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Full Description of Ramanujan Cubic Polynomials

Roman Wituła
Institute of Mathematics
Silesian University of Technology
Kaszubska 23
Gliwice 44-100
Poland
roman.witula@polsl.pl

Dedicated to Vladimir Shevelev – for his inspiration

Abstract

We give a full description of the Ramanujan cubic polynomials, introduced and first investigated by V. Shevelev. We also present some applications of this result.

1 Introduction

Shevelev [2] called the cubic polynomial

$$x^3 + p\,x^2 + q\,x + r \tag{1}$$

a *Ramanujan cubic polynomial* (RCP), if it has real roots x_1, x_2, x_3 and the condition

$$p\,r^{1/3} + 3\,r^{2/3} + q = 0 \tag{2}$$

is satisfied. It should be noticed, that if x_1, x_2, x_3 are roots of RCP of the form (1), then the following formulas hold (see [2, 5]):

$$x_1^{1/3} + x_2^{1/3} + x_3^{1/3} = \left(-p - 6\,r^{1/3} + 3\,(9\,r - p\,q)^{1/3}\right)^{1/3}, \tag{3}$$

$$(x_1\,x_2)^{1/3} + (x_1\,x_3)^{1/3} + (x_2\,x_3)^{1/3} = \left(q + 6\,r^{2/3} - 3\,(9\,r^2 - p\,q\,r)^{1/3}\right)^{1/3}, \tag{4}$$

and Shevelev's formula [2]:

$$\left(\frac{x_1}{x_2}\right)^{1/3}+\left(\frac{x_2}{x_1}\right)^{1/3}+\left(\frac{x_1}{x_3}\right)^{1/3}+\left(\frac{x_3}{x_1}\right)^{1/3}+\left(\frac{x_2}{x_3}\right)^{1/3}+\left(\frac{x_3}{x_2}\right)^{1/3}=\left(\frac{p\,q}{r}-9\right)^{1/3}. \tag{5}$$

We note that (3) easily implies all three Ramanujan equalities

$$\left(\frac{1}{9}\right)^{1/3}-\left(\frac{2}{9}\right)^{1/3}+\left(\frac{4}{9}\right)^{1/3}=\left(\sqrt[3]{2}-1\right)^{1/3}, \tag{6}$$

$$\left(\cos\frac{2\,\pi}{7}\right)^{1/3}+\left(\cos\frac{4\,\pi}{7}\right)^{1/3}+\left(\cos\frac{8\,\pi}{7}\right)^{1/3}=\left(\frac{5-3\,\sqrt[3]{7}}{2}\right)^{1/3}, \tag{7}$$

$$\left(\cos\frac{2\,\pi}{9}\right)^{1/3}+\left(\cos\frac{4\,\pi}{9}\right)^{1/3}+\left(\cos\frac{8\,\pi}{9}\right)^{1/3}=\left(\frac{3\,\sqrt[3]{9}-6}{2}\right)^{1/3}, \tag{8}$$

since the following decompositions of polynomials hold: (19), which implies (6) after some algebraic transformations for every $r\in\mathbb{R}\setminus\{0\}$ (the equality (6) we obtain by setting $r=8/729$), (28), which implies (7) and at last (10), which implies (8).

In [2] many interesting and fundamental properties of RCP's are presented.

The object of this paper is to prove the following fact

Theorem 1. *All RCP's have the following form*

$$\begin{aligned} x^3-\frac{P(\gamma-1)}{(\gamma-1)\,(\gamma-2)}\,r^{1/3}\,x^2-\frac{P(2-\gamma)}{(1-\gamma)\,(2-\gamma)}\,r^{2/3}\,x+r&=\\ =\left(x-\frac{r^{1/3}}{2-\gamma}\right)\left(x-(\gamma-1)\,r^{1/3}\right)&\left(x-\frac{2-\gamma}{1-\gamma}\,r^{1/3}\right), \end{aligned} \tag{9}$$

where $r\in\mathbb{R}\setminus\{0\}$, $\gamma\in\mathbb{R}\setminus\{1,2\}$, *and*

$$P(\gamma):=\gamma^3-3\,\gamma+1=\left(\gamma-2\,\cos\frac{2\,\pi}{9}\right)\left(\gamma-2\,\cos\frac{4\,\pi}{9}\right)\left(\gamma-2\,\cos\frac{8\,\pi}{9}\right). \tag{10}$$

Corollary 2. *From formulas (3), (4) and (5) for the sums of the real cube root of the roots of polynomial (9), the following equalities can be generated*

$$\begin{aligned} &\gamma^3-9\left(\gamma-1\right)^2+3\left(\gamma^2-3\,\gamma+3\right)\sqrt[3]{(\gamma-1)\,(\gamma-2)}=\\ &\qquad=\left(\sqrt[3]{1-\gamma}-\sqrt[3]{(2-\gamma)\,(1-\gamma)^2}+\sqrt[3]{(2-\gamma)^2}\right)^3, \end{aligned} \tag{11}$$

$$\begin{aligned} &\gamma^3-9\,\gamma+9-3\left(\gamma^2-3\,\gamma+3\right)\sqrt[3]{(\gamma-1)\,(\gamma-2)}=\\ &\qquad=\left(\sqrt[3]{2-\gamma}-\sqrt[3]{(1-\gamma)\,(2-\gamma)^2}-\sqrt[3]{(1-\gamma)^2}\right)^3, \end{aligned} \tag{12}$$

which, after replacing $\gamma:=3-\gamma$, *is equivalent to (11);*

$$\begin{aligned} &\left(\frac{1}{\sqrt[3]{(2-\gamma)\,(1-\gamma)}}+\sqrt[3]{(2-\gamma)\,(1-\gamma)}-\sqrt[3]{\frac{1-\gamma}{(2-\gamma)^2}}+\sqrt[3]{\frac{2-\gamma}{(1-\gamma)^2}}+\right.\\ &\qquad\left.+\sqrt[3]{\frac{(1-\gamma)^2}{2-\gamma}}-\sqrt[3]{\frac{(2-\gamma)^2}{1-\gamma}}\right)^3=9-\frac{P(\gamma-1)\,P(2-\gamma)}{(\gamma-1)^2\,(2-\gamma)^2}, \end{aligned} \tag{13}$$

i.e.,

$$\left(\gamma^2 - 3\,\gamma + 3\right)^3 = 9\,(\gamma-1)^2\,(2-\gamma)^2 - P(\gamma-1)\,P(2-\gamma). \tag{14}$$

The above relations essentially supplement the set of identities presented in [1]. Furthermore, (11)–(14) entail Ramanujan's equalities (6)–(8), as well as all the other expressions of this type discussed in [2, 4, 5].

In the second part of this paper we will discuss an important Shevelev parameter $\frac{p\,q}{r}$ of RCP's having the form (1). We note, that from (17) the following Shevelev inequality follows:

$$\frac{p\,q}{r} \le \frac{9}{4}. \tag{15}$$

We remark that for every $a \in \mathbb{R}$, $a \le \frac{9}{4}$, there exist at most six different sets of RCP's, depending only on values r and having the same value of $\frac{p\,q}{r}$, equal to a. In the sequel, there exist only two sets of RCP's, depending on $r \in \mathbb{R}$, having the value $\frac{p\,q}{r} = 2$ (see the descriptions (40) and (41)). However, there is only one family of RCP's, depending on $r \in \mathbb{R}$ with $\frac{p\,q}{r} = \frac{9}{4}$ (see the descriptions (19)). This fact is proven in Section 2, but it independently results from (31), (9), (14) and from the following identity

$$\frac{p\,q}{r} = 9 - \frac{\big((\gamma-1)\,(\gamma-2)+1\big)^3}{\big((\gamma-1)\,(\gamma-2)\big)^2}. \tag{16}$$

From (16) we get

$$\frac{p\,q}{r} = \frac{9}{4} \quad \Leftrightarrow \quad t := (\gamma-1)\,(\gamma-2) \in \Big\{-\frac{1}{4}, 2\Big\} \quad \Leftrightarrow \quad \gamma \in \Big\{0, \frac{3}{2}, 3\Big\},$$

since we have

$$\frac{d}{dt}\Big(9 - \frac{(t+1)^3}{t^2}\Big) = t\,(2-t)\,\frac{(t+1)^2}{t^4}.$$

All three values $\gamma \in \big\{0, \frac{3}{2}, 3\big\}$ generate the same set of RCP's of the form (19).

2 Proof of Theorem 1

Let us indicate that from condition (2) the following equality follows (see [2]):

$$\frac{9}{4} - \frac{p\,q}{r} = \Big(\frac{3}{2} + \frac{p}{r^{1/3}}\Big)^2. \tag{17}$$

Let

$$p := \Big(\alpha - \frac{3}{2}\Big)\,r^{1/3}.$$

By (17) we have

$$\frac{p\,q}{r} = \frac{9}{4} - \alpha^2,$$
$$q = -\Big(\alpha + \frac{3}{2}\Big)\,r^{2/3}.$$

In other words, an RCP has the form

$$x^3 + \Big(\alpha - \frac{3}{2}\Big)\, r^{1/3}\, x^2 - \Big(\alpha + \frac{3}{2}\Big)\, r^{2/3}\, x + r \tag{18}$$

for some $\alpha, r \in \mathbb{R}$. If $\alpha = 0$, the following decomposition holds

$$x^3 - \frac{3}{2}\, r^{1/3}\, x^2 - \frac{3}{2}\, r^{2/3}\, x + r = \Big(x - \frac{1}{2}\, r^{1/3}\Big)\,\big(x + r^{1/3}\big)\,\big(x - 2\, r^{1/3}\big). \tag{19}$$

Accordingly, the roots x_1, x_2, x_3 of the polynomial (18) have the form ($r \neq 0$):

$$x_1 = \Big(\frac{1}{2} + \beta\Big)\, r^{1/3}, \quad x_2 = \big(-1 + \gamma\big)\, r^{1/3}, \quad x_3 = \big(2 + \delta\big)\, r^{1/3} \tag{20}$$

for certain $\beta, \gamma, \delta \in \mathbb{R}$. Then from Vieta's formulae the following equations can be obtained

$$\alpha = -\big(\beta + \gamma + \delta\big), \tag{21}$$

$$\Big(\frac{1}{2} + \beta\Big)\,\big(-1 + \gamma\big) + \Big(\frac{1}{2} + \beta\Big)\,\big(2 + \delta\big) + \big(-1 + \gamma\big)\,\big(2 + \delta\big) = -\alpha - \frac{3}{2}, \tag{22}$$

$$\Big(\frac{1}{2} + \beta\Big)\,\big(-1 + \gamma\big)\,\big(2 + \delta\big) = 1. \tag{23}$$

From (21) and (22) we receive

$$\beta = \frac{\frac{3}{2}\,(\delta - \gamma) - \delta\,\gamma}{\delta + \gamma}, \tag{24}$$

which, by (23), implies

$$\delta^2\,\big(\gamma^2 - 3\,\gamma + 2\big) + \delta\,\big(3\,\gamma^2 - 7\,\gamma + 3\big) + 2\,\gamma^2 - 3\,\gamma = 0.$$

Hence, after some manipulations, we get

$$\Delta_\delta = \big(\gamma^2 - 3\,\gamma + 3\big)^2,$$

and next

$$\delta = \frac{\gamma}{1 - \gamma} \qquad \text{or} \qquad \delta = \frac{3 - 2\,\gamma}{\gamma - 2}. \tag{25}$$

If we choose $\delta = \frac{\gamma}{1-\gamma}$, then by (24) we have $\beta = \frac{\gamma}{2\,(2-\gamma)}$, and by (20) we obtain

$$\begin{aligned} x_1 &= \frac{r^{1/3}}{2 - \gamma}, \quad x_2 = (\gamma - 1)\, r^{1/3}, \quad x_3 = \frac{2 - \gamma}{1 - \gamma}\, r^{1/3}, \\ \alpha &= -\Big(\frac{\gamma}{2\,(2 - \gamma)} + \gamma + \frac{\gamma}{1 - \gamma}\Big) = \frac{-2\,\gamma^3 + 9\,\gamma^2 - 9\,\gamma}{2\,(\gamma - 1)\,(\gamma - 2)}. \end{aligned} \tag{26}$$

Finally

$$\begin{aligned} x^3 + \frac{-\gamma^3 + 3\,\gamma^2 - 3}{(\gamma - 1)\,(\gamma - 2)}\, r^{1/3}\, x^2 + \frac{\gamma^3 - 6\,\gamma^2 + 9\,\gamma - 3}{(\gamma - 1)\,(\gamma - 2)}\, r^{2/3}\, x + r = \\ = \Big(x - \frac{r^{1/3}}{2 - \gamma}\Big)\,\Big(x - (\gamma - 1)\, r^{1/3}\Big)\,\Big(x - \frac{2 - \gamma}{1 - \gamma}\, r^{1/3}\Big), \end{aligned} \tag{27}$$

which is compatible with (9).

On the other hand, if we choose $\delta = \frac{3-2\,\gamma}{\gamma-2}$, then $\beta = \frac{\gamma-3}{2\,(\gamma-1)}$, and we obtain the same values of x_1, x_2, x_3 and α as in (26) above. □

Example 3. Since

$$\left(x - 2\,\cos\frac{2\,\pi}{7}\right)\left(x - 2\,\cos\frac{4\,\pi}{7}\right)\left(x - 2\,\cos\frac{8\,\pi}{7}\right) = x^3 + x^2 - 2\,x - 1 \tag{28}$$

is the RCP [4], then, from (9) the following relations can be deduced

$$\gamma = 1 - 2\,\cos\frac{2\,\pi}{7}, \quad r = -1,$$
$$\frac{P(\gamma-1)}{(1-\gamma)\,(2-\gamma)} = 1 \quad \text{and} \quad \frac{P(2-\gamma)}{(1-\gamma)\,(2-\gamma)} = 2,$$

which implies the equalities

$$\frac{1}{\gamma-2} = 2\,\cos\frac{4\,\pi}{7}, \qquad \frac{\gamma-2}{1-\gamma} = 2\,\cos\frac{8\,\pi}{7},$$

$$\frac{\left(\cos\frac{2\pi}{7}+\cos\frac{2\pi}{9}\right)\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{9}\right)\left(\cos\frac{2\pi}{7}+\cos\frac{8\pi}{9}\right)}{\cos\frac{2\pi}{7}\left(1+2\,\cos\frac{2\pi}{7}\right)} = -\frac{1}{4}, \tag{29}$$

and the equivalent one

$$\frac{\left(\frac{1}{2}+\cos\frac{2\pi}{7}-\cos\frac{2\pi}{9}\right)\left(\frac{1}{2}+\cos\frac{2\pi}{7}-\cos\frac{4\pi}{9}\right)\left(\frac{1}{2}+\cos\frac{2\pi}{7}-\cos\frac{8\pi}{9}\right)}{\cos\frac{2\pi}{7}\left(1+2\,\cos\frac{2\pi}{7}\right)} = \frac{1}{2}. \tag{30}$$

3 Values of $\frac{p\,q}{r}$ for RCP's

By (9) we obtain

$$\frac{p\,q}{r} = \frac{P(\gamma-1)\,P(2-\gamma)}{(\gamma-1)^2\,(2-\gamma)^2}. \tag{31}$$

The examples of RCP's, which are given in [4, 5] (see also [2]), are produced by $\frac{p\,q}{r}$ equal only to 2, −40, −180.

The following theorem holds.

Theorem 4. *For every $a \le \frac{9}{4}$ there exist at most six different sets of RCP's, depending on $r \in \mathbb{R}$, having the same value of $\frac{p\,q}{r}$, equal to a.*

Proof. The proof of this theorem results easily from inequality (15) and from relation (31). □

We will present now a series of remarks, connected with the parameter $a = \frac{p\,q}{r}$.

Remark 5. Let us consider the following equation

$$\frac{P(\gamma-1)\,P(2-\gamma)}{(\gamma-1)^2\,(2-\gamma)^2} = a \qquad (a \in \mathbb{R}). \tag{32}$$

This equation, by (16), after substitution $t := (\gamma-1)\,(\gamma-2)$, is equivalent to the following one

$$R(t) := t^3 + (a-6)\,t^2 + 3\,t + 1 = 0. \tag{33}$$

If we replace t in (33) by $\tau - \frac{a-6}{3}$, then the canonical form of $R(t)$ can be generated

$$\tau^3 + \big(3 - \frac{1}{3}\,\big(a-6\big)^2\,\big)\,\tau + \frac{2}{27}\,(a-6)^3 - (a-6) + 1. \tag{34}$$

But the polynomial (34) has only one real root, if and only if

$$\frac{1}{4}\,\big(\frac{2}{27}\,(a-6)^3 - (a-6) + 1\big)^2 + \frac{1}{27}\,\big(3 - \frac{1}{3}\,\big(a-6\big)^2\,\big)^3 > 0 \iff$$

$$\iff \frac{4}{27}\,(a-6)^3 - \frac{1}{3}\,(a-6)^2 - 2\,(a-6) + 5 > 0 \iff (a-9)^2\Big(a - \frac{9}{4}\Big) > 0.$$

Since the case $a = \frac{9}{4}$ was discussed in (19), the polynomial $R(t)$ has three real roots for every $a \le \frac{9}{4}$.

Remark 6. If $\gamma_0 \in \mathbb{C}$ is a root of equation (32) (for fixed $a \in \mathbb{C}$) then also $\gamma = 3 - \gamma_0$ and $\gamma = \frac{\gamma_0}{1-\gamma_0}$ are roots of this one. We note, that the last fact derives from the following identities

$$(1-\gamma)^3\,P\Big(\frac{1}{\gamma-1}\Big) = P(2-\gamma)$$

and

$$(1-\gamma)^3\,P\Big(\frac{\gamma-2}{\gamma-1}\Big) = P(\gamma-1).$$

Consequently, the roots of (32) are also

$$\gamma = \frac{3-\gamma_0}{1-(3-\gamma_0)} = \frac{3-\gamma_0}{\gamma_0-2}, \quad \gamma = 3 - \frac{\gamma_0}{1-\gamma_0} = \frac{3-4\,\gamma_0}{1-\gamma_0}, \quad \gamma = 3 - \frac{3-\gamma_0}{\gamma_0-2} = \frac{4\,\gamma_0-9}{\gamma_0-2}.$$

Remark 7. Let us separately discuss equation (32) for $a = 2$. After substitution $t = 1-\tau$ in (33), the following equation is derived

$$\tau^3 + \tau^2 - 2\,\tau - 1 = 0, \tag{35}$$

i.e. (see [4]):

$$\Big(\tau - 2\,\cos\frac{2\,\pi}{7}\Big)\,\Big(\tau - 2\,\cos\frac{4\,\pi}{7}\Big)\,\Big(\tau - 2\,\cos\frac{8\,\pi}{7}\Big) = 0. \tag{36}$$

Hence, equation (32) for $a = 2$ is equivalent to each of the following three equations

$$(\gamma-1)\,(\gamma-2) = 1 - 2\,\cos\frac{2\,\pi}{7} \iff \gamma - 1 = -2\,\cos\frac{4\,\pi}{7} \;\;\vee\;\; \gamma - 2 = 2\,\cos\frac{4\,\pi}{7}, \tag{37}$$

or

$$(\gamma-1)\,(\gamma-2) = 1 - 2\,\cos\frac{4\,\pi}{7} \iff \gamma - 1 = -2\,\cos\frac{8\,\pi}{7} \;\;\vee\;\; \gamma - 2 = 2\,\cos\frac{8\,\pi}{7}, \tag{38}$$

or

$$(\gamma-1)\,(\gamma-2) = 1 - 2\,\cos\frac{8\,\pi}{7} \iff \gamma - 1 = -2\,\cos\frac{2\,\pi}{7} \quad\vee\quad \gamma - 2 = 2\,\cos\frac{2\,\pi}{7}. \tag{39}$$

For the values

$$\gamma \in \Big\{1 - 2\,\cos\frac{2^k\,\pi}{7}:\ k = 1,2,3\Big\},$$

we obtain the same set of RCP's

$$x^3 + r^{1/3}\,x^2 - 2\,r^{2/3}\,x - r, \qquad r \in \mathbb{R}. \tag{40}$$

On the other hand, for values

$$\gamma \in \Big\{2 + 2\,\cos\frac{2^k\,\pi}{7}:\ k = 1,2,3\Big\},$$

we obtain the following set of RCP's

$$x^3 - 2\,r^{1/3}\,x^2 - r^{2/3}\,x + r, \qquad r \in \mathbb{R}. \tag{41}$$

We note, that RCP of the form (see [2, 4]):

$$\begin{aligned} x^3 + 7\,x^2 - 98\,x - 343 &= \\ &= \Big(x - 128\,\cos\frac{2\,\pi}{7}\Big(\sin\frac{2\,\pi}{7}\,\sin\frac{8\,\pi}{7}\Big)^3\Big)\Big(x - 128\,\cos\frac{4\,\pi}{7}\Big(\sin\frac{2\,\pi}{7}\,\sin\frac{4\,\pi}{7}\Big)^3\Big)\cdot \\ &\qquad\cdot\Big(x - 128\,\cos\frac{8\,\pi}{7}\Big(\sin\frac{4\,\pi}{7}\,\sin\frac{8\,\pi}{7}\Big)^3\Big) \end{aligned}$$

belongs to the set (40) of RCP's with $\frac{p\,q}{r} = 2$ for $r = 7^3$, because of the following remark.

Remark 8. Suppose, that $\alpha \in \{\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{8\pi}{7}\}$. Then, we have $\sin\alpha = \sin 8\alpha$, which implies

$$\begin{aligned} 14\,\cos\alpha = 7\,\frac{\sin 2\alpha}{\sin\alpha} &= \big(8\,\sin\alpha\,\sin 2\alpha\,\sin 4\alpha\big)^2\,\frac{\sin 2\alpha}{\sin\alpha} = 64\,\frac{\sin\alpha}{\sin 4\alpha}\big(\sin 2\alpha\big)^3\big(\sin 4\alpha\big)^3 = \\ &= 64\,\frac{\sin 8\alpha}{\sin 4\alpha}\big(\sin 2\alpha\big)^3\big(\sin 4\alpha\big)^3 = 128\,\cos 4\alpha\,\big(\sin 2\alpha\,\sin 4\alpha\big)^3. \end{aligned}$$

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