x_2}{2}\right)^2 - \left(\frac{a^{(r_4-r_2)/2}}{2}\right)^2 \equiv 1 \pmod 4$. This follows that $2|\frac{a^{r_4-r_2}}{4}$. Thus by (6) and Remark 10 of Lemma 9, we know that $(\frac{x_1}{2}, a^{\frac{r_2}{2}})$ is the minimal positive solution of the Diophantine equation

$$kx^2 - \frac{la^{r_4-r_2}}{4}y^2 = C/4. \tag{10}$$

By (7), $(\frac{x_2}{2}, b^{\frac{r_2}{2}})$ is a positive integer solution of (10). By Lemma 11, we obtain $a^{\frac{r_2}{2}}|b^{\frac{r_2}{2}}$, which contradicts the assumption that $a > b$.

Similarly we have that $2 \nmid r_3$ from (8).

We now suppose that $2 \nmid r_2$ and $2 \nmid r_3$. Then, by (7), (8), both $(\frac{x_2}{2}, a^{\frac{r_3-r_2}{2}}b^{\frac{r_2-1}{2}})$ and $(\frac{x_3}{2}, b^{\frac{r_3-1}{2}})$ are positive integer solutions of the Diophantine equation

$$kx^2 - \frac{la^{r_4-r_3}b}{4}y^2 = C/4.$$

Noting that $x_2 > x_3$, by Lemmas 6 and 7, $\frac{x_3}{2}\sqrt{k} + b^{\frac{r_3-1}{2}}\sqrt{\frac{la^{r_4-r_3}b}{4}} = \varepsilon$ must be the minimal positive solution. Therefore again by Lemmas 6 and 7, $\frac{x_3}{2}\sqrt{k} + a^{\frac{r_3-r_2}{2}}b^{\frac{r_2-1}{2}}\sqrt{\frac{la^{r_4-r_3}b}{4}} = \varepsilon^3$ and $a^{\frac{r_3-r_2}{2}}b^{\frac{r_2-1}{2}} = 3^s b^{\frac{r_3-1}{2}}$, $3^s \mp 3 = la^{r_4-r_3}b^{r_3}$. It follows that $a^{(r_3-r_2)/2} = 3^s b^{(r_3-r_2)/2}$, and thus $b = 1,\ a = 3$, which contradicts the assumption that $2|a$. This concludes the analysis of Case 1.

Case 2: $2 \nmid r_4$. We can prove that $2 \nmid r_2 r_3$ is impossible by using the same method of proving Case 1.

If $2|r_2$, then, since $r_4 - r_2 > 2$, we have $2|\frac{a^{r_4-r_2}}{4}$. Therefor by (6) and Remark 10 of Lemma 9, we know that $(\frac{x_1}{2}, a^{\frac{r_2}{2}})$ is the minimal positive solution of the Diophantine equation

$$kx^2 - \frac{la^{r_4-r_2}}{4}y^2 = C/4. \tag{11}$$

By (7), $(\frac{x_2}{2}, b^{\frac{r_2}{2}})$ is a positive integer solution of (11). So by Lemma 11, we obtain $a^{\frac{r_2}{2}}|b^{\frac{r_2}{2}}$, which contradicts the assumption that $a > b$.

If $2 \nmid r_2$, then $2|r_3$, noting that $b^{\frac{r_4-1}{2}} \neq 5$, by (9) and Lemma 2.5, $(x_4, b^{\frac{r_4-1}{2}})$ is the minimal positive solution of the Diophantine equation $kx^2 - lby^2 = C$. By (7) , $(x_2, a^{\frac{r_4-r_2}{2}} b^{\frac{r_2-1}{2}})$ is a positive integer solution of $kx^2 - lby^2 = C$. Thus by Lemma 11, we obtain $b^{\frac{r_4-1}{2}}|a^{\frac{r_4-r_2}{2}} b^{\frac{r_2-1}{2}}$. This follows that $b^{\frac{r_4-r_2}{2}}|a^{\frac{r_4-r_2}{2}}$, and so, since $\gcd(a,b) = 1$, $b = 1$. We have $2|\frac{a^{r_4-r_3}}{4}$ as shown at the beginning of Case 1. Proceeding as before, we can prove $2 \nmid r_3$, which is a contradiction. This concludes the analysis of Case 2. The proof of $2 \nmid a$ and $2|b$ is similar. This completes the proof of Lemma 12. □

3 Proof of Theorem 1

Proof. Suppose that there is a geometric progression $\{a_n\}$ which contains four distinct integers $Dm_1^2 + C = a_1 q^{t_1}$, $Dm_2^2 + C = a_1 q^{t_2}$, $Dm_3^2 + C = a_1 q^{t_3}$, $Dm_4^2 + C = a_1 q^{t_4}$ with $0 \le t_1 < t_2 < t_3 < t_4$, where $q = b/a$ is the common ratio such that $a \ge 1$ and $\gcd(a,b) = 1$. It is easy to see that both a_1 and q are not zero and that $|q| \neq 1$. Without loss of generality, we may assume that $0 < |q| < 1$, so $a > |b| > 0$. Let $Dm_1^2 + C = A$, $t_2 - t_1 = r_2$, $t_3 - t_1 = r_3$, $t_4 - t_1 = r_4$, then $A \neq 0$ and $0 < r_2 < r_3 < r_4$ satisfying

$$Dm_1^2 + C = A,\ Dm_2^2 + C = Aq^{r_2},\ Dm_3^2 + C = Aq^{r_3},\ Dm_4^2 + C = Aq^{r_4}. \tag{12}$$

Since Aq^{r_4} is an integer, then $a^{r_4}|Ab^{r_4}$, and so $a^{r_4}|A$ since $\gcd(a,b) = 1$. Let $A = a_0 a^{r_4}$. We can derive that all the numbers $Dm_i + C, i \in \{1,2,3,4\}$ are positive integers. If not, then since $a > 1,\ r_4 \ge 3,\ r_4 - r_2 \ge 2,\ r_4 - r_3 \ge 1$, we must have either

$$Dm_3^2 + C = a_0 a^{r_4-r_3} b^{r_3} = -2, -3$$

or

$$Dm_4^2 + C = a_0 b^{r_4} = -1, -2 \quad or \quad -3.$$

This follows either

$$(D, m_3, C, a_0, a, b) = (2, 1, -4, 1, 2, -1), (1, 1, -4, 1, 3, -1)$$

such that r_4 is even since $Dm_4^2 + C = a_0 b^{r_4}$ is a positive integer, or

$$(D, m_4, C, a_0, b) = (3, 1, -4, 1, -1), (2, 1, -4, 2, -1), \quad or \quad (1, 1, -4, 3, -1)$$

such that r_4 is odd integer and such that both r_2 and r_3 are even integers since both $Dm_2^2 + C = a_0 a^{r_4 - r_2} b^{r_2}$ and $Dm_3^2 + C = a_0 a^{r_4 - r_3} b^{r_3}$ are positive integers.

If $(D, m_3, C, a_0, a, b) = (2, 1, -4, 1, 2, -1)$, we will get $m_1^2 \equiv 2 \pmod 4$ by considering equation $m_1^2 - 2 = 2^{r_4 - 1}$ mod 4, which is impossible.

If $(D, m_3, C, a_0, a, b) = (1, 1, -4, 1, 3, -1)$, we will get $m_1^2 - (3^{r_4/2})^2 = 4$, which is impossible.

If $(D, m_4, C, a_0, b) = (3, 1, -4, 1, -1)$, we have that both $(m_2, a^{(r_4 - r_2 - 1)/2})$ and $(m_3, a^{(r_4 - r_3 - 1)/2})$ are positive integer solutions of Diophantine equation

$$3x^2 - ay^2 = 4.$$

Thus by Lemma 8, $(m_2, a^{\frac{r_4 - r_2 - 1}{2}}) = (m_3, a^{\frac{r_4 - r_3 - 1}{2}})$ is the minimal positive solution of $3x^2 - ay^2 = 4$, which contradicts the assumption that $m_2 \neq m_3$.

If $(D, m_4, C, a_0, b) = (2, 1, -4, 1, -1)$, we have that both $(m_1, a^{(r_4 - 3)/2})$ and $(m_2, a^{(r_4 - r_2 - 3)/2})$ are positive integer solutions of Diophantine equation

$$x^2 - a^3 y^2 = 2.$$

Thus by Lemma 9, $(m_1, a^{\frac{r_4 - 3}{2}}) = (m_2, a^{\frac{r_4 - r_2 - 3}{2}})$ is the minimal positive solution of $x^2 - a^3 y^2 = 2$, which contradicts the assumption that $m_1 \neq m_2$.

If $(D, m_4, C, a_0, b) = (1, 1, -4, 3, -1)$ and $2 \nmid a$, then, both $(m_1, a^{(r_4 - 3)/2})$ and $(m_2, a^{(r_4 - r_2 - 3)/2})$ are positive integer solutions of Diophantine equation

$$x^2 - 3a^3 y^2 = 4.$$

Thus by Lemma 9, $(m_1, a^{\frac{r_4 - 3}{2}}) = (m_2, a^{\frac{r_4 - r_2 - 3}{2}})$ is the minimal positive solution, which contradicts the assumption that $m_1 \neq m_2$.

If $(D, m_4, C, a_0, b) = (1, 1, -4, 3, -1)$ and $2 | a$, then, both $(m_1/2, a^{(r_4 - 3)/2})$ and $(m_2/2, a^{(r_4 - r_2 - 3)/2})$ are positive integer solutions of Diophantine equation

$$x^2 - \frac{3a^3}{4} y^2 = 1.$$

Thus by Remark 4 of Lemma 9, $(m_1/2, a^{\frac{r_4 - 3}{2}}) = (m_2/2, a^{\frac{r_4 - r_2 - 3}{2}})$ is the minimal positive solution, which contradicts the assumption that $m_1 \neq m_2$.

Therefore we can assume that $0 < q < 1$, which follows that $0 < b < a$ and $m_1 > m_2 > m_3 > m_4 > 0$. It follows from (12) that

$$Dm_1^2 - a_0 a^{r_4} = -C, \tag{13}$$

$$Dm_2^2 - a_0 a^{r_4 - r_2} b^{r_2} = -C, \tag{14}$$

$$Dm_3^2 - a_0 a^{r_4 - r_3} b^{r_3} = -C, \tag{15}$$

$$Dm_4^2 - a_0 b^{r_4} = -C. \tag{16}$$

It is easy to see that $\gcd(a_0 a, a_0 b) = a_0$ since $\gcd(a, b) = 1$. We will consider three cases according to the divisibility of a_0 by 2.

If $2 \nmid a_0$, we must have either $2 \nmid a_0 a$ or $2 \nmid a_0 b$. Suppose now that $2 \nmid a_0 a$, then D is odd, by (13). Thus by Lemma 12, we have that $(D, a_0 ab) = 1$ and $2 \nmid Da_0 ab$. The case of $2 \nmid a_0 b$ is similar, using (16).

If $2 || a_0$, we can derive that $2 \nmid a$ and $2 \nmid b$. Assume to the contrary, we let $2|a$ (the case that $2|b$ is similar), then $2 \nmid b$. We get $Dm_4^2 \equiv 2 \pmod 4$ by considering equation (16) mod 4, which implies that $2||D$ since $2|m_4$ would imply $Dm_4^2 \not\equiv 2 \pmod 4$. Therefore we obtain from (13) that either $2m_1^2 \equiv \pm 4 \pmod 8$ or $6m_1^2 \equiv \pm 4 \pmod 8$ which is impossible. Hence $2 \nmid a$ and $2 \nmid b$, and so $2||D$. Let $a_0 = 2l_1, D = 2D_1$, where l_1 and D_1 are odd positive integers. Thus we have from (13), (14), (15), (16) that

$$D_1 m_1^2 - l_1 a^{r_4} = -C/2, \tag{17}$$

$$D_1 m_2^2 - l_1 a^{r_4 - r_2} b^{r_2} = -C/2, \tag{18}$$

$$D_1 m_3^2 - l_1 a^{r_4 - r_3} b^{r_3} = -C/2, \tag{19}$$

$$D_1 m_4^2 - l_1 b^{r_4} = -C/2, \tag{20}$$

where l_1, a, b and D_1 are odd positive integers.

If $4|a_0$, we must have either $4|D$ or $2|\gcd(m_1, m_2, m_3, m_4)$. Let $D = 4D_2$, $a_0 = 4l_2$ and $(n_1, n_2, n_3, n_4) = (m_1, m_2, m_3, m_4)$ when $4|D$, and let $D_2 = D$, $a_0 = 4l_2$ and $(n_1, n_2, n_3, n_4) = (m_1/2, m_2/2, m_3/2, m_4/2)$ when $2|\gcd(m_1, m_2, m_3, m_4)$. Thus we have from (13), (14), (15), (16) that

$$D_2 n_1^2 - l_2 a^{r_4} = -C/4, \tag{21}$$

$$D_2 n_2^2 - l_2 a^{r_4 - r_2} b^{r_2} = -C/4, \tag{22}$$

$$D_2 n_3^2 - l_2 a^{r_4 - r_3} b^{r_3} = -C/4, \tag{23}$$

$$D_2 n_4^2 - l_2 b^{r_4} = -C/4. \tag{24}$$

From consideration of these three cases, let $(k, l, u_i, C_1) = (D, a_0, m_i, -C)$ or $(D_1, l_1, m_i, -C/2)$ or $(D_2, l_2, n_i, -C/4)$, then one can easily see that the problem is equivalent to proving that the following questions

$$k u_1^2 - l a^{r_4} = C_1, \tag{25}$$

$$k u_2^2 - l a^{r_4 - r_2} b^{r_2} = C_1, \tag{26}$$

$$k u_3^2 - l a^{r_4 - r_3} b^{r_3} = C_1, \tag{27}$$

$$k u_4^2 - l b^{r_4} = C_1, \tag{28}$$

where $C_1 \in \{-1, 1, -2, 2, -4, 4\}$, $u_1 > u_2 > u_3 > u_4 > 0$, $\gcd(k, lab) = 1$ and $2 \nmid klab$ if $2|C_1$, cannot be simultaneously satisfied.

Case 1: $2|r_4$. It is easy to see that kl is not a square. Otherwise both k and l are squares. And so $(\sqrt{k}u_1, \sqrt{l}a^{r_4/2})$ is a positive integer solution of equation $X^2 - Y^2 = C_1$ by (25), which is impossible.

If $2|r_2$, then, by (25) and Lemma 9, we know that $(u_1, a^{r_2/2})$ is the minimal positive solution of the Diophantine equation

$$k x^2 - l a^{r_4 - r_2} y^2 = C_1. \tag{29}$$

By (26), $(u_2, b^{r_2/2})$ is a positive integer solution of (29). So by Lemma 11, we obtain $a^{r_2/2} | b^{r_2/2}$, which contradicts the assumption that $a > b$.

Similarly we have $2 \nmid r_3$ from (27).

We now suppose that $2 \nmid r_2$ and $2 \nmid r_3$.

If $C_1 = 4$, then, by (26), (27) and Lemma 8, $(u_2, a^{\frac{r_4 - r_2 - 1}{2}} b^{\frac{r_2 - 1}{2}}) = (u_3, a^{\frac{r_4 - r_3 - 1}{2}} b^{\frac{r_3 - 1}{2}})$ is the minimal positive solution of the Diophantine equation

$$k x^2 - laby^2 = 4,$$

which contradicts the assumption that $u_2 > u_3$.

If $C_1 = -4$, then, by (26), (27) and Lemma 8, we have that $(a^{\frac{r_4 - r_3 - 1}{2}} b^{\frac{r_3 - 1}{2}}, u_3)$ is the minimal positive solution of the Diophantine equation $labx^2 - ky^2 = 4$, and $(lab, k, a^{\frac{r_4 - r_2 - 1}{2}} b^{\frac{r_2 - 1}{2}}, u_2) = (5, 1, 5, 11)$, and so $l = b = k = 1, a = 5$. Thus $(1, 1) = (a^{\frac{r_4 - r_3 - 1}{2}} b^{\frac{r_3 - 1}{2}}, u_3)$ is the minimal positive solution of $5x^2 - y^2 = 4$, which follows $a = 1$, which is a contradiction.

If $C_1 = \pm 2$ or $C_1 = \pm 1$, then, by (26), (27), both $(u_2,\ a^{\frac{r_4-r_2-1}{2}} b^{\frac{r_2-1}{2}})$ and $(u_3,\ a^{\frac{r_4-r_3-1}{2}} b^{\frac{r_3-1}{2}})$ are positive integer solutions of the Diophantine equation

$$kx^2 - laby^2 = C_1.$$

Noting that $u_2 > u_3$, by Lemmas 6 and 7,

$$u_3\sqrt{k} + a^{(r_4-r_3-1)/2} b^{(r_3-1)/2} \sqrt{lab} = \varepsilon$$

must be the minimal positive solution. Therefore again by Lemmas 6 and 7, $u_2\sqrt{k} + a^{(r_4-r_2-1)/2} b^{(r_2-1)/2} \sqrt{lab} = \varepsilon^3$ and

$$a^{(r_4-r_2-1)/2} b^{(r_2-1)/2} = 3^s a^{(r_4-r_3-1)/2} b^{(r_3-1)/2},\ 3^s \mp 3 = \frac{4}{|C_1|} l a^{r_4-r_3} b^{r_3}.$$

It follows that $a^{(r_3-r_2)/2} = 3^s b^{(r_3-r_2)/2}$, and thus

$$b = 1,\ a = 3,\ r_3 = 2s + r_2,\ l = (3^{s-1} \mp 1)|C_1|/4,$$

since $\gcd(a, b) = 1$. By (28), we get $4ku_4^2/|C_1| = 3^{s-1} \pm 3$, and so $3|k$ and $3|\gcd(k, a)$, which is impossible since $\gcd(k, a) = 1$. This concludes the analysis of Case 1.

Case 2: $2 \nmid r_4$.

Subcase 2.1: $C_1 = \pm 4$. Similarly, by (25) and Lemma 8, we can derive that

$$u_1\sqrt{k} + a^{(r_4-1)/2}\sqrt{la} = \varepsilon \tag{30}$$

is the minimal positive solution of the Diophantine equation

$$kx^2 - lay^2 = C_1. \tag{31}$$

If not, we must have $(la, k, a^{(r_4-1)/2}, u_1, C_1) = (5, 1, 5, 11, -4)$. This follows that

$$l = k = 1, a = 5, r_4 = 3,$$

and thus

$$r_2 = 1, r_3 = 2, b = 1 \quad or \quad 3.$$

Hence by (26), we get either $u_2^2 = 21$ or $u_2^2 = 71$, which is impossible.

If $2|r_2$, then, by (26), $(u_2, a^{(r_4-r_2-1)/2} b^{r_2/2})$ is a positive integer solutions of (31). We have by Lemma 11 that $a^{(r_4-1)/2} | a^{(r_4-r_2-1)/2} b^{r_2/2}$. Therefore $a^{r_2/2} | b^{r_2/2}$, contradicting with $a > b$.

Similarly we have $2 \nmid r_3$ from (27).

Now we assume that $2 \nmid r_2$ and $2 \nmid r_3$, then since $la^2b \neq 5$, by (26), (27) and Lemma 8, $(u_2, a^{\frac{r_4-r_2-2}{2}} b^{\frac{r_2-1}{2}}) = (u_3, a^{\frac{r_4-r_3-2}{2}} b^{\frac{r_3-1}{2}})$ is the minimal positive solution of the Diophantine equation

$$kx^2 - la^2by^2 = C_1,$$

which contradicts the assumption that $u_2 > u_3$.

Subcase 2.2: $C_1 = \pm 1$ or $C_1 = \pm 2$.

If $2|r_2$, then, by (26), $(u_2,\, a^{\frac{r_4-r_2-3}{2}} b^{\frac{r_2}{2}})$ is a positive integer solution of Diophantine equation

$$kx^2 - la^3y^2 = C_1. \tag{32}$$

By (25) and Lemma 9, $u_1\sqrt{k} + a^{(r_4-3)/2}\sqrt{la^3}$ must be the minimal positive solution of (32). Therefore we have by Lemma 11 that $a^{(r_4-3)/2} | a^{\frac{r_4-r_2-3}{2}} b^{\frac{r_2}{2}}$. So $a^{r_2/2} | b^{r_2/2}$, which contradicts the assumption that $a > b$.

If $2 \nmid r_2$ and $2 \nmid r_3$, then, by (26) and (27), both $(u_2,\, a^{\frac{r_4-r_2-2}{2}} b^{\frac{r_2-1}{2}})$ and $(u_3,\, a^{\frac{r_4-r_3-2}{2}} b^{\frac{r_3-1}{2}})$ are positive integer solutions of Diophantine equation

$$kx^2 - la^2by^2 = C_1. \tag{33}$$

Noting that $u_2 > u_3$, by Lemmas 6 and 7,

$$u_3\sqrt{k} + a^{(r_4-r_3-2)/2} b^{(r_3-1)/2} \sqrt{la^2b} = \varepsilon$$

must be the minimal positive solution. Therefore again by Lemmas 6 and 7, $u_2\sqrt{k} + a^{(r_4-r_2-2)/2} b^{(r_2-1)/2} \sqrt{la^2b} = \varepsilon^3$ and that

$$a^{\frac{r_4-r_2-2}{2}} b^{\frac{r_2-1}{2}} = 3^s a^{\frac{r_4-r_3-2}{2}} b^{\frac{r_3-1}{2}}, \quad 3^s \mp 3 = \frac{4}{|C_1|} l a^{r_4-r_3} b^{r_3}.$$

This follows that $a^{\frac{r_3-r_2}{2}} = 3^s b^{\frac{r_3-r_2}{2}}$, and so $3|a$. Since $r_4 - r_3$ is even, $3^s \mp 3 = \frac{4}{|C_1|} l a^{r_4-r_3} b^{r_3}$ is also divisible by 9. Hence $9|3$, which is a contradiction.

If $2 \nmid r_2$ and $2|r_3$, then, by (25), $(u_1,\, a^{(r_4-1)/2})$ is a positive integer solution of Diophantine equation

$$kx^2 - lay^2 = C_1, \tag{34}$$

and by (28), $(u_4,\, b^{(r_4-1)/2})$ is a positive integer solution of Diophantine equation

$$kx^2 - lby^2 = C_1. \tag{35}$$

We have by Lemmas 6 and 7 that either

$$u_1\sqrt{k} + a^{(r_4-1)/2}\sqrt{la} = \varepsilon,$$

or

$$\frac{u_1\sqrt{k}+a^{(r_4-1)/2}\sqrt{la}}{\sqrt{|C_1|}}=\left(\frac{\varepsilon}{\sqrt{|C_1|}}\right)^3,\ a^{(r_4-1)/2}=3^s y_1,\ 3^s\mp 3=\frac{4}{|C_1|}lay_1^2,\ s\geq 2,$$

where $\varepsilon=x_1\sqrt{k}+y_1\sqrt{la}$ is the minimal positive solution of (34), and that either

$$u_4\sqrt{k}+b^{(r_4-1)/2}\sqrt{lb}=\delta,$$

or

$$\frac{u_4\sqrt{k}+b^{(r_4-1)/2}\sqrt{lb}}{\sqrt{|C_1|}}=\left(\frac{\delta}{\sqrt{|C_1|}}\right)^3,\ b^{(r_4-1)/2}=3^{s_1}v_1,\ 3^{s_1}\mp 3=\frac{4}{|C_1|}lbv_1^2,\ s_1\geq 2,$$

where $\delta=d_1\sqrt{k}+v_1\sqrt{lb}$ is the minimal positive solution of (35).

If $\frac{u_1\sqrt{k}+a^{(r_4-1)/2}\sqrt{la}}{\sqrt{|C_1|}}=\left(\frac{\varepsilon}{\sqrt{|C_1|}}\right)^3$ and $\frac{u_4\sqrt{k}+b^{(r_4-1)/2}\sqrt{lb}}{\sqrt{C_1}}=\left(\frac{\delta}{\sqrt{|C_1|}}\right)^3$, then $a^{(r_4-1)/2}=3^s y_1^2,\ b^{(r_4-1)/2}=3^{s_1}v_1^2$. Thus $3|a$ and $3|b$, which contradicts the assumption that $\gcd(a,\,b)=1$.

If $u_1\sqrt{k}+a^{(r_4-1)/2}\sqrt{la}=\varepsilon$, then, by (27), $(u_3,\,a^{\frac{r_4-r_3-1}{2}}b^{\frac{r_3}{2}})$ is a positive integer solution of (34). By Lemma 4, we obtain

$$\frac{u_3\sqrt{k}+a^{(r_4-r_3-1)/2}b^{r_3/2}\sqrt{la}}{\sqrt{|C_1|}}=\left(\frac{u_1\sqrt{k}+a^{(r_4-1)/2}\sqrt{la}}{\sqrt{|C_1|}}\right)^n$$

for some positive integer n. This implies that $u_3\geq u_1$, which is a contradiction.

If $u_4\sqrt{k}+b^{(r_4-1)/2}\sqrt{lb}=\delta$, then, by (26), $(u_2,\,a^{\frac{r_4-r_2}{2}}b^{\frac{r_2-1}{2}})$ is a positive integer solution of (35). We have by Lemma 11 that $b^{(r_4-1)/2}|a^{(r_4-r_2)/2}b^{(r_2-1)/2}$. This implies $b^{(r_4-r_2)/2}|a^{(r_4-r_2)/2}$, and so, since $\gcd(a,b)=1$, $b=1$. By (25) and (27), both $(u_1,\,a^{\frac{r_4-1}{2}})$ and $(u_3,\,a^{\frac{r_4-r_3-1}{2}})$ are positive integer solutions of Diophantine equation

$$kx^2-lay^2=C_1.$$

Noting that $u_1>u_3$, by Lemmas 6 and 7,

$$u_3\sqrt{k}+a^{\frac{r_4-r_3-1}{2}}\sqrt{la}=\varepsilon$$

must be the minimal positive solution. Therefore again by Lemmas 6 and 7, $u_1\sqrt{k}+a^{(r_4-1)/2}\sqrt{la}=\varepsilon^3$ and that $a^{\frac{r_4-1}{2}}=3^s a^{\frac{r_4-r_3-1}{2}},\quad 3^s\mp 3=\frac{4}{|C_1|}la^{r_4-r_3}$. It follows that

$$a=3,\ l=(3^{s-1}\mp 1)|C_1|/4,$$

which is impossible as shown at the end of Case 1. This concludes the analysis of Case 2.

This completes the proof of Theorem 1.

□

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