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On s-Fibonomials

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Abstract

For a given natural number s, we study s-Fibonacci sequences F_{sn} and the corresponding s-Fibonomial coefficients $\binom{n}{p}_{F_s} = \frac{F_{sn}F_{s(n-1)}\cdots F_{s(n-p+1)}}{F_sF_{2s}\cdots F_{ps}}$. We obtain the Z transform of products of powers of s-Fibonacci sequences. Since the s-Fibonomials are involved in this Z transform, we obtain from it some new results involving products of sequences of the type F_{sn+m}^k together with s-Fibonomials.

1 Introduction

We use $\mathbb{N}$ for the natural numbers and $\mathbb{N}'$ for $\mathbb{N} \cup \{0\}$. We will be using without further comments the basic formulas for the Fibonacci sequence F_n and Lucas sequence L_n (A000045 and A000032 of Sloane's *Encyclopedia*, respectively), such as $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1}{2}\left(1+\sqrt{5}\right)$ and $\beta = \frac{1}{2}\left(1-\sqrt{5}\right)$, together with the well-known relations involving α and β. What we use about Fibonacci identities is contained in the references [7] and [14].

Throughout this work, s will denote a natural number.

For a given Fibonacci number F_n, $n \in \mathbb{N}$, the *s-Fibonacci factorial* of F_n, denoted by $(F_n!)_s$, is defined as $(F_n!)_s = F_{sn}F_{s(n-1)}\cdots F_s$. Given $n \in \mathbb{N}'$ and $k \in \{0, 1, \ldots, n\}$, the *$s$-Fibonomial coefficient* $\binom{n}{k}_{F_s}$ is defined by $\binom{n}{0}_{F_s} = \binom{n}{n}_{F_s} = 1$, and

$$\binom{n}{k}_{F_s} = \frac{(F_n!)_s}{(F_k!)_s\,(F_{n-k}!)_s}, \qquad k = 1, 2, \ldots, n-1,$$

that is

$$\binom{n}{k}_{F_s} = \frac{F_{sn}F_{s(n-1)}\cdots F_{s(n-k+1)}}{F_sF_{2s}\cdots F_{ks}}. \tag{1}$$

(The 1-Fibonomials are called simply Fibonomials.) It is clear that $\binom{n}{k}_{F_s} = \binom{n}{n-k}_{F_s}$. From the identity $F_{s(n-k)+1}F_{sk} + F_{sk-1}F_{s(n-k)} = F_{sn}$ one can see at once that

$$\binom{n}{k}_{F_s} = F_{s(n-k)+1}\binom{n-1}{k-1}_{F_s} + F_{sk-1}\binom{n-1}{k}_{F_s}, \tag{2}$$

which shows (with a simple induction argument) that s-Fibonomials are integers. (See [5].)

Some examples are the following (as triangular arrays, where the lines correspond to $n \in \mathbb{N}'$, the columns to k, $0 \le k \le n$, and the array is filled-out with formula (1) and/or rule (2):

(a) $s = 1$ (the Fibonomials: A010048 of Sloane's *Encyclopedia*):

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1_{\bigstar} \quad 1_{\flat} \\
1 \quad 2_{\triangle} \quad 2_{\flat} \quad 1 \\
1 \quad 3_{\maltese} \quad 6_{\flat} \quad 3 \quad 1 \\
1 \quad 5 \quad 15_{\flat} \quad 15 \quad 5 \quad 1 \\
1 \quad 8 \quad 40_{\flat} \quad 60 \quad 40 \quad 8 \quad 1 \\
\cdot \quad \cdot \quad \cdot
\end{array}$$

(b) $s = 2$ (A034801 of Sloane's *Encyclopedia*):

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 3_{\bigstar} \quad 1_{\flat} \\
1 \quad 8_{\triangle} \quad 8_{\flat} \quad 1 \\
1 \quad 21_{\maltese} \quad 56_{\flat} \quad 21 \quad 1 \\
1 \quad 55 \quad 385_{\flat} \quad 385 \quad 55 \quad 1 \\
1 \quad 144 \quad 2640_{\flat} \quad 6930 \quad 2640 \quad 144 \quad 1 \\
\cdot \quad \cdot \quad \cdot
\end{array}$$

(c) $s = 3$ (A034802 of Sloane's *Encyclopedia*):

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 4_{\bigstar} \quad 1_{\flat} \\
1 \quad 17_{\triangle} \quad 17_{\flat} \quad 1 \\
1 \quad 72_{\maltese} \quad 306_{\flat} \quad 72 \quad 1 \\
1 \quad 305 \quad 5490_{\flat} \quad 5490 \quad 305 \quad 1 \\
1 \quad 1292 \quad 98515_{\flat} \quad 417240 \quad 98515 \quad 1292 \quad 1 \\
\cdot \quad \cdot \quad \cdot
\end{array}$$

We have marked some patterns that can be recognized as general relations among the s-Fibonomials $\binom{n}{1}_{F_s}$ (that is, among the quotients $\frac{F_{sn}}{F_s}$, as s-sequences —with fixed n) as follows:

- (a)
$$\binom{2^n}{1}_{F_s} = \frac{F_{2(2^{n-1}s)}}{F_s} = L_{2^{n-1}s}\frac{F_{2^{n-1}s}}{F_s} = L_{2^{n-1}s}\binom{2^{n-1}}{1}_{F_s},$$
and then
$$\binom{2^n}{1}_{F_s} = L_sL_{2s}\cdots L_{2^{n-1}s}. \tag{3}$$
Thus, for $n = 1$ we have the sequence of Lucas numbers $\binom{2}{1}_{F_s} = \frac{F_{2s}}{F_s} = L_s$ (corresponding to the mark ★ in previous tables), and for $n = 2$ we have the sequence $\binom{4}{1}_{F_s} = \frac{F_{4s}}{F_s} = L_sL_{2s} = (3, 21, 72, \ldots)$ (A083564 of Sloane's *Encyclopedia*; corresponding to the mark ✠ in previous tables).

- (b)
$$\binom{3^n}{1}_{F_s} = \frac{F_{3^ns}}{F_s} = \frac{\left(\alpha^{3^{n-1}s}\right)^3 - \left(\beta^{3^{n-1}s}\right)^3}{F_s} = \left(\alpha^{3^{n-1}2s} + \beta^{3^{n-1}2s} + (-1)^{3^{n-1}s}\right)\frac{F_{3^{n-1}s}}{F_s}$$
$$= \left(L_{3^{n-1}2s} + (-1)^s\right)\binom{3^{n-1}}{1}_{F_s},$$
and then
$$\binom{3^n}{1}_{F_s} = \left(L_{3^{n-1}2s} + (-1)^s\right)\left(L_{3^{n-2}2s} + (-1)^s\right)\cdots\left(L_{2s} + (-1)^s\right). \tag{4}$$
Thus for $n = 1$ we have the sequence $\binom{3}{1}_{F_s} = \frac{F_{3s}}{F_s} = L_{2s} + (-1)^s = (2, 8, 17, \ldots)$ (A047946 of Sloane's *Encyclopedia*; corresponding to the mark △ in previous tables).

- (c)
$$\binom{2^m3^n}{1}_{F_s} = \frac{F_{2(2^{m-1}3^ns)}}{F_s} = L_{2^{m-1}3^ns}\frac{F_{2^{m-1}3^ns}}{F_s} = L_{2^{m-1}3^ns}\binom{2^{m-1}3^n}{1}_{F_s},$$
and then
$$\binom{2^m3^n}{1}_{F_s} = L_{2^{m-1}3^ns}L_{2^{m-2}3^ns}\cdots L_{3^ns}\binom{3^n}{1}_{F_s}. \tag{5}$$

Finally, note that for fixed s, the sequence (1, ★, ▲, ✠, ...) corresponds to $\binom{n}{1}_{F_s} = \frac{F_{ns}}{F_s}$, $n \geq 1$. Thus, for $s = 1$ we have the sequence $\frac{F_n}{F_1} = F_n$. For $s = 2$ we have the sequence $\frac{F_{2n}}{F_2} = F_{2n} = (1, 3, 8, 21, \ldots)$ (A001906 of Sloane's *Encyclopedia*), and for $s = 3$ we have the sequence $\frac{F_{3n}}{F_3} = \frac{1}{2}F_{3n} = (1, 4, 17, 72, \ldots)$ (A001076 of Sloane's *Encyclopedia*).

Remark 1. For fixed s, the sequence $\binom{n}{2}_{F_s} = \frac{F_{sn}F_{s(n-1)}}{F_sF_{2s}}$, $n \geq 2$, (marked with ♭ for each $s = 1, 2, 3$ in previous tables) is, in the case of Fibonomials ($s = 1$), the famous golden rectangle sequence $\binom{n}{2}_F = F_nF_{n-1} = (1, 2, 6, 15, \ldots)$ (A001654 of Sloane's *Encyclopedia*). The corresponding sequences for the cases $s = 2$ and $s = 3$ are also included in *the mentioned Encyclopedia* (A092521 and A156085, respectively), but there are not references to the formulas in which they appear in this work, namely $\binom{n}{2}_{F_2} = \frac{1}{3}F_{2n}F_{2(n-1)} = (1, 8, 56, 385, \ldots)$ and $\binom{n}{2}_{F_3} = \frac{1}{16}F_{3n}F_{3(n-1)} = (1, 17, 306, 5490, \ldots)$, respectively.

We have found just a few references on s-Fibonomials, even though Hoggatt [5] considered them in 1967. Besides the sequences of s-Fibonomials for $s=1$, $s=2$ and $s=3$ included in Sloane's *Encyclopedia* mentioned above, Gould [4] studied divisibility properties of s-Fibonomials.

The context in which s-Fibonomials are studied in this article is related to Z transforms (and then to generating functions) of certain Fibonacci sequences. This story begins with the works of Riordan [9], Carlitz [2] and Horadam [6] first, and Shannon [12] later. From all these works we know that the Z transform of the sequence F_n^k is

$$\mathcal{Z}\left(F_n^k\right) = z\frac{\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{\frac{j(j+1)}{2}}\binom{k+1}{j}_F F_{i-j}^k z^{k-i}}{\sum_{i=0}^{k+1}(-1)^{\frac{i(i+1)}{2}}\binom{k+1}{i}_F z^{k+1-i}}, \tag{6}$$

(See also [10] and [11].) In a recent work [8], we proved the following formula for the Z transform of the sequence $F_{n+m_1}^{k_1}F_{n+m_2}^{k_2}$:

$$\mathcal{Z}\left(F_{n+m_1}^{k_1}F_{n+m_2}^{k_2}\right) = z\frac{\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{j(j+1)}{2}}\binom{k_1+k_2+1}{j}_F F_{m_1+i-j}^{k_1}F_{m_2+i-j}^{k_2}z^{k_1+k_2-i}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{i(i+1)}{2}}\binom{k_1+k_2+1}{i}_F z^{k_1+k_2+1-i}}, \tag{7}$$

where $m_1, m_2 \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{N}'$ are given. So (6) became a particular case of (7). After our work [8], we were able to find that (7) is in fact a particular case of a more general result: it turns out that for given $m_1, m_2 \in \mathbb{Z}, s \in \mathbb{N}$ and $k_1, k_2, t_1, t_2 \in \mathbb{N}'$, the Z transform of the sequence $F_{t_1sn+m_1}^{k_1}F_{t_2sn+m_2}^{k_2}$ is

$$\begin{aligned} &\mathcal{Z}\left(F_{t_1sn+m_1}^{k_1}F_{t_2sn+m_2}^{k_2}\right) \\ = \ & z\frac{\sum_{i=0}^{t_1k_1+t_2k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1k_1+t_2k_2+1}{j}_{F_s} F_{m_1+t_1s(i-j)}^{k_1}F_{m_2+t_2s(i-j)}^{k_2}z^{t_1k_1+t_2k_2-i}}{\sum_{i=0}^{t_1k_1+t_2k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1k_1+t_2k_2+1}{i}_{F_s} z^{t_1k_1+t_2k_2+1-i}}, \end{aligned} \tag{8}$$

which means that for $z \in \mathbb{C}$ outside the disk $\overline{D} = \left\{z \in \mathbb{C} : |z| \leq \alpha^{s(t_1k_1+t_2k_2)}\right\}$, the right-hand side of this formula equals to

$$\sum_{n=0}^{\infty}\frac{F_{t_1sn+m_1}^{k_1}F_{t_2sn+m_2}^{k_2}}{z^n}.$$

We have to mention that Fibonomials can be seen as *combinatorial objects* (see [1]), and it is natural to expect that s-Fibonomials could enclose more general combinatorial interpretations. Formula (8) is the main result in this work. But we will see that some interesting results are obtained as consequences of (8), so its proof is not our final goal. In section 2 we introduce some basic facts about Z transform that will be used in the rest of the work, together with some preliminary results to be used mainly in section 3. Is in section 3 where we prove (8). In section 4 we establish some corollaries of (8), and finally in section 5 we do some comments on the representation of s-Fibonomials as linear combinations of certain "homogeneous terms".

2 Preliminaries

We will be working with the so called "Z transform", of which we recall some basic facts in this section. (For more details see [3] and [15].) The Z transform maps complex sequences $(a_0, a_1, a_2, \ldots)$ into complex (holomorphic) functions $A : U \subset \mathbb{C} \to \mathbb{C}$ given by the Laurent series $A(z) = \sum_{n=0}^{\infty} a_n z^{-n}$. This function A is defined outside the closure $\overline{D}$ of the disk D of convergence of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$. We will also denote the Z transform of the sequence $(a_n)_{n=0}^{\infty}$ by $\mathcal{Z}(a_n)$. Some properties of the Z transform which we will be using throughout this work are the following: (avoiding the details of regions of convergence)

(a) $\mathcal{Z}$ is linear and injective.

(b) *Advance-shifting property.* For $k \in \mathbb{N}$ we have

$$\mathcal{Z}(a_{n+k}) = z^k \left(\mathcal{Z}(a_n) - a_0 - \frac{a_1}{z} - \cdots - \frac{a_{k-1}}{z^{k-1}} \right). \tag{9}$$

Here a_{n+k} is the sequence $a_{n+k} = (a_k, a_{k+1}, \ldots)$.

(c) *Multiplication by the sequence λ^n.* If $\mathcal{Z}(a_n) = A(z)$, then

$$\mathcal{Z}(\lambda^n a_n) = A\left(\frac{z}{\lambda}\right). \tag{10}$$

(d) *Convolution theorem.* If a_n and b_n are two given sequences, then

$$\mathcal{Z}(a_n * b_n) = \mathcal{Z}(a_n)\,\mathcal{Z}(b_n), \tag{11}$$

where $a_n * b_n = \sum_{t=0}^{n} a_t b_{n-t}$ is the convolution of the sequences a_n and b_n.

If $A(z) = \mathcal{Z}(a_n)$, we also write $a_n = \mathcal{Z}^{-1}(A(z))$, and we say that the sequence a_n is the *inverse Z transform* of $A(z)$. Clearly $\mathcal{Z}^{-1}$ is also linear and injective.

Observe that according to (10), if $\mathcal{Z}(a_n) = A(z)$ then

$$\mathcal{Z}((-1)^n a_n) = A(-z), \tag{12}$$

and

$$\mathcal{Z}(L_{sn+m} a_n) = \alpha^m A\left(\frac{z}{\alpha^s}\right) + \beta^m A\left(\frac{z}{\beta^s}\right). \tag{13}$$

For given $\lambda \in \mathbb{C}$, $\lambda \neq 0$, the Z transform of the sequence λ^n is plainly

$$\mathcal{Z}(\lambda^n) = \sum_{n=0}^{\infty} \frac{\lambda^n}{z^n} = \frac{z}{z-\lambda}, \tag{14}$$

(defined for $|z| > |\lambda|$). This is an important formula in this article since Fibonacci sequences are combinations of sequences of the form λ^n. In particular we have that the Z transform of the constant sequence 1 is

$$\mathcal{Z}(1) = \frac{z}{z-1}. \tag{15}$$

(Observe that if $t_1 = t_2 = 0$, formula (8) says that $\mathcal{Z}\left(F_{m_1}^{k_1} F_{m_2}^{k_2}\right) = z\frac{(-1)^{s+1} F_{m_1}^{k_1} F_{m_2}^{k_2}}{(-1)^{s+1} z + (-1)^s}$, which is essentially (15).)

For given $m \in \mathbb{Z}$ one has

$$\mathscr{Z}\left(F_{sn+m}\right) = \frac{z\left(F_m z + (-1)^m F_{s-m}\right)}{z^2 - L_s z + (-1)^s}, \tag{16}$$

and from this expression one can see that

$$F_s F_{sn+m} = F_m F_{s(n+1)} + (-1)^m F_{s-m} F_{sn}. \tag{17}$$

(See [8].)

Now we begin with a list of preliminary results that will be used in sections 3 and 4.

Proposition 2. *For given $k \in \mathbb{N}'$ we have*

$$(-1)^{s+1} \prod_{j=0}^{k} \left(z - \alpha^{sj} \beta^{s(k-j)}\right) = \sum_{i=0}^{k+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{k+1}{i}_{F_s} z^{k+1-i}. \tag{18}$$

Proof. We proceed by induction on k. For $k = 0$ the result is clearly true (both sides are equal to $(-1)^{s+1}(z-1)$). Let us suppose the formula is true for a given k. Then

$$\begin{aligned}
& (-1)^{s+1} \prod_{j=0}^{k+1} \left(z - \alpha^{sj} \beta^{s(k+1-j)}\right) \\
= \; & (-1)^{s+1} \left(z - \alpha^{s(k+1)}\right) \beta^{s(k+1)} \prod_{j=0}^{k} \left(\frac{z}{\beta^s} - \alpha^{sj} \beta^{s(k-j)}\right) \\
= \; & (-1)^{s+1} \left(z - \alpha^{s(k+1)}\right) \beta^{s(k+1)} (-1)^{s+1} \sum_{i=0}^{k+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{k+1}{i}_{F_s} \left(\frac{z}{\beta^s}\right)^{k+1-i} \\
= \; & \left(z - \alpha^{s(k+1)}\right) \sum_{i=0}^{k+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{k+1}{i}_{F_s} \beta^{si} z^{k+1-i} \\
= \; & \sum_{i=0}^{k+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{k+1}{i}_{F_s} \beta^{si} z^{k+2-i} - \alpha^{s(k+1)} \sum_{i=1}^{k+2} (-1)^{\frac{(s(i-1)+2(s+1))i}{2}} \binom{k+1}{i-1}_{F_s} \beta^{s(i-1)} z^{k+2-i} \\
= \; & \sum_{i=0}^{k+2} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{k+2}{i}_{F_s} \frac{1}{F_{s(k+2)}} \left(\beta^{si} F_{s(k+2-i)} + (-1)^{-s(i+1)} \alpha^{s(k+1)} \beta^{s(i-1)} F_{si}\right) z^{k+2-i} \\
= \; & \sum_{i=0}^{k+2} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{k+2}{i}_{F_s} z^{k+2-i}
\end{aligned}$$

as wanted. Here we used that

$$\beta^{si} F_{s(k+2-i)} + (-1)^{-s(i+1)} \alpha^{s(k+1)} \beta^{s(i-1)} F_{si} = F_{s(k+2)},$$

which can be proved easily by writing the F's in terms of α and β. ∎

We will denote the $(k+1)$-th degree polynomial $(-1)^{s+1}\prod_{j=0}^{k}\left(z-\alpha^{sj}\beta^{s(k-j)}\right)$ as $D_{s,k+1}(z)$. By combining pairs of adequate linear factors, it is possible to express $D_{s,k+1}(z)$ as product of "Lucas factors" (quadratic factors in which the coefficient of the linear term is a Lucas number, like in the denominator of (16); see also [13]). We have two cases:

(a) If k is even, $k=2p$ say, then

$$D_{s,2p+1}(z)=(-1)^{s+1}\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right). \tag{19}$$

(b) If k is odd, $k=2p-1$ say, then

$$D_{s,2p}(z)=(-1)^{s+1}\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s\right). \tag{20}$$

Indeed, we have

$$\begin{aligned}
D_{s,2p+1}(z) &= (-1)^{s+1}\prod_{j=0}^{2p}\left(z-\alpha^{sj}\beta^{s(2p-j)}\right)\\
&= (-1)^{s+1}\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z-\alpha^{sj}\beta^{s(2p-j)}\right)\prod_{j=p+1}^{2p}\left(z-\alpha^{sj}\beta^{s(2p-j)}\right)\\
&= (-1)^{s+1}\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z-\alpha^{sj}\beta^{s(2p-j)}\right)\prod_{j=0}^{p-1}\left(z-\alpha^{s(2p-j)}\beta^{sj}\right)\\
&= (-1)^{s+1}\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right),
\end{aligned}$$

which proves (19). Similarly we have

$$\begin{aligned}
D_{s,2p}(z) &= (-1)^{s+1}\prod_{j=0}^{2p-1}\left(z-\alpha^{sj}\beta^{s(2p-1-j)}\right)\\
&= (-1)^{s+1}\prod_{j=0}^{p-1}\left(z-\alpha^{sj}\beta^{s(2p-1-j)}\right)\prod_{j=p}^{2p-1}\left(z-\alpha^{sj}\beta^{s(2p-1-j)}\right)\\
&= (-1)^{s+1}\prod_{j=0}^{p-1}\left(z-\alpha^{sj}\beta^{s(2p-1-j)}\right)\prod_{j=0}^{p-1}\left(z-\alpha^{s(2p-1-j)}\beta^{sj}\right)\\
&= (-1)^{s+1}\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s\right),
\end{aligned}$$

which proves (20).

Before we establish the next proposition, we would like to see how (18) and (19) have some interesting identities (involving s-Fibonomials) hidden. First observe that we can write (18) as follows:

$$\prod_{j=0}^{k}\left(z-\alpha^{sj}\beta^{s(k-j)}\right)=\sum_{i=0}^{k+1}(-1)^{\frac{i(s(i+1)+2(s+1))}{2}}\binom{k+1}{i}_{F_s}z^{k+1-i}. \tag{21}$$

Since

$$\alpha^{sj}\beta^{s(k-j)}=(-1)^{s(k+j)}\alpha^{s(2j-k)}=\frac{(-1)^{s(k+j)}}{2}\left(L_{s(2j-k)}+\sqrt{5}F_{s(2j-k)}\right),$$

we see that the numbers $a_{s,j,k}=\frac{(-1)^{s(k+j)}}{2}\left(L_{s(2j-k)}+\sqrt{5}F_{s(2j-k)}\right)$, $j,k\in\mathbb{Z}$, form an abelian multiplicative group $a_{s,j_1,k_1}a_{s,j_2,k_2}=a_{s,j_1+j_2,k_1+k_2}$, (with identity $a_{s,0,0}$ and inverses $a_{s,j,k}^{-1}=a_{s,-j,-k}$). Thus, (21) together with Viète's formulas tell us that

$$\begin{aligned}&(-1)^{\frac{r(s(r+1)+2(s+1))}{2}-ksr+r}2\binom{k+1}{r}_{F_s} \\ =&\sum_{0\le j_1<j_2<\cdots<j_r\le k}(-1)^{s(j_1+j_2+\cdots+j_r)}\left(L_{2s(j_1+j_2+\cdots+j_r)-ksr}+\sqrt{5}F_{2s(j_1+j_2+\cdots+j_r)-ksr}\right),\end{aligned} \tag{22}$$

where $r\in\{1,2,\ldots,k+1\}$ is given. Moreover, since the left-hand side of (22) is an integer, we have

$$\sum_{0\le j_1<j_2<\cdots<j_r\le k}(-1)^{s(j_1+j_2+\cdots+j_r)}L_{2s(j_1+j_2+\cdots+j_r)-ksr}=(-1)^{\frac{r(s(r+1)+2(s+1))}{2}-ksr+r}2\binom{k+1}{r}_{F_s}, \tag{23}$$

and

$$\sum_{0\le j_1<j_2<\cdots<j_r\le k}(-1)^{s(j_1+j_2+\cdots+j_r)}F_{2s(j_1+j_2+\cdots+j_r)-ksr}=0. \tag{24}$$

In particular we have the following identities, corresponding to (23) and (24) with $r=1$:

$$\sum_{j=0}^{k}(-1)^{sj}L_{s(2j-k)}=\frac{(-1)^{ks}}{F_s}2F_{s(k+1)} \quad , \quad \sum_{j=0}^{k}(-1)^{sj}F_{s(2j-k)}=0, \tag{25}$$

and with $r=2$:

$$\sum_{j=1}^{k}\sum_{i=0}^{j-1}(-1)^{s(i+j)}L_{2s(i+j-k)}=\frac{2(-1)^{s}}{F_sF_{2s}}F_{sk}F_{s(k+1)} \quad , \quad \sum_{j=1}^{k}\sum_{i=0}^{j-1}(-1)^{s(i+j)}F_{2s(i+j-k)}=0. \tag{26}$$

On the other hand, observe that according to (18) and (19) we have

$$\sum_{i=0}^{2p+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{2p+1}{i}_{F_s}z^{2p+1-i}=(-1)^{s+1}\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right),$$

which can be written as follows:

$$\sum_{i=0}^{p}\left((-1)^{\frac{(si+2(s+1))(i+1)}{2}}z^{2p+1-i}+(-1)^{\frac{(s(2p+1-i)+2(s+1))(2p+2-i)}{2}}z^{i}\right)\binom{2p+1}{i}_{F_s} \tag{27}$$
$$= (-1)^{s+1}(z-(-1)^{sp})\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right).$$

If s is even, $s=2\sigma$ say ($\sigma\in\mathbb{N}$), and we set $z=-1$ we get from (27) that

$$\sum_{i=0}^{p}\left((-1)^{\sigma i(i+1)}+(-1)^{\sigma i(i-1)}\right)\binom{2p+1}{i}_{F_{2\sigma}}=2\prod_{j=0}^{p-1}\left(2+L_{4\sigma(p-j)}\right).$$

That is, we have the identity

$$\sum_{i=0}^{p}\binom{2p+1}{i}_{F_{2\sigma}}=\prod_{j=1}^{p}L_{2\sigma j}^{2}. \tag{28}$$

Similarly, if s is odd, $s=2\sigma-1$ say ($\sigma\in\mathbb{N}$), formula (27) can be written as follows:

$$\sum_{i=0}^{p}\left((-1)^{\frac{i(i+1)}{2}}z^{2p+1-i}+(-1)^{\frac{(2p+1-i)(2p+2-i)}{2}}z^{i}\right)\binom{2p+1}{i}_{F_{2\sigma-1}} \tag{29}$$
$$= (z-(-1)^{p})\prod_{j=0}^{p-1}\left(z^2-(-1)^{j}L_{2(2\sigma-1)(p-j)}z+1\right).$$

If in (29) p is replaced by $2p-1$ and we set $z=1$, then we get

$$\sum_{i=0}^{2p-1}\left((-1)^{\frac{i(i+1)}{2}}+(-1)^{\frac{i(i+1)}{2}}\right)\binom{2(2p-1)+1}{i}_{F_{2\sigma-1}}=2\prod_{j=0}^{2p-2}\left(2-(-1)^{j}L_{2(2\sigma-1)(2p-1-j)}\right),$$

from where we obtain the identity

$$\sum_{i=0}^{2p-1}(-1)^{\frac{i(i+1)}{2}}\binom{4p-1}{i}_{F_{2\sigma-1}}=(-1)^{p}\prod_{j=1}^{2p-1}L_{(2\sigma-1)j}^{2}. \tag{30}$$

If now in (29) p is replaced by $2p$ and we set $z=-1$, then we get

$$\sum_{i=0}^{2p}2(-1)^{\frac{i(i-1)}{2}+1}\binom{4p+1}{i}_{F_{2\sigma-1}}=-2\prod_{j=0}^{2p-1}\left(2+(-1)^{j}L_{2(2\sigma-1)(2p-j)}\right),$$

from where we obtain the identity

$$\sum_{i=0}^{2p}(-1)^{\frac{i(i-1)}{2}}\binom{4p+1}{i}_{F_{2\sigma-1}}=(-1)^{p}\prod_{j=1}^{2p}L_{(2\sigma-1)j}^{2}. \tag{31}$$

Proposition 3. *Let $t,k\in\mathbb{N}'$, $m\in\mathbb{Z}$ be given. Then*

(a)

$$\frac{\alpha^{sk}}{\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\alpha^{si}z^{t+1-i}}+\frac{\beta^{sk}}{\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\beta^{si}z^{t+1-i}}$$
$$=\frac{L_{sk}z+(-1)^{sk+1}L_{s(t-k+1)}}{\sum_{i=0}^{t+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+2}{i}_{F_s}z^{t+2-i}}. \tag{32}$$

(b)

$$\frac{\alpha^{m+sk}}{\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\alpha^{si}z^{t+1-i}}-\frac{\beta^{m+sk}}{\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\beta^{si}z^{t+1-i}}$$
$$=\frac{\sqrt{5}\left(F_{sk+m}z+(-1)^{sk+m}F_{s(t-k+1)-m}\right)}{\sum_{i=0}^{t+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+2}{i}_{F_s}z^{t+2-i}}. \tag{33}$$

Proof. Observe that, according to proposition 2, we have that

$$\begin{aligned}\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\alpha^{si}z^{t+1-i} &= \alpha^{s(t+1)}\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\left(\frac{z}{\alpha^s}\right)^{t+1-i}\\ &= (-1)^{s+1}\alpha^{s(t+1)}\prod_{j=0}^{t}\left(\frac{z}{\alpha^s}-\alpha^{sj}\beta^{s(t-j)}\right)\\ &= (-1)^{s+1}\prod_{j=0}^{t}\left(z-\alpha^{s(j+1)}\beta^{s(t-j)}\right),\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\beta^{si}z^{t+1-i} &= (-1)^{s+1}\beta^{s(t+1)}\prod_{j=0}^{t}\left(\frac{z}{\beta^s}-\alpha^{sj}\beta^{s(t-j)}\right)\\ &= (-1)^{s+1}\prod_{j=0}^{t}\left(z-\alpha^{sj}\beta^{s(t-j+1)}\right).\end{aligned}$$

Then

$$\frac{\alpha^{sk}}{\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\alpha^{si}z^{t+1-i}}+\frac{\beta^{sk}}{\sum_{i=0}^{t+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+1}{i}_{F_s}\beta^{si}z^{t+1-i}}$$

$$= \frac{\alpha^{sk}}{(-1)^{s+1}\prod_{j=0}^{t}\left(z-\alpha^{s(j+1)}\beta^{s(t-j)}\right)}+\frac{\beta^{sk}}{(-1)^{s+1}\prod_{j=0}^{t}\left(z-\alpha^{sj}\beta^{s(t-(j-1))}\right)}$$

$$= \frac{\alpha^{sk}}{(-1)^{s+1}\prod_{j=1}^{t+1}\left(z-\alpha^{sj}\beta^{s(t-(j-1))}\right)}+\frac{\beta^{sk}}{(-1)^{s+1}\prod_{j=0}^{t}\left(z-\alpha^{sj}\beta^{s(t-(j-1))}\right)}$$

$$= \frac{\alpha^{sk}\left(z-\beta^{s(t+1)}\right)+\beta^{sk}\left(z-\alpha^{s(t+1)}\right)}{(-1)^{s+1}\prod_{j=0}^{t+1}\left(z-\alpha^{sj}\beta^{s(t+1-j)}\right)}$$

$$= \frac{L_{sk}z+(-1)^{sk+1}L_{s(t-k+1)}}{\sum_{i=0}^{t+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+2}{i}_{F_s}z^{t+2-i}},$$

which proves (a). The proof of (b) is similar. ∎

Lemma 4. *For given $i,t\in\mathbb{N}'$, the following identity holds*

$$(-1)^{si}F_{s(t+2-i)}F_{s(t+1-i)}+L_{s(t+1)}F_{s(t+2-i)}F_{si}+(-1)^{s(t+i)}F_{si}F_{s(i-1)}=F_{s(t+2)}F_{s(t+1)}. \quad (34)$$

Proof. Leaving the details for the reader, the proof is as follows: first, use α's and β's to prove that $F_{s(t+1-i)}(-1)^{si}+L_{s(t+1)}F_{si}=F_{s(t+1+i)}$. Then write the left-hand side of (34) as $F_{s(t+2-i)}F_{s(t+1+i)}+(-1)^{s(t+i)}F_{si}F_{s(i-1)}$. Finally use the standard identity $F_mF_n-F_{m+k}F_{n-k}=(-1)^{n-k}F_{m+k-n}F_k$ to obtain (34). ∎

Proposition 5. *Let $t\in\mathbb{N}'$ be given. Then*

$$\sum_{i=0}^{t+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+2}{i}_{F_s}z^{t+2-i} \quad (35)$$

$$= \left(z^2-L_{s(t+1)}z+(-1)^{s(t+1)}\right)\sum_{i=0}^{t}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t}{i}_{F_s}(-1)^{si}z^{t-i}.$$

Proof. We have

$$
\begin{aligned}
&\left(z^2 - L_{s(t+1)}z + (-1)^{s(t+1)}\right)\sum_{i=0}^{t}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t}{i}_{F_s}(-1)^{si}z^{t-i}\\
=\ &\sum_{i=0}^{t}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t}{i}_{F_s}(-1)^{si}z^{t+2-i} - L_{s(t+1)}\sum_{i=1}^{t+1}(-1)^{\frac{(si-s+2(s+1))i}{2}}\binom{t}{i-1}_{F_s}(-1)^{s(i-1)}z^{t+2-i}\\
&+(-1)^{s(t+1)}\sum_{i=2}^{t+2}(-1)^{\frac{(si-2s+2(s+1))(i-1)}{2}}\binom{t}{i-2}_{F_s}(-1)^{si}z^{t+2-i}\\
=\ &\sum_{i=0}^{t+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+2}{i}_{F_s}\frac{1}{F_{s(t+2)}F_{s(t+1)}}\begin{pmatrix}(-1)^{si}F_{s(t+2-i)}F_{s(t+1-i)}\\ +L_{s(t+1)}F_{s(t+2-i)}F_{si}\\ +(-1)^{s(t+i)}F_{si}F_{s(i-1)}\end{pmatrix}z^{t+2-i}\\
=\ &\sum_{i=0}^{t+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t+2}{i}_{F_s}z^{t+2-i},
\end{aligned}
$$

as wanted. In the last step we used lemma 4. ∎

Proposition 6. *For given $i, t \in \mathbb{N}'$ the following identity holds*

$$\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{t+1}{i-j}_{F_s}F_{stj+m} = (-1)^{\frac{is}{2}(i-1)+i+s+m}\binom{t}{i}_{F_s}F_{is-m}. \tag{36}$$

Proof. We proceed by induction on t. For $t = 0$ we need to check that

$$\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{1}{i-j}_{F_s}F_m = (-1)^{\frac{is}{2}(i-1)+i+s+m}\binom{0}{i}_{F_s}F_{is-m}.$$

If $i = 0$ we have a trivial equality (both sides are equal to $(-1)^{s+1}F_m$). If $i \geq 1$ we have

$$\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{1}{i-j}_{F_s}F_m = (-1)^{s+1}F_m + (-1)^sF_m = 0,$$

as expected. Suppose now that (36) is valid for a given t. We have (by using (2))

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{t+2}{i-j}_{F_s}F_{s(t+1)j+m}\\
=&\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}F_{s(t+1)j+m}\left(F_{s(t+2-i+j)+1}\binom{t+1}{i-j-1}_{F_s}+F_{s(i-j)-1}\binom{t+1}{i-j}_{F_s}\right)\\
=&\sum_{j=1}^{i+1}(-1)^{\frac{(s(i-j+1)+2(s+1))(i-j+2)}{2}}F_{s(t+1)(j-1)+m}F_{s(t+1-i+j)+1}\binom{t+1}{i-j}_{F_s}\\
&+\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}F_{s(t+1)j+m}F_{s(i-j)-1}\binom{t+1}{i-j}_{F_s}\\
=&\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{t+1}{i-j}_{F_s}\left(\begin{array}{c}(-1)^{s(i-j)+1}F_{s(t+1)(j-1)+m}F_{s(t+1-i+j)+1}\\ +F_{s(t+1)j+m}F_{s(i-j)-1}\end{array}\right)\\
&-(-1)^{\frac{(s(i+1)+2(s+1))(i+2)}{2}}F_{s(t+1)(-1)+m}F_{s(t+1-i)+1}\binom{t+1}{i}_{F_s}.
\end{aligned}$$

Next we will use the identity

$$\begin{aligned}
&(-1)^{s(i-j)+1}F_{s(t+1)(j-1)+m}F_{s(t+1-i+j)+1}+F_{s(t+1)j+m}F_{s(i-j)-1}\\
=&\frac{F_{s(t+1)}}{F_{st}}\left(F_{s(i-1)-1}F_{stj+m}+(-1)^{s(i-1)-1}F_{s(t-i+1)+1}F_{st(j-1)+m}\right),
\end{aligned}\tag{37}$$

which can be proved in two steps (we leave the details for the reader): first use α's and β's to prove that

$$(-1)^{s(i-j)+1}F_{s(t+1)(j-1)+m}F_{s(t+1-i+j)+1}+F_{s(t+1)j+m}F_{s(i-j)-1}=F_{s(t+1)}F_{st(j-1)+m+s(i-1)-1},$$

and then use (17) to write

$$F_{st(j-1)+m+s(i-1)-1}=\frac{1}{F_{st}}\left(F_{s(i-1)-1}F_{stj+m}+(-1)^{s(i-1)-1}F_{s(t-i+1)+1}F_{st(j-1)+m}\right),$$

obtaining in this way (37).

Thus, by using (37) we can write

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t+2}{j}_{F_s}F_{s(t+1)(i-j)+m}\\
=&\frac{F_{s(t+1)}}{F_{st}}\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{t+1}{i-j}_{F_s}\left(\begin{array}{c}F_{s(i-1)-1}F_{stj+m}\\ +(-1)^{s(i-1)-1}F_{s(t-i+1)+1}F_{st(j-1)+m}\end{array}\right)\\
&-(-1)^{\frac{(s(i+1)+2(s+1))(i+2)}{2}}F_{s(t+1)(-1)+m}F_{s(t+1-i)+1}\binom{t+1}{i},
\end{aligned}$$

and by using the induction hypothesis (together with some simplifications), we get

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t+2}{j}_{F_s}F_{s(t+1)(i-j)+m}\\
=\ &\frac{F_{s(t+1)}}{F_{st}}F_{s(i-1)-1}\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}F_{stj+m}\binom{t+1}{i-j}_{F_s}\\
&+\frac{F_{s(t+1)}}{F_{st}}F_{s(t-i+1)+1}(-1)^{s(i-1)-1}\sum_{j=0}^{i-1}(-1)^{\frac{(s(i-j-1)+2(s+1))(i-j)}{2}}F_{stj+m}\binom{t+1}{i-1-j}_{F_s}\\
&-(-1)^{\frac{(s(i+1)+2(s+1))(i+2)}{2}}F_{s(t+1)(-1)+m}F_{s(t+1-i)+1}\binom{t+1}{i}\\
&+\frac{F_{s(t+1)}}{F_{st}}F_{s(t-i+1)+1}(-1)^{s(i-1)-1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}F_{-st+m}\binom{t+1}{i}_{F_s}\\
=\ &\frac{F_{s(t+1)}}{F_{st}}F_{s(i-1)-1}(-1)^{\frac{is}{2}(i-1)+i+s+m}\binom{t}{i}_{F_s}F_{is-m}\\
&+\frac{F_{s(t+1)}}{F_{st}}F_{s(t-i+1)+1}(-1)^{s(i-1)-1}(-1)^{\frac{(i-1)s}{2}i+i-1+(2-i)s+m}\binom{t}{i-1}_{F_s}F_{(i-1)s-m}\\
&+\binom{t+1}{i}\left(\begin{array}{c}\frac{F_{s(t+1)}}{F_{st}}F_{s(t-i+1)+1}(-1)^{s(i-1)-1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}F_{-st+m}\\ -(-1)^{\frac{(s(i+1)+2(s+1))(i+2)}{2}}F_{s(t+1)(-1)+m}F_{s(t+1-i)+1}\end{array}\right)\\
=\ &(-1)^{\frac{is}{2}(i-1)+i+s+m}\binom{t+1}{i}_{F_s}\frac{1}{F_{st}}\left(F_{is-m}F_{s(i-1)-1}F_{s(t-i+1)}+F_{is}F_{(i-1)s-m}F_{s(t-i+1)+1}\right)\\
&+\binom{t+1}{i}(-1)^{\frac{is}{2}(i-1)+i+s+m}\frac{(-1)^{si+m}}{F_{st}}F_{s(t-i+1)+1}\left(\begin{array}{c}(-1)^{s}F_{s(t+1)}F_{-st+m}\\ -(-1)^{si+m}F_{st}F_{s(t+1)(-1)+m}\end{array}\right)\\
=\ &(-1)^{\frac{is}{2}(i-1)+i+s+m}\binom{t+1}{i}_{F_s}F_{is-m},
\end{aligned}$$

as wanted. In the last step we used the identity

$$\begin{aligned}
&F_{is-m}F_{s(i-1)-1}F_{s(t-i+1)}+F_{is}F_{(i-1)s-m}F_{s(t-i+1)+1}\\
&+(-1)^{si+m}F_{s(t-i+1)+1}\left((-1)^{s}F_{s(t+1)}F_{-st+m}-F_{st}F_{-s(t+1)+m}\right)\\
=\ &F_{st}F_{is-m},
\end{aligned}$$

which proof is an easy exercise left to the reader. ∎

The proof of the following proposition is similar to the proof of proposition 6, with some changes in signs and, of course, some F's substituted by L's. We leave it for the reader.

Proposition 7. *For given $i,t\in\mathbb{N}'$ the following identity holds*

$$\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{t+1}{i-j}_{F_s}L_{stj+m}=(-1)^{(i+1)\left(\frac{is}{2}+1\right)-is+m+s}\binom{t}{i}_{F_s}L_{is-m}.\tag{38}$$

3 The main results

We begin this section by noting that we can write explicitly the sequence $F_{sn+m_1}^{k_1}F_{sn+m_2}^{k_2}$ (where $m_1, m_2 \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{N}'$ are given) as follows:

$$\begin{aligned}
&F_{sn+m_1}^{k_1}F_{sn+m_2}^{k_2}\\
= &\left(\frac{\alpha^{sn+m_1}-\beta^{sn+m_1}}{\sqrt{5}}\right)^{k_1}\left(\frac{\alpha^{sn+m_2}-\beta^{sn+m_2}}{\sqrt{5}}\right)^{k_2}\\
= &5^{-\frac{k_1+k_2}{2}}\sum_{i=0}^{k_1}\binom{k_1}{i}\left(\alpha^{sn+m_1}\right)^i\left(-\beta^{sn+m_1}\right)^{k_1-i}\sum_{j=0}^{k_2}\binom{k_2}{j}\left(\alpha^{sn+m_2}\right)^j\left(-\beta^{sn+m_2}\right)^{k_2-j}\\
= &5^{-\frac{k_1+k_2}{2}}\beta^{m_1k_1+m_2k_2}\sum_{j=0}^{k_1+k_2}\sum_{i=0}^{k_1}(-1)^{k_1+k_2-j}\binom{k_1}{i}\binom{k_2}{j-i}\left(\frac{\alpha}{\beta}\right)^{(m_1-m_2)i+m_2j}\left(\alpha^{sj}\beta^{s(k_1+k_2-j)}\right)^n.
\end{aligned}$$

Then the Z transform of $F_{sn+m_1}^{k_1}F_{sn+m_2}^{k_2}$ is

$$\begin{aligned}
&\mathcal{Z}\left(F_{sn+m_1}^{k_1}F_{sn+m_2}^{k_2}\right) \qquad (39)\\
= &5^{-\frac{k_1+k_2}{2}}\beta^{m_1k_1+m_2k_2}\sum_{j=0}^{k_1+k_2}\sum_{i=0}^{k_1}(-1)^{k_1+k_2-j}\binom{k_1}{i}\binom{k_2}{j-i}\left(\frac{\alpha}{\beta}\right)^{(m_1-m_2)i+m_2j}\frac{z}{z-\alpha^{sj}\beta^{s(k_1+k_2-j)}}.
\end{aligned}$$

Our first main result says that this expression can be written in a special form.

Theorem 8. *Let $m_1, m_2 \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{N}'$ be given. The Z transform of the sequence $F_{sn+m_1}^{k_1}F_{sn+m_2}^{k_2}$ is*

$$\mathcal{Z}\left(F_{sn+m_1}^{k_1}F_{sn+m_2}^{k_2}\right) = z\frac{\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}z^{k_1+k_2-i}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{i}_{F_s}z^{k_1+k_2+1-i}}. \qquad (40)$$

Proof. We have to show that

$$\begin{aligned}
&5^{-\frac{k_1+k_2}{2}}\beta^{m_1k_1+m_2k_2}\sum_{j=0}^{k_1+k_2}\sum_{i=0}^{k_1}\binom{k_1}{i}\binom{k_2}{j-i}(-1)^{k_1+k_2-j}\left(\frac{\alpha}{\beta}\right)^{(m_1-m_2)i+m_2j}\frac{z}{z-\alpha^{sj}\beta^{s(k_1+k_2-j)}}\\
= &z\frac{\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}z^{k_1+k_2-i}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{i}_{F_s}z^{k_1+k_2+1-i}}. \qquad (41)
\end{aligned}$$

We will proceed by induction on k_1 and/or k_2. Observe that the case $k_1 = k_2 = 0$ is (15). Let us consider the case $k_1 = k_2 = 1$ and then we construct the induction argument on k_1

(note that (40) is symmetric with respect to k_1 and k_2). For the case $k_1 = k_2 = 1$ we use that $L_{2s} + (-1)^s = \frac{F_{3s}}{F_s}$ to write

$$
\begin{aligned}
& 5^{-\frac{1+1}{2}} \beta^{m_1+m_2} \sum_{i=0}^{1} \sum_{j=0}^{1} \binom{1}{i} \binom{1}{j} (-1)^{i+j} \left(\frac{\alpha}{\beta}\right)^{m_1 i + m_2 j} \frac{z}{z - \alpha^{s(i+j)} \beta^{s(2-i-j)}} \\
= \;& 5^{-1} z \left(\frac{\left(\alpha^{m_1+m_2} + \beta^{m_1+m_2}\right) z - \beta^{m_1+m_2}\alpha^{2s} - \alpha^{m_1+m_2}\beta^{2s}}{z^2 - L_{2s} z + 1} - \frac{\alpha^{m_1}\beta^{m_2} + \alpha^{m_2}\beta^{m_1}}{z - (-1)^s} \right) \\
= \;& 5^{-1} \frac{z}{z^3 - \left(L_{2s} + (-1)^s\right) z^2 + \left(1 + (-1)^s L_{2s}\right) z - (-1)^s} \times \\
& \times \left(\begin{array}{c} \left(\alpha^{m_1+m_2} + \beta^{m_1+m_2} - \alpha^{m_1}\beta^{m_2} - \alpha^{m_2}\beta^{m_1}\right) z^2 \\ - \left(\beta^{m_1+m_2}\alpha^{2s} + \alpha^{m_1+m_2}\beta^{2s} - \left(\alpha^{m_1}\beta^{m_2} + \alpha^{m_2}\beta^{m_1}\right)\left(\alpha^{2s} + \beta^{2s}\right) + (-1)^s \left(\alpha^{m_1+m_2} + \beta^{m_1+m_2}\right)\right) z \\ + (-1)^s \beta^{m_1+m_2}\alpha^{2s} + (-1)^s \alpha^{m_1+m_2}\beta^{2s} - \alpha^{m_1}\beta^{m_2} - \alpha^{m_2}\beta^{m_1} \end{array} \right) \\
= \;& \frac{z}{(-1)^{s+1} z^3 + (-1)^s \frac{F_{3s}}{F_s} z^2 - \frac{F_{3s}}{F_s} z + 1} \times \\
& \times \left(\begin{array}{c} (-1)^{s+1} F_{m_1} F_{m_2} z^2 + (-1)^{s+1} \left(F_{m_1+s} F_{m_2+s} - \frac{F_{3s}}{F_s} F_{m_1} F_{m_2}\right) z \\ + (-1)^{s+1} F_{m_1+2s} F_{m_2+2s} + (-1)^s \frac{F_{3s}}{F_s} F_{m_1+s} F_{m_2+s} - \frac{F_{3s}}{F_s} F_{m_1} F_{m_2} \end{array} \right) \\
= \;& z \frac{\sum_{i=0}^{2} \sum_{j=0}^{i} (-1)^{\frac{(sj + 2(s+1))(j+1)}{2}} \binom{3}{j}_{F_s} F_{m_1 + s(i-j)} F_{m_2 + s(i-j)} z^{2-i}}{\sum_{i=0}^{3} (-1)^{\frac{(si + 2(s+1))(i+1)}{2}} \binom{3}{i}_{F_s} z^{3-i}},
\end{aligned}
$$

which is (41) with $k_1 = k_2 = 1$. Suppose now that (41) is true for a given k_1. We will show that it is also true for $k_1 + 1$. We have

$$
\begin{aligned}
\mathcal{Z}\left(F_{sn+m_1}^{k_1+1} F_{sn+m_2}^{k_2}\right) \;=\;& 5^{-\frac{k_1+k_2+1}{2}} \beta^{m_1(k_1+1) + m_2 k_2} \sum_{j=0}^{k_1+k_2+1} \sum_{i=0}^{k_1+1} (-1)^{k_1+k_2+1-j} \binom{k_1+1}{i} \binom{k_2}{j-i} \times \\
& \times \left(\frac{\alpha}{\beta}\right)^{(m_1-m_2)i + m_2 j} \frac{z}{z - \alpha^{sj} \beta^{s(k_1+k_2+1-j)}}.
\end{aligned}
$$

If we use that $\binom{k_1+1}{i} = \binom{k_1}{i} + \binom{k_1}{i-1}$, separate in the corresponding two terms, and shift the indices of the second term, we get

$$
\begin{aligned}
& \mathcal{Z}\left(F_{sn+m_1}^{k_1+1} F_{sn+m_2}^{k_2}\right) \qquad (42) \\
= \;& -5^{-\frac{k_1+k_2+1}{2}} \beta^{m_1 k_1 + m_2 k_2} \beta^{m_1} \sum_{j=0}^{k_1+k_2} \sum_{i=0}^{k_1} (-1)^{k_1+1+k_2-j} \binom{k_1}{i} \binom{k_2}{j-i} \left(\frac{\alpha}{\beta}\right)^{(m_1-m_2)i + m_2 j} \frac{\frac{z}{\beta^s}}{\frac{z}{\beta^s} - \alpha^{sj} \beta^{s(k_1+k_2-j)}} \\
& + 5^{-\frac{k_1+k_2+1}{2}} \beta^{m_1 k_1 + m_2 k_2} \alpha^{m_1} \sum_{j=0}^{k_1+k_2} \sum_{i=0}^{k_1} (-1)^{k_1+k_2-j} \binom{k_1}{i} \binom{k_2}{j-i} \left(\frac{\alpha}{\beta}\right)^{(m_1-m_2)i + m_2 j} \frac{\frac{z}{\alpha^s}}{\frac{z}{\alpha^s} - \alpha^{sj} \beta^{s(k_1+k_2-j)}}.
\end{aligned}
$$

By using the induction hypothesis and proposition 3 (b), we can write (42) as

$$
\begin{aligned}
&\mathcal{Z}\left(F_{sn+m_1}^{k_1+1}F_{sn+m_2}^{k_2}\right)\\
=&\ \frac{\alpha^{m_1}}{5^{\frac{1}{2}}}\frac{\frac{z}{\alpha^s}\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}\left(\frac{z}{\alpha^s}\right)^{k_1+k_2-i}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{i}_{F_s}\left(\frac{z}{\alpha^s}\right)^{k_1+k_2+1-i}}\\
&-\frac{\beta^{m_1}}{5^{\frac{1}{2}}}\frac{\frac{z}{\beta^s}\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}\left(\frac{z}{\beta^s}\right)^{k_1+k_2-i}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{i}_{F_s}\left(\frac{z}{\beta^s}\right)^{k_1+k_2+1-i}}\\
=&\ \frac{z}{5^{\frac{1}{2}}}\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}\times\\
&\times\left(\begin{array}{c}\dfrac{\alpha^{m_1+si}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{i}_{F_s}\alpha^{si}z^{k_1+k_2+1-i}}\\ -\dfrac{\beta^{m_1+si}}{\sum_{i=0}^{k_1+k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+1}{i}_{F_s}\beta^{si}z^{k_1+k_2+1-i}}\end{array}\right)z^{k_1+k_2-i}\\
=&\ \frac{z\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}}{\sum_{i=0}^{k_1+k_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+2}{i}_{F_s}z^{k_1+k_2+2-i}}\times\\
&\times\left(F_{m_1+si}z+(-1)^{si+m_1}F_{s(k_1+k_2-i+1)-m_1}\right)z^{k_1+k_2-i}\\
=&\ \frac{z}{\sum_{i=0}^{k_1+k_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+2}{i}_{F_s}z^{k_1+k_2+2-i}}\times\\
&\times\left(\begin{array}{c}\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}\times\\ \times F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}F_{m_1+si}z^{k_1+k_2+1-i}\\ +\\ \sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}\times\\ \times F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}(-1)^{si+m_1}F_{s(k_1+k_2-i+1)-m_1}z^{k_1+k_2-i}\end{array}\right)
\end{aligned}
$$

Some further simplifications give us

$$
\begin{aligned}
&\mathcal{Z}\left(F_{sn+m_1}^{k_1+1}F_{sn+m_2}^{k_2}\right)\\
=\;& \frac{z}{\sum_{i=0}^{k_1+k_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+2}{i}_{F_s}z^{k_1+k_2+2-i}}\times\\
&\times\left(\begin{array}{c}\sum_{i=0}^{k_1+k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}\times\\
\times F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}F_{m_1+si}z^{k_1+k_2+1-i}\\
+\\
\sum_{i=1}^{k_1+k_2+1}\sum_{j=1}^{i}(-1)^{\frac{(sj-s+2(s+1))j}{2}}\binom{k_1+k_2+1}{j-1}_{F_s}\times\\
\times F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}(-1)^{si-s+m_1}F_{s(k_1+k_2-i+2)-m_1}z^{k_1+k_2+1-i}\end{array}\right)\\
=\;& \frac{z}{\sum_{i=0}^{k_1+k_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+2}{i}_{F_s}z^{k_1+k_2+2-i}}\times\\
&\times\sum_{i=0}^{k_1+k_2+1}\sum_{j=0}^{i}\left(\begin{array}{c}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}F_{m_1+s(i-j)}^{k_1}F_{m_2+s(i-j)}^{k_2}\binom{k_1+k_2+2}{j}_{F_s}\frac{1}{F_{s(k_1+k_2+2)}}\times\\
\times\left(F_{s(k_1+k_2+2-j)}F_{m_1+si}+(-1)^{m_1+1+s(i-j)}F_{sj}F_{s(k_1+k_2-i+2)-m_1}\right)\end{array}\right)z^{k_1+k_2-i+1}\\
=\;& z\frac{\sum_{i=0}^{k_1+k_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}F_{m_1+s(i-j)}^{k_1+1}F_{m_2+s(i-j)}^{k_2}\binom{k_1+k_2+2}{j}_{F_s}z^{k_1+k_2-i+1}}{\sum_{i=0}^{k_1+k_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+k_2+2}{i}_{F_s}z^{k_1+k_2+2-i}},
\end{aligned}
$$

as wanted. We used that

$$
\sum_{j=0}^{k_1+k_2+1}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+k_2+1}{j}_{F_s}F_{m_1+s(k_1+k_2+1-j)}^{k_1}F_{m_2+s(k_1+k_2+1-j)}^{k_2}=0,
$$

(which comes from the induction hypothesis). We also used in the last step that

$$
F_{s(k_1+k_2+2-j)}F_{m_1+si}+(-1)^{m_1+1+s(i-j)}F_{sj}F_{s(k_1+k_2-i+2)-m_1}=F_{s(k_1+k_2+2)}F_{m_1+s(i-j)},
$$

which is a direct consequence of the known identity $F_mF_n-F_{m+k}F_{n-k}=(-1)^{n-k}F_{m+k-n}F_k$. ∎

Our second important result of this section is the following theorem.

Theorem 9. *Let $m_1,m_2\in\mathbb{Z}$ and $t_1,t_2\in\mathbb{N}'$ be given. The Z transform of the sequence $F_{t_1sn+m_1}^{k_1}F_{t_2sn+m_2}^{k_2}$ is given by*

$$
\begin{aligned}
&\mathcal{Z}\left(F_{t_1sn+m_1}F_{t_2sn+m_2}\right) \qquad (43)\\
=\;& z\frac{\sum_{i=0}^{t_1+t_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+1}{j}_{F_s}F_{m_1+t_1s(i-j)}F_{m_2+t_2s(i-j)}z^{t_1+t_2-i}}{\sum_{i=0}^{t_1+t_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1+t_2+1}{i}_{F_s}z^{t_1+t_2+1-i}}.
\end{aligned}
$$

Proof. We will proceed by induction on the parameters t_1 and/or t_2. Observe that case $t_1 = t_2 = 0$ is trivial (it is essentially (15)) and in the case $t_1 = t_2 = 1$ the result is true by theorem 8. Suppose now the result is true for a given t_1 together with all its values $\leq t_1$, and let us prove it is also true for $t_1 + 1$. (As in the proof of 8, we have a symmetry property that allows us to proceed in this way.) We will use that

$$F_{(t_1+1)sn+m_1} = F_{t_1 sn+m_1} L_{sn} - (-1)^{sn} F_{(t_1-1)sn+m_1},$$

(easy to prove). Then we have that

$$\begin{aligned} \mathcal{Z}\left(F_{(t_1+1)sn+m_1} F_{t_2 sn+m_2}\right) &= \mathcal{Z}\left(\left(F_{t_1 sn+m_1} L_{sn} - (-1)^{sn} F_{(t_1-1)sn+m_1}\right) F_{t_2 sn+m_2}\right) \\ &= \mathcal{Z}\left(L_{sn} F_{t_1 sn+m_1} F_{t_2 sn+m_2}\right) - \mathcal{Z}\left((-1)^{sn} F_{(t_1-1)sn+m_1} F_{t_2 sn+m_2}\right). \end{aligned} \tag{44}$$

Now we use (10) and (13), together with the induction hypothesis to write

$$\begin{aligned} &\mathcal{Z}\left(F_{(t_1+1)sn+m_1} F_{t_2 sn+m_2}\right) \\ = \;& \frac{z}{\alpha^s} \frac{\sum_{i=0}^{t_1+t_2} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1+t_2+1}{j}_{F_s} F_{m_1+t_1 s(i-j)} F_{m_2+t_2 s(i-j)} \left(\frac{z}{\alpha^s}\right)^{t_1+t_2-i}}{\sum_{i=0}^{t_1+t_2+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{t_1+t_2+1}{i}_{F_s} \left(\frac{z}{\alpha^s}\right)^{t_1+t_2+1-i}} \\ &+ \frac{z}{\beta^s} \frac{\sum_{i=0}^{t_1+t_2} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1+t_2+1}{j}_{F_s} F_{m_1+t_1 s(i-j)} F_{m_2+t_2 s(i-j)} \left(\frac{z}{\beta^s}\right)^{t_1+t_2-i}}{\sum_{i=0}^{t_1+t_2+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{t_1+t_2+1}{i}_{F_s} \left(\frac{z}{\beta^s}\right)^{t_1+t_2+1-i}} \\ &- \frac{z}{(-1)^s} \frac{\sum_{i=0}^{t_1+t_2-1} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1+t_2}{j}_{F_s} F_{m_1+(t_1-1)s(i-j)} F_{m_2+t_2 s(i-j)} \left(\frac{z}{(-1)^s}\right)^{t_1-1+t_2-i}}{\sum_{i=0}^{t_1+t_2} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{t_1+t_2}{i}_{F_s} \left(\frac{z}{(-1)^s}\right)^{t_1+t_2-i}} \\ = \;& z \sum_{i=0}^{t_1+t_2} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1+t_2+1}{j}_{F_s} F_{m_1+t_1 s(i-j)} F_{m_2+t_2 s(i-j)} \times \\ &\times \left(\frac{\alpha^{si}}{\sum_{i=0}^{t_1+t_2+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{t_1+t_2+1}{i}_{F_s} \alpha^{si} z^{t_1+t_2+1-i}} + \frac{\beta^{si}}{\sum_{i=0}^{t_1+t_2+1} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{t_1+t_2+1}{i}_{F_s} \beta^{si} z^{t_1+t_2+1-i}} \right) z^{t_1+t_2-i} \\ &- z \frac{\sum_{i=0}^{t_1+t_2-1} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1+t_2}{j}_{F_s} F_{m_1+(t_1-1)s(i-j)} F_{m_2+t_2 s(i-j)} (-1)^{si} z^{t_1-1+t_2-i}}{\sum_{i=0}^{t_1+t_2} (-1)^{\frac{(si+2(s+1))(i+1)}{2}} \binom{t_1+t_2}{i}_{F_s} (-1)^{si} z^{t_1+t_2-i}}. \end{aligned}$$

Next we use proposition 3 (a) to obtain that

$$
\begin{aligned}
&= \frac{\frac{\mathcal{Z}\left(F_{(t_1+1)sn+m_1}F_{t_2sn+m_2}\right)}{z}}{\sum_{i=0}^{t_1+t_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1+t_2+2}{i}_{F_s} z^{t_1+t_2+2-i}} \times \\
&\quad \times \sum_{i=0}^{t_1+t_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+1}{j}_{F_s} F_{m_1+t_1s(i-j)}F_{m_2+t_2s(i-j)} \times \\
&\quad \times \left(L_{si}z+(-1)^{si+1}L_{s(t_1+t_2-i+1)}\right)z^{t_1+t_2-i} \\
&\quad - z\frac{\sum_{i=0}^{t_1+t_2-1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2}{j}_{F_s} F_{m_1+(t_1-1)s(i-j)}F_{m_2+t_2s(i-j)}(-1)^{si}z^{t_1-1+t_2-i}}{\sum_{i=0}^{t_1+t_2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1+t_2}{i}_{F_s}(-1)^{si}z^{t_1+t_2-i}},
\end{aligned}
$$

and then, by proposition 5 we have

$$
\begin{aligned}
&= \frac{\frac{\mathcal{Z}\left(F_{(t_1+1)sn+m_1}F_{t_2sn+m_2}\right)}{z}}{\sum_{i=0}^{t_1+t_2+2}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1+t_2+2}{i}_{F_s} z^{t_1+t_2+2-i}} \times \\
&\times \left(\begin{array}{c}
\sum_{i=0}^{t_1+t_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+1}{j}_{F_s} F_{m_1+t_1s(i-j)}F_{m_2+t_2s(i-j)} \times \\
\times \left(L_{si}z+(-1)^{si+1}L_{s(t_1+t_2-i+1)}\right)z^{t_1+t_2-i} \\
\\
-\sum_{i=0}^{t_1+t_2-1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2}{j}_{F_s} F_{m_1+(t_1-1)s(i-j)}F_{m_2+t_2s(i-j)} \times \\
\times (-1)^{si}z^{t_1-1+t_2-i}\left(z^2-L_{s(t_1+t_2+1)}z+(-1)^{s(t_1+t_2+1)}\right)
\end{array}\right).
\end{aligned} \tag{45}
$$

We have now the expected denominator (of (43) with t_1 replaced by t_1+1). Let us work with the numerator of (45), $z(A-B)$ say, where

$$
\begin{aligned}
A &= \sum_{i=0}^{t_1+t_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+1}{j}_{F_s} F_{m_1+t_1s(i-j)}F_{m_2+t_2s(i-j)} \times \\
&\quad \times \left(L_{si}z+(-1)^{si+1}L_{s(t_1+t_2-i+1)}\right)z^{t_1+t_2-i},
\end{aligned}
$$

and

$$
\begin{aligned}
B &= \sum_{i=0}^{t_1+t_2-1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2}{j}_{F_s} F_{m_1+(t_1-1)s(i-j)}F_{m_2+t_2s(i-j)} \times \\
&\quad \times (-1)^{si}z^{t_1-1+t_2-i}\left(z^2-L_{s(t_1+t_2+1)}z+(-1)^{s(t_1+t_2+1)}\right).
\end{aligned}
$$

We have that

$$\begin{aligned} A &= \sum_{i=0}^{t_1+t_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+1}{j}_{F_s}F_{m_1+t_1s(i-j)}F_{m_2+t_2s(i-j)}L_{si}z^{t_1+t_2+1-i} \\ &\quad+\sum_{i=1}^{t_1+t_2+1}\sum_{j=1}^{i}(-1)^{\frac{(sj-s+2(s+1))j}{2}}\binom{t_1+t_2+1}{j-1}_{F_s}\times \\ &\quad\times F_{m_1+t_1s(i-j)}F_{m_2+t_2s(i-j)}(-1)^{s(i-1)+1}L_{s(t_1+t_2-i+2)}z^{t_1+t_2+1-i} \\ &= \sum_{i=0}^{t_1+t_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+2}{j}_{F_s}\frac{F_{m_2+t_2s(i-j)}F_{m_1+t_1s(i-j)}}{F_{s(t_1+t_2+2)}}\times \\ &\quad\times\left(F_{s(t_1+t_2+2-j)}L_{si}+(-1)^{s(i-j)}F_{sj}L_{s(t_1+t_2-i+2)}\right)z^{t_1+1+t_2-i}. \end{aligned}$$

But we have the identity

$$F_{s(t_1+t_2+2-j)}L_{si}+(-1)^{s(i-j)}F_{sj}L_{s(t_1+t_2-i+2)}=L_{s(i-j)}F_{s(t_1+t_2+2)},$$

thus

$$A=\sum_{i=0}^{t_1+t_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+2}{j}_{F_s}F_{m_2+t_2s(i-j)}F_{m_1+t_1s(i-j)}L_{s(i-j)}z^{t_1+1+t_2-i}.$$

Now let us work with B :

$$\begin{aligned} B &= \sum_{i=0}^{t_1+t_2-1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2}{j}_{F_s}F_{m_1+(t_1-1)s(i-j)}F_{m_2+t_2s(i-j)}(-1)^{si}z^{t_1+1+t_2-i} \\ &\quad-\sum_{i=1}^{t_1+t_2}\sum_{j=1}^{i}(-1)^{\frac{(sj-s+2(s+1))j}{2}}\binom{t_1+t_2}{j-1}_{F_s}\begin{pmatrix} F_{m_1+(t_1-1)s(i-j)}F_{m_2+t_2s(i-j)}\times \\ \times L_{s(t_1+t_2+1)}(-1)^{s(i-1)}z^{t_1+t_2+1-i}\end{pmatrix} \\ &\quad+\sum_{i=2}^{t_1+t_2+1}\sum_{j=2}^{i}(-1)^{\frac{(sj-2s+2(s+1))(j-1)}{2}}\binom{t_1+t_2}{j-2}_{F_s}\begin{pmatrix} F_{m_1+(t_1-1)s(i-j)}F_{m_2+t_2s(i-j)}\times \\ \times(-1)^{si}z^{t_1+1+t_2-i}(-1)^{s(t_1+t_2+1)}\end{pmatrix} \\ &= \sum_{i=0}^{t_1+t_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+2}{j}_{F_s}\frac{F_{m_2+t_2s(i-j)}F_{m_1+(t_1-1)s(i-j)}}{F_{s(t_1+t_2+2)}F_{s(t_1+t_2+1)}}\times \\ &\quad\times(-1)^{s(i-j)}\begin{pmatrix}(-1)^{sj}F_{s(t_1+t_2+1-j)}F_{s(t_1+t_2+2-j)} \\ +F_{sj}F_{s(t_1+t_2+2-j)}L_{s(t_1+t_2+1)} \\ +(-1)^{s(t_1+t_2+j)}F_{sj}F_{s(j-1)}\end{pmatrix}z^{t_1+1+t_2-i} \\ &= \sum_{i=0}^{t_1+t_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+2}{j}_{F_s}(-1)^{s(i-j)}F_{m_2+t_2s(i-j)}F_{m_1+(t_1-1)s(i-j)}z^{t_1+1+t_2-i}. \end{aligned}$$

(We used lemma 4 in the last step.)

Thus, the numerator of (45) is

$$z\left(A-B\right) = z\sum_{i=0}^{t_1+t_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+2}{j}_{F_s}F_{m_2+t_2s(i-j)}\times \times\left(F_{m_1+t_1s(i-j)}L_{s(i-j)}-(-1)^{s(i-j)}F_{m_1+(t_1-1)s(i-j)}\right)z^{t_1+1+t_2-i}. \tag{46}$$

Finally, with a simple calculation with α's and β's (left to the reader), we see that

$$F_{m_1+t_1s(i-j)}L_{s(i-j)}-(-1)^{s(i-j)}F_{m_1+(t_1-1)s(i-j)}=F_{m_1+(t_1+1)s(i-j)},$$

so (46) is

$$z\left(A-B\right)=z\sum_{i=0}^{t_1+t_2+1}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+t_2+2}{j}_{F_s}F_{m_1+(t_1+1)s(i-j)}F_{m_2+t_2s(i-j)}z^{t_1+1+t_2-i},$$

which is the expected numerator (of (43) with t_1 replaced by t_1+1). This ends our induction argument. ∎

The natural generalization of (40) (theorem 8) is clearly as follows:

$$\mathcal{Z}\left(F_{sn+m_1}^{k_1}\cdots F_{sn+m_l}^{k_l}\right) = z\frac{\sum_{i=0}^{k_1+\cdots+k_l}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1+\cdots+k_l+1}{j}_{F_s}F_{m_1+s(i-j)}^{k_1}\cdots F_{m_l+s(i-j)}^{k_l}z^{k_1+\cdots+k_l-i}}{\sum_{i=0}^{k_1+\cdots+k_l+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1+\cdots+k_l+1}{i}_{F_s}z^{k_1+\cdots+k_l+1-i}}, \tag{47}$$

(which can be proved with the same sort of arguments used in the proof of (40)), and the natural generalization of (43) (theorem 9) is

$$\mathcal{Z}\left(F_{t_1sn+m_1}\cdots F_{t_lsn+m_l}\right) = z\frac{\sum_{i=0}^{t_1+\cdots+t_l}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1+\cdots+t_l+1}{j}_{F_s}F_{m_1+t_1s(i-j)}\cdots F_{m_l+t_ls(i-j)}z^{t_1+\cdots+t_l-i}}{\sum_{i=0}^{t_1+\cdots+t_l+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1+\cdots+t_l+1}{i}_{F_s}z^{t_1+\cdots+t_l+1-i}}. \tag{48}$$

But then, from (48) we see that

$$\mathcal{Z}\left(F_{t_1sn+m_1}^{k_1}\cdots F_{t_lsn+m_l}^{k_l}\right) = z\frac{\sum_{i=0}^{k_1t_1+\cdots+k_lt_l}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{k_1t_1+\cdots+k_lt_l+1}{j}_{F_s}F_{m_1+t_1s(i-j)}^{k_1}\cdots F_{m_l+t_ls(i-j)}^{k_l}z^{k_1t_1+\cdots+k_lt_l-i}}{\sum_{i=0}^{k_1t_1+\cdots+k_lt_l+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{k_1t_1+\cdots+k_lt_l+1}{i}_{F_s}z^{k_1t_1+\cdots+k_lt_l+1-i}}. \tag{49}$$

That is, the generalization (47) is indeed a particular case of (49). This is the result we wanted to prove in this section.

4 Some corollaries

Our starting point in this section is formula (49). We want to present here some consequences of it.

Corollary 10. *For $p \in \mathbb{N}'$ given, the Z transform of the sequence $\binom{n}{p}_{F_s}$ is*

$$\mathcal{Z}\left(\binom{n}{p}_{F_s}\right) = \frac{(-1)^{s+1} z}{D_{s,p+1}(z)}. \tag{50}$$

Proof. We use (49) to write

$$\begin{aligned}\mathcal{Z}\left(\binom{n}{p}_{F_s}\right) &= \frac{1}{(F_p!)_s}\mathcal{Z}\left(F_{s(n-p+1)}F_{s(n-p+2)}\cdots F_{sn}\right) \\ &= \frac{1}{(F_p!)_s} z \frac{\sum_{i=0}^{p}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{p+1}{j}_{F_s}F_{s(1-p+i-j)}F_{s(2-p+i-j)}\cdots F_{s(i-j)}z^{p-i}}{\sum_{i=0}^{p+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{p+1}{i}_{F_s}z^{p+1-i}}.\end{aligned}$$

But the product $F_{s(1-p+i-j)}F_{s(2-p+i-j)}\cdots F_{s(i-j)}$ is different from zero if and only if $i = p$ and $j = 0$. In such a case that product is $(F_p!)_s$ and the numerator reduces to $(-1)^{s+1}(F_p!)_s$. Thus (50) follows. ∎

(Formula (50) corresponding to $s = 1$ was proved by a different method in [13].)

Observe that according to the advance-shifting property (9) and (50), if $0 \leq p_0 \leq p$, then

$$\mathcal{Z}\left(\binom{n+p_0}{p}_{F_s}\right) = z^{p_0}\frac{(-1)^{s+1} z}{\sum_{i=0}^{p+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{p+1}{i}_{F_s}z^{p+1-i}}. \tag{51}$$

By using that $G_{sn+m} = G_m F_{sn-1} + G_{m+1} F_{sn}$, we can see that formula (49) is valid for Gibonacci sequences G_n replacing the Fibonacci ones. In fact (it suffices to check the case

$G^{k_1}_{st_1n+m_1}G^{k_2}_{st_2n+m_2}$), we have

$$
\begin{aligned}
&\mathcal{Z}\left(G^{k_1}_{st_1n+m_1}G^{k_2}_{st_2n+m_2}\right)\\
=&\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2}G^{l_1}_{m_1}G^{k_1-l_1}_{m_1+1}G^{l_2}_{m_2}G^{k_2-l_2}_{m_2+1}\mathcal{Z}\left(F^{l_1}_{st_1n-1}F^{k_1-l_1}_{st_1n}F^{l_2}_{st_2n-1}F^{k_2-l_2}_{st_2n}\right)\\
=&\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2}G^{l_1}_{m_1}G^{k_1-l_1}_{m_1+1}G^{l_2}_{m_2}G^{k_2-l_2}_{m_2+1}z\times\\
&\times\frac{\sum_{i=0}^{t_1k_1+t_2k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1k_1+t_2k_2+1}{j}_{F_s}F^{l_1}_{st_1(i-j)-1}F^{k_1-l_1}_{st_1(i-j)}F^{l_2}_{st_2(i-j)-1}F^{k_2-l_2}_{st_2(i-j)}z^{t_1k_1+t_2k_2-i}}{\sum_{i=0}^{t_1k_1+t_2k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1k_1+t_2k_2+1}{i}_{F_s}z^{t_1k_1+t_2k_2+1-i}}\\
=&\frac{z}{\sum_{i=0}^{t_1k_1+t_2k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1k_1+t_2k_2+1}{i}_{F_s}z^{t_1k_1+t_2k_2+1-i}}\times\\
&\times\left(\begin{array}{c}\sum_{i=0}^{t_1k_1+t_2k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1k_1+t_2k_2+1}{j}_{F_s}\times\\
\times\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2}G^{l_1}_{m_1}G^{k_1-l_1}_{m_1+1}G^{l_2}_{m_2}G^{k_2-l_2}_{m_2+1}F^{l_1}_{st_1(i-j)-1}F^{k_1-l_1}_{st_1(i-j)}F^{l_2}_{st_2(i-j)-1}F^{k_2-l_2}_{st_2(i-j)}z^{t_1k_1+t_2k_2-i}\end{array}\right)\\
=&z\frac{\sum_{i=0}^{t_1k_1+t_2k_2}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1k_1+t_2k_2+1}{j}_{F_s}G^{k_1}_{st_1(i-j)+m_1}G^{k_2}_{st_2(i-j)+m_2}z^{t_1k_1+t_2k_2-i}}{\sum_{i=0}^{t_1k_1+t_2k_2+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1k_1+t_2k_2+1}{j}_{F_s}z^{t_1k_1+t_2k_2+1-i}},
\end{aligned}
$$

as expected. In general we have

$$
\begin{aligned}
&\mathcal{Z}\left(G^{k_1}_{st_1n+m_1}\cdots G^{k_l}_{st_ln+m_l}\right) \qquad (52)\\
=&z\frac{\sum_{i=0}^{t_1k_1+\cdots+t_lk_l}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1k_1+\cdots+t_lk_l+1}{j}_{F_s}G^{k_1}_{m_1+st_1(i-j)}\cdots G^{k_l}_{m_l+st_l(i-j)}z^{t_1k_1+\cdots+t_lk_l-i}}{\sum_{i=0}^{t_1k_1+\cdots+t_lk_l+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1k_1+\cdots+t_lk_l+1}{i}_{F_s}z^{t_1k_1+\cdots+t_lk_l+1-i}}.
\end{aligned}
$$

Corollary 11. *Let $m_1,\ldots,m_l\in\mathbb{Z}$ and $t_1,\ldots,t_l,k_1,\ldots,k_l\in\mathbb{N}'$ be given. The sequence $\prod_{i=1}^{l}G^{k_i}_{t_isn+m_i}$ can be expressed as a linear combination of s-Fibonomials $\binom{n+t_1k_1+\cdots+t_lk_l-i}{t_1k_1+\cdots+t_lk_l}_{F_s}$, $i=0,1,\ldots,t_1k_1+\cdots+t_lk_l$, according to*

$$
\begin{aligned}
G^{k_1}_{t_1sn+m_1}\cdots G^{k_l}_{t_lsn+m_l} \;=\;& (-1)^{s+1}\sum_{i=0}^{t_1k_1+\cdots+t_lk_l}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1k_1+\cdots+t_lk_l+1}{j}_{F_s}\times\\
&\times G^{k_1}_{m_1+t_1s(i-j)}\cdots G^{k_l}_{m_l+t_ls(i-j)}\binom{n+t_1k_1+\cdots+t_lk_l-i}{t_1k_1+\cdots+t_lk_l}_{F_s}. \qquad (53)
\end{aligned}
$$

Proof. This follows directly from (52) and (51). ∎

Some examples of (53) are the following (after simplifications on the coefficients of the s-Fibonomials sequences of the right-hand side):

$$F_{2sn+m}=F_m\binom{n+2}{2}_{F_s}+(-1)^m F_{s-m}L_s\binom{n+1}{2}_{F_s}+(-1)^{s+m+1}F_{2s-m}\binom{n}{2}_{F_s}. \tag{54}$$

$$F_{sn}^3=F_s^3\left(\binom{n+2}{3}_{F_s}+2(-1)^s L_s\binom{n+1}{3}_{F_s}+(-1)^s\binom{n}{3}_{F_s}\right). \tag{55}$$

$$F_{2sn}^2=F_{2s}^2\left(\binom{n+3}{4}_{F_s}+\binom{n}{4}_{F_s}+(-1)^{s+1}\frac{L_{3s}}{L_s}\left(\binom{n+2}{4}_{F_s}+\binom{n+1}{4}_{F_s}\right)\right). \tag{56}$$

$$L_{sn}^2=4\binom{n+2}{2}_{F_s}-\left(3L_s^2+4(-1)^{s+1}\right)\binom{n+1}{2}_{F_s}+(-1)^s L_s^2\binom{n}{2}_{F_s}. \tag{57}$$

$$F_{sn}^4=F_s^4\left(\binom{n+3}{4}_{F_s}+\binom{n}{4}_{F_s}+\left(3(-1)^s\frac{F_{3s}}{F_s}+2\right)\left(\binom{n+2}{4}_{F_s}+\binom{n+1}{4}_{F_s}\right)\right). \tag{58}$$

$$L_{2sn}F_{sn}=F_s\left(L_{2s}\binom{n+2}{3}_{F_s}-2L_s\binom{n+1}{3}_{F_s}+(-1)^s L_{2s}\binom{n}{3}_{F_s}\right). \tag{59}$$

$$L_{2sn}F_{sn}^2=F_s^2\left(L_{2s}\left(\binom{n+3}{4}_{F_s}+\binom{n}{4}_{F_s}\right)+(-1)^s(5F_sF_{3s}-2)\left(\binom{n+2}{4}_{F_s}+\binom{n+1}{4}_{F_s}\right)\right). \tag{60}$$

From (53) we see that

$$G_{tsn+m}=(-1)^{s+1}\sum_{i=0}^{t}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t+1}{j}_{F_s}G_{ts(i-j)+m}\binom{n+t-i}{t}_{F_s}. \tag{61}$$

But when $G=F$ or $G=L$, this formula can be written in a simpler form.

Corollary 12. *Let $t\in\mathbb{N}'$ be given. The following identities hold*

(a)

$$F_{tsn+m}=\sum_{i=0}^{t}(-1)^{\frac{is(i-1)}{2}+i+m+1}\binom{t}{i}_{F_s}F_{is-m}\binom{n+t-i}{t}_{F_s}. \tag{62}$$

(b)

$$L_{tsn+m}=\sum_{i=0}^{t}(-1)^{\frac{is(i-1)}{2}+i+m}\binom{t}{i}_{F_s}L_{is-m}\binom{n+t-i}{t}_{F_s}. \tag{63}$$

Proof. We have that

$$\begin{aligned} F_{tsn+m} &= (-1)^{s+1}\sum_{i=0}^{t}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t+1}{j}_{F_s}F_{ts(i-j)+m}\binom{n+t-i}{t}_{F_s} \\ &= (-1)^{s+1}\sum_{i=0}^{t}\sum_{j=0}^{i}(-1)^{\frac{(s(i-j)+2(s+1))(i-j+1)}{2}}\binom{t+1}{i-j}_{F_s}F_{tsj+m}\binom{n+t-i}{t}_{F_s}. \end{aligned}$$

According to proposition 6 we can write

$$\begin{aligned} F_{tsn+m} &= (-1)^{s+1}\sum_{i=0}^{t}(-1)^{\frac{is}{2}(i-1)+i+s+m}\binom{t}{i}_{F_s}F_{is-m}\binom{n+t-i}{t}_{F_s} \\ &= \sum_{i=0}^{t}(-1)^{\frac{is}{2}(i-1)+i+m+1}\binom{t}{i}_{F_s}F_{is-m}\binom{n+t-i}{t}_{F_s}, \end{aligned}$$

which proves (62).

The proof of (63) is similar (using proposition 7). ∎

In the following corollary we consider sequences involving "s-Gibonomials" $\binom{n}{p}_{G_s}=\frac{G_{sn}\cdots G_{s(n-p+1)}}{G_s\cdots G_{sp}}$.

Corollary 13. *Let $t_1,\ldots,t_l\in\mathbb{N}$ and $r_1\ldots,r_l,p_1\ldots,p_l\in\mathbb{N}'$ be given. Then the Z transform of the sequence $\binom{n}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{n}{p_k}_{G_{st_k}}^{r_k}$ is given by*

$$\begin{aligned} &\mathcal{Z}\left(\binom{n}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{n}{p_k}_{G_{st_k}}^{r_k}\right) \qquad (64)\\ =&\frac{z}{\sum\limits_{i=0}^{t_1r_1p_1+\cdots+t_kr_kp_k+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1r_1p_1+\cdots+t_kr_kp_k+1}{i}_{F_s}z^{t_1r_1p_1+\cdots+t_kr_kp_k+1-i}}\times \\ &\times\sum_{i=0}^{t_1r_1p_1+\cdots+t_kr_kp_k}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1r_1p_1+\cdots+t_kr_kp_k+1}{j}_{F_s}\times \\ &\times\binom{i-j}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{i-j}{p_k}_{G_{st_k}}^{r_k}z^{t_1r_1p_1+\cdots+t_kr_kp_k-i}. \end{aligned}$$

Proof. First we write

$$\begin{aligned} &\mathcal{Z}\left(\binom{n}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{n}{p_k}_{G_{st_k}}^{r_k}\right) \\ =&\frac{1}{G_{st_1}^{r_1}\cdots G_{st_1p_1}^{r_1}\cdots G_{st_k}^{r_k}\cdots G_{st_kp_k}^{r_k}}\mathcal{Z}\left(G_{st_1n}^{r_1}\cdots G_{st_1(n-p_1+1)}^{r_1}\cdots G_{st_kn}^{r_k}\cdots G_{st_k(n-p_k+1)}^{r_k}\right), \end{aligned}$$

and then we use (52) to get

$$
\begin{aligned}
&\mathcal{Z}\left(\binom{n}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{n}{p_k}_{G_{st_k}}^{r_k}\right)\\
=\;& \frac{1}{G_{st_1}^{r_1}\cdots G_{st_1p_1}^{r_1}\cdots G_{st_k}^{r_k}\cdots G_{st_kp_k}^{r_k}}\times\\
&\times\frac{z}{\sum_{i=0}^{t_1r_1p_1+\cdots+t_kr_kp_k+1}(-1)^{\frac{(si+2(s+1))(i+1)}{2}}\binom{t_1r_1p_1+\cdots+t_kr_kp_k+1}{i}_{F_s}z^{t_1r_1p_1+\cdots+t_kr_kp_k+1-i}}\times\\
&\times\sum_{i=0}^{t_1r_1p_1+\cdots+t_kr_kp_k}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1r_1p_1+\cdots+t_kr_kp_k+1}{j}_{F_s}\times\\
&\times G_{st_1(i-j)}^{r_1}\cdots G_{st_1(i-j-p_1+1)}^{r_1}\cdots G_{st_k(i-j)}^{r_k}\cdots G_{st_k(i-j-p_k+1)}^{r_k}z^{t_1r_1p_1+\cdots+t_kr_kp_k-i},
\end{aligned}
$$

which implies the desired formula (64). ∎

Corollary 14. *Let* $t_1,\ldots,t_l\in\mathbb{N}$ *and* $r_1\ldots,r_l,p_1\ldots,p_l\in\mathbb{N}'$ *be given. The sequence* $\prod_{i=1}^{k}\binom{n}{p_i}_{G_{st_i}}^{r_i}$ *can be expressed as a linear combination of s-Fibonomials* $\binom{n+t_1r_1p_1+\cdots+t_kr_kp_k-i}{t_1r_1p_1+\cdots+t_kr_kp_k}_{F_s}$, $i=0,1,\ldots,t_1r_1p_1+\cdots+t_kr_kp_k$, *according to*

$$
\begin{aligned}
\prod_{i=1}^{k}\binom{n}{p_i}_{G_{st_i}}^{r_i} =\;& (-1)^{s+1}\sum_{i=0}^{t_1r_1p_1+\cdots+t_kr_kp_k}\sum_{j=0}^{i}(-1)^{\frac{(sj+2(s+1))(j+1)}{2}}\binom{t_1r_1p_1+\cdots+t_kr_kp_k+1}{j}_{F_s}\times\\
&\times\binom{i-j}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{i-j}{p_k}_{G_{st_k}}^{r_k}\binom{n+t_1r_1p_1+\cdots+t_kr_kp_k-i}{t_1r_1p_1+\cdots+t_kr_kp_k}_{F_s}. \qquad (65)
\end{aligned}
$$

Proof. This comes directly from (64) and (51). ∎

Some examples of (65) are the following (after simplifications on the coefficients of the s-Fibonomials sequences of the right-hand side):

$$\binom{n}{2}_{F_{2s}}=\binom{n+2}{4}_{F_s}+(-1)^{s+1}L_{2s}\binom{n+1}{4}_{F_s}+\binom{n}{4}_{F_s}. \qquad (66)$$

$$\binom{n}{2}_{F_s}\binom{n}{3}_{F_s}=\frac{F_{3s}}{F_s}\binom{n+2}{5}_{F_s}+\frac{L_sF_{3s}}{F_s}\binom{n+1}{5}_{F_s}+\binom{n}{5}_{F_s}. \qquad (67)$$

$$\binom{n}{2}_{F_s}^2=\binom{n+2}{4}_{F_s}+(-1)^{s}L_s^2\binom{n+1}{4}_{F_s}+\binom{n}{4}_{F_s}. \qquad (68)$$

$$\binom{n}{2}_{L_s}=\frac{2(-1)^s}{L_{2s}}\binom{n+2}{2}_{F_s}+2(-1)^{s+1}\binom{n+1}{2}_{F_s}+\binom{n}{2}_{F_s}. \qquad (69)$$

$$\binom{n}{2}_{L_s}\binom{n}{2}_{F_s}=\binom{n+2}{4}_{F_s}+(-1)^{s+1}L_{2s}\binom{n+1}{4}_{F_s}+\binom{n}{4}_{F_s}. \qquad (70)$$

$$\binom{n}{2}_{L_s}\binom{n}{3}_{F_s} = \frac{L_{3s}}{L_s}\binom{n+2}{5}_{F_s} - L_{3s}\binom{n+1}{5}_{F_s} + \binom{n}{5}_{F_s}. \tag{71}$$

$$\begin{aligned}\binom{n}{2}_{L_s}\binom{n}{2}_{F_s}^2 &= \binom{n+4}{6}_{F_s} + L_{2s}\binom{n+3}{6}_{F_s} + (-1)^s L_{2s}\left(L_{2s}^2-1\right)\binom{n+2}{6}_{F_s} \\ &\quad + L_{2s}\binom{n+1}{6}_{F_s} + \binom{n}{6}_{F_s}.\end{aligned} \tag{72}$$

Corollary 15. *(a) Let $m_1,\ldots,m_l \in \mathbb{Z}$ and $t_1,\ldots,t_l,k_1\ldots,k_l \in \mathbb{N}'$ be given. For $n \geq k_1+\cdots+k_l+1$ we have that*

$$\sum_{j=0}^{t_1k_1+\cdots+t_lk_l+1} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1k_1+\cdots+t_lk_l+1}{j}_{F_s} G^{k_1}_{m_1+st_1(n-j)}\cdots G^{k_l}_{m_l+t_ls(n-j)} = 0. \tag{73}$$

(b) Let $t_1,\ldots,t_l \in \mathbb{N}$ and $r_1\ldots,r_l,p_1\ldots,p_l \in \mathbb{N}'$ be given. For $n \geq t_1r_1p_1+\cdots+t_kr_kp_k+1$ we have that

$$\sum_{j=0}^{t_1r_1p_1+\cdots+t_kr_kp_k+1} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{t_1r_1p_1+\cdots+t_kr_kp_k+1}{j}_{F_s} \binom{n-j}{p_1}_{G_{st_1}}^{r_1}\cdots\binom{n-j}{p_k}_{G_{st_k}}^{r_k} = 0. \tag{74}$$

Proof. These results are direct consequences of (the numerators in) formulas (52) and (64). ∎

Corollary 16. *Let $p \in \mathbb{N}'$ be given. The following identities hold*

(a)

$$\binom{n+1}{p+2}_{F_s} = \frac{1}{F_{s(p+2)}} F_{s(p+2)n} * (-1)^{s(n+p)}\binom{n}{p}_{F_s}. \tag{75}$$

(b)

$$\binom{n+2}{p+4}_{F_s} = \frac{1}{F_{s(p+4)}F_{s(p+2)}} F_{s(p+4)n} * (-1)^{s(n+1)} F_{s(p+2)n} * \binom{n}{p}_{F_s}. \tag{76}$$

Proof. (a) First observe that

$$\begin{aligned} & D_{s,p+3}(z) \\ =\ & \prod_{j=0}^{p+2}\left(z-\alpha^{sj}\beta^{s(p+2-j)}\right) = \prod_{j=-1}^{p+1}\left(z-\alpha^{s(j+1)}\beta^{s(p+1-j)}\right) = \prod_{j=-1}^{p+1}\left(z-(-1)^s\alpha^{sj}\beta^{s(p-j)}\right) \\ =\ & \left(z-(-1)^s\alpha^{-s}\beta^{s(p+1)}\right)\left(z-(-1)^s\alpha^{s(p+1)}\beta^{-s}\right)\prod_{j=0}^{p}(-1)^s\left((-1)^s z-\alpha^{sj}\beta^{s(p-j)}\right) \\ =\ & (-1)^{s(p+1)}\left(z^2-L_{s(p+2)}z+(-1)^{sp}\right)D_{s,p+1}\left((-1)^s z\right). \end{aligned}$$

Then

$$\begin{aligned}
\mathcal{Z}\left(\binom{n+1}{p+2}_{F_s}\right) &= \frac{(-1)^{s+1} z^2}{D_{s,p+3}(z)} \\
&= \frac{(-1)^{s+1} z^2}{(-1)^{s(p+1)}\left(z^2 - L_{s(p+2)}z + (-1)^{sp}\right) D_{s,p+1}\left((-1)^s z\right)} \\
&= (-1)^{s(p+1)} (-1)^s \frac{1}{F_{s(p+2)}} \frac{F_{s(p+2)}z}{z^2 - L_{s(p+2)}z + (-1)^{sp}} \frac{(-1)^{s+1}(-1)^s z}{D_{s,p+1}\left((-1)^s z\right)},
\end{aligned}$$

from where (according to (16) and the convolution theorem)

$$\begin{aligned}
\binom{n+1}{p+2}_{F_s} &= (-1)^{sp} \frac{1}{F_{s(p+2)}} F_{s(p+2)n} * (-1)^{sn} \binom{n}{p}_{F_s} \\
&= \frac{1}{F_{s(p+2)}} F_{s(p+2)n} * (-1)^{s(n+p)} \binom{n}{p}_{F_s},
\end{aligned}$$

as wanted.

(b) Let us consider the polynomial $D_{s,p+5}(z)$ and observe that

$$\begin{aligned}
& D_{s,p+5}(z) \\
&= \prod_{j=0}^{p+4}\left(z - \alpha^{sj}\beta^{s(p+4-j)}\right) = \prod_{j=-2}^{p+2}\left(z - \alpha^{s(j+2)}\beta^{s(p+2-j)}\right) = \prod_{j=-2}^{p+2}\left(z - \alpha^{sj}\beta^{s(p-j)}\right) \\
&= \left(z - \alpha^{-2s}\beta^{s(p+2)}\right)\left(z - \alpha^{s(p+2)}\beta^{-2s}\right)\left(z - \alpha^{-s}\beta^{s(p+1)}\right)\left(z - \alpha^{s(p+1)}\beta^{-s}\right)\prod_{j=0}^{p}\left(z - \alpha^{sj}\beta^{s(p-j)}\right) \\
&= \left(z^2 - L_{s(p+4)}z + (-1)^{sp}\right)\left(z^2 - (-1)^s L_{s(p+2)}z + (-1)^{sp}\right) D_{s,p+1}(z).
\end{aligned}$$

Then

$$\begin{aligned}
& \mathcal{Z}\left(\binom{n+2}{p+4}_{F_s}\right) \\
&= \frac{(-1)^{s+1} z^3}{D_{s,p+5}(z)} \\
&= \frac{(-1)^{s+1} z^3}{\left(z^2 - L_{s(p+4)}z + (-1)^{sp}\right)\left(z^2 - (-1)^s L_{s(p+2)}z + (-1)^{sp}\right) D_{s,p+1}(z)} \\
&= \frac{1}{F_{s(p+4)}} \frac{F_{s(p+4)}z}{z^2 - L_{s(p+4)}z + (-1)^{sp}} \frac{1}{F_{s(p+2)}} (-1)^s \frac{F_{s(p+2)}(-1)^s z}{\left(z^2 - (-1)^s L_{s(p+2)}z + (-1)^{sp}\right)} \frac{(-1)^{s+1} z}{D_{s,p+1}(z)},
\end{aligned}$$

from where (according to (16) and the convolution theorem)

$$\begin{aligned}
\binom{n+2}{p+4}_{F_s} &= \frac{1}{F_{s(p+4)}} F_{s(p+4)n} * \frac{1}{F_{s(p+2)}} (-1)^s (-1)^{ns} F_{s(p+2)n} * \binom{n}{p}_{F_s} \\
&= \frac{1}{F_{s(p+4)}F_{s(p+2)}} F_{s(p+4)n} * (-1)^{s(n+1)} F_{s(p+2)n} * \binom{n}{p}_{F_s},
\end{aligned}$$

as wanted. ∎

Some examples are

$$\binom{n+1}{4}_{F_s} = \frac{1}{F_{4s}} \sum_{i=0}^{n} (-1)^{si} F_{4s(n-i)} \binom{i}{2}_{F_s}. \tag{77}$$

$$\binom{n+2}{6}_{F_s} = \frac{1}{F_{6s}F_{4s}} \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{s(j+1)} F_{4sj} F_{6s(i-j)} \binom{n-i}{2}_{F_s}. \tag{78}$$

In the following corollary we will deal with k given sequences $(a_n)_0, (a_n)_1, \cdots, (a_n)_k$ and the convolution of them $(a_n)_0 * (a_n)_1 * \cdots * (a_n)_k$, which we will denote as $*_{j=0}^{k} (a_n)_j$.

Corollary 17. *Let $p \in \mathbb{N}$ be given. The following identities hold*

(a)

$$\binom{n+p}{2p}_{F_s} = (-1)^{spn} *_{j=0}^{p-1} \frac{(-1)^{sj(n+1)}}{F_{2s(p-j)}} F_{2s(p-j)n}. \tag{79}$$

(b)

$$\binom{n+p-1}{2p-1}_{F_s} = *_{j=0}^{p-1} \frac{(-1)^{sj(n+1)}}{F_{s(2p-1-2j)}} F_{s(2p-1-2j)n}. \tag{80}$$

Proof. (a) According to (50) and (19) we have that

$$z^p \mathcal{Z}\left(\binom{n}{2p}_{F_s}\right) = \frac{z^{p+1}}{(z-(-1)^{sp}) \prod_{j=0}^{p-1} \left(z^2 - (-1)^{sj} L_{2s(p-j)} z + 1\right)},$$

or (by using (51))

$$\mathcal{Z}\left(\binom{n+p}{2p}_{F_s}\right) = \frac{z}{z-(-1)^{sp}} \prod_{j=0}^{p-1} \frac{z}{z^2 - (-1)^{sj} L_{2s(p-j)} z + 1},$$

from where (79) follows (by using (16) and the convolution theorem).

(b) According to (50) and (20) we have that

$$\begin{aligned} z^{p-1} \mathcal{Z}\left(\binom{n}{2p-1}_{F_s}\right) &= \frac{z^p}{\prod_{j=0}^{p-1} \left(z^2 - (-1)^{sj} L_{s(2p-1-2j)} z + (-1)^s\right)} \\ &= \prod_{j=0}^{p-1} \frac{z}{z^2 - (-1)^{sj} L_{s(2p-1-2j)} z + (-1)^s}, \end{aligned}$$

from where (80) follows (with (51), (16) and the convolution theorem). ∎

Some examples are

$$\binom{n+1}{2}_{F_s} = \frac{1}{F_{2s}} \sum_{j=0}^{n} (-1)^{s(n-j)} F_{2sj}. \tag{81}$$

$$\binom{n+2}{4}_{F_s} = \frac{1}{F_{4s}F_{2s}} \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{s(j+1)} F_{2sj} F_{4s(i-j)}. \tag{82}$$

In the following corollary we present decompositions of some sequences of s-Fibonomials as linear combinations of certain Fibonacci sequences.

Corollary 18. *We have the following decompositions of the s-Fibonomials:*

$$\begin{aligned} &\binom{n+p}{2p}_{F_s} \\ &= \sum_{j=0}^{p-1} \frac{1}{F_{2s(p-j)} \prod_{i=0, i\neq j}^{p-1} \left((-1)^{sj} L_{2s(p-j)} - (-1)^{si} L_{2s(p-i)} \right)} \sum_{t=0}^{n} (-1)^{s(p(n-t)+j(t+1))} F_{2s(p-j)t}. \quad (p \geq 1) \end{aligned} \tag{83}$$

$$\begin{aligned} &\binom{n+p-1}{2p-1}_{F_s} \\ &= \sum_{j=0}^{p-1} \frac{(-1)^{js(n+1)}}{F_{s(2p-1-2j)} \prod_{i=0, i\neq j}^{p-1} \left((-1)^{sj} L_{s(2p-1-2j)} - (-1)^{si} L_{s(2p-1-2i)} \right)} F_{s(2p-1-2j)n}. \quad (p \geq 1) \end{aligned} \tag{84}$$

$$\begin{aligned} \binom{n+p-1}{2p}_{F_s} &= \sum_{t=0}^{n} \sum_{j=0}^{p-1} \frac{(-1)^{s(p(n-t)+jt)+1}}{\prod_{i=0, i\neq j}^{p-1} \left((-1)^{sj} L_{2s(p-j)} - (-1)^{si} L_{2s(p-i)} \right)} \times \\ &\quad \times \left(F_{2s(p-j)t+1} + \frac{F_{2s(p-j)-1} - L_{2s(p-j)}}{F_{2s(p-j)}} F_{2s(p-j)t} \right). \quad (p \geq 2) \end{aligned} \tag{85}$$

$$\begin{aligned} \binom{n+p-2}{2p-1}_{F_s} &= \sum_{j=0}^{p-1} \frac{(-1)^{s(jn+1)+1}}{\prod_{i=0, i\neq j}^{p-1} \left((-1)^{sj} L_{s(2p-1-2j)} - (-1)^{si} L_{s(2p-1-2i)} \right)} \times \\ &\quad \times \left(F_{s(2p-1-2j)n+1} + \frac{F_{s(2p-1-2j)-1} - L_{s(2p-1-2j)}}{F_{s(2p-1-2j)}} F_{s(2p-1-2j)n} \right). \quad (p \geq 2) \end{aligned} \tag{86}$$

$$\begin{aligned} \binom{n+p-2}{2p}_{F_s} &= \sum_{t=0}^{n} \sum_{j=0}^{p-1} \frac{(-1)^{s(p(n-t)+j(t+1))+1} L_{2s(p-j)}}{\prod_{i=0, i\neq j}^{p-1} \left((-1)^{sj} L_{2s(p-j)} - (-1)^{si} L_{2s(p-i)} \right)} \times \\ &\quad \times \left(F_{2s(p-j)t+1} + \frac{F_{2s(p-j)-1} - \frac{L_{4s(p-j)}+1}{L_{2s(p-j)}}}{F_{2s(p-j)}} F_{2s(p-j)t} \right). \quad (p \geq 3) \end{aligned} \tag{87}$$

$$\binom{n+p-3}{2p-1}_{F_s} = \sum_{j=0}^{p-1} \frac{(-1)^{sj(n+1)+1} L_{s(2p-1-2j)}}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj} L_{s(2p-1-2j)} - (-1)^{si} L_{s(2p-1-2i)}\right)} \times \tag{88}$$

$$\times \left(F_{s(2p-1-2j)n+1} + \frac{F_{s(2p-1-2j)-1} - \frac{L_{2s(2p-1-2j)}+(-1)^s}{L_{s(2p-1-2j)}}}{F_{s(2p-1-2j)}} F_{s(2p-1-2j)n} \right). \quad (p \geq 3)$$

$$\binom{n+p-3}{2p}_{F_s} = \sum_{t=0}^{n}\sum_{j=0}^{p-1} \frac{(-1)^{s(p(n-t)+jt)+1} \left(L_{4s(p-j)}+1\right)}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj} L_{2s(p-j)} - (-1)^{si} L_{2s(p-i)}\right)} \times \tag{89}$$

$$\times \left(F_{2s(p-j)t+1} + \frac{F_{2s(p-j)-1} - \frac{L_{6s(p-j)}+L_{2s(p-j)}}{L_{4s(p-j)}+1}}{F_{2s(p-j)}} F_{2s(p-j)t} \right). \quad (p \geq 4)$$

$$\binom{n+p-4}{2p-1}_{F_s} \tag{90}$$

$$= \sum_{j=0}^{p-1} \frac{(-1)^{s(jn+1)+1} \left(L_{2s(2p-1-2j)} + (-1)^s\right)}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj} L_{s(2p-1-2j)} - (-1)^{si} L_{s(2p-1-2i)}\right)} \times$$

$$\times \left(F_{s(2p-1-2j)n+1} + \frac{F_{s(2p-1-2j)-1} - \frac{L_{3s(2p-1-2j)}+(-1)^s L_{s(2p-1-2j)}}{L_{2s(2p-1-2j)}+(-1)^s}}{F_{s(2p-1-2j)}} F_{s(2p-1-2j)n} \right). \quad (p \geq 4)$$

Proof. The proof is similar in all the four cases. It depends on an adequate partial fractions decomposition. We show all the steps of the proof only for the case (a), and indicate the corresponding decompositions used in the remaining cases. For the case (a) we use that for $p \geq 1$ one has the following partial fractions decompositions:

$$\frac{z^p}{\prod_{j=0}^{p-1}\left(z^2 - (-1)^{sj} L_{2s(p-j)} z + 1\right)} \tag{91}$$

$$= \sum_{j=0}^{p-1} \frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj} L_{2s(p-j)} - (-1)^{si} L_{2s(p-i)}\right)} \frac{z}{z^2 - (-1)^{sj} L_{2s(p-j)} z + 1},$$

and

$$\frac{z^p}{\prod_{j=0}^{p-1}\left(z^2 - (-1)^{js} L_{s(2p-1-2j)} z + (-1)^s\right)} \tag{92}$$

$$= \sum_{j=0}^{p-1} \frac{1}{\prod_{i=0,i\neq j}^{p-i}\left((-1)^{sj} L_{s(2p-1-2j)} - (-1)^{si} L_{s(2p-1-2i)}\right)} \frac{z}{z^2 - (-1)^{js} L_{s(2p-1-2j)} z + (-1)^s}.$$

We begin by noting that with (50) and (19) we can write

$$z^p \mathcal{Z}\left(\binom{n}{2p}_{F_s}\right) = \frac{z^{p+1}}{\left(z-(-1)^{sp}\right)\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right)}$$
$$= \frac{z}{z-(-1)^{sp}}\frac{z^p}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right)},$$

and then, by using (91) we have

$$z^p \mathcal{Z}\left(\binom{n}{2p}_{F_s}\right)$$
$$= \frac{z}{z-(-1)^{sp}}\sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}\frac{z}{z^2-(-1)^{sj}L_{2s(p-j)}z+1}$$
$$\frac{z}{z-(-1)^{sp}}\sum_{j=0}^{p-1}\frac{(-1)^{sj}}{F_{2s(p-j)}\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}\frac{F_{2s(p-j)}(-1)^{sj}z}{z^2-(-1)^{sj}L_{2s(p-j)}z+1}.$$

Thus, according to (51), convolution theorem and (16) we have that

$$\binom{n+p}{2p}_{F_s}$$
$$= (-1)^{spn}*\sum_{j=0}^{p-1}\frac{(-1)^{sj}}{F_{2s(p-j)}\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}(-1)^{sjn}F_{2s(p-j)n}$$
$$= \sum_{t=0}^{n}(-1)^{sp(n-t)}\sum_{j=0}^{p-1}\frac{1}{F_{2s(p-j)}\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}(-1)^{sj(t+1)}F_{2s(p-j)t}$$
$$= \sum_{j=0}^{p-1}\frac{1}{F_{2s(p-j)}\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}(-1)^{s(p(n-t)+j(t+1))}\sum_{t=0}^{n}F_{2s(p-j)t},$$

which proves (83). Also, by using (50), (20) and (92) we have that

$$
\begin{aligned}
& z^{p-1}\mathcal{Z}\left(\binom{n}{2p-1}_{F_s}\right) \\
= \; & \frac{z^p}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s\right)} \\
= \; & \sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-i}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}\frac{z}{z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s} \\
= \; & \sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-i}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}\frac{(-1)^{js}}{F_{s(2p-1-2j)}}\frac{F_{s(2p-1-2j)}(-1)^{js}z}{z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s}
\end{aligned}
$$

from where (by using (51) and (16))

$$
\begin{aligned}
& \binom{n+p-1}{2p-1}_{F_s} \\
= \; & \sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-i}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}\frac{(-1)^{js}}{F_{s(2p-1-2j)}}(-1)^{jsn}F_{s(2p-1-2j)n} \\
= \; & \sum_{j=0}^{p-1}\frac{(-1)^{js(n+1)}}{F_{s(2p-1-2j)}\prod_{i=0,i\neq j}^{p-i}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}F_{s(2p-1-2j)n},
\end{aligned}
$$

which proves (84).

For the case (b), use the partial fractions decomposition (valid for $p \geq 2$):

$$
\begin{aligned}
& \frac{z^{p-2}}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right)} \\
= \; & -\sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}\frac{z-(-1)^{sj}L_{2s(p-j)}}{z^2-(-1)^{sj}L_{2s(p-j)}z+1},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{z^{p-2}}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s\right)} \\
= \; & \sum_{j=0}^{p-1}\frac{(-1)^{s+1}}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}\frac{z-(-1)^{s}L_{s(2p-1-2j)}}{z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s}.
\end{aligned}
$$

For the case (c), use the partial fractions decomposition (valid for $p \geq 3$):

$$\frac{z^{p-3}}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right)} = \sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}\times \times\frac{(-1)^{sj+1}L_{2s(p-j)}z+L_{4s(p-j)}+1}{z^2-(-1)^{sj}L_{2s(p-j)}z+1},$$

and

$$\frac{z^{p-3}}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s\right)} = \sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}\times \times\frac{(-1)^{sj+1}L_{s(2p-1-2j)}z+L_{2s(2p-1-2j)}+(-1)^s}{z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s},$$

Finally, For the case (d) use the partial fractions decomposition (valid for $p \geq 4$):

$$\frac{z^{p-4}}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{2s(p-j)}z+1\right)} = -\sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{2s(p-j)}-(-1)^{si}L_{2s(p-i)}\right)}\times \times\frac{\left(L_{4s(p-j)}+1\right)z-(-1)^{sj}\left(L_{6s(p-j)}+L_{2s(p-j)}\right)}{z^2-(-1)^{sj}L_{2s(p-j)}z+1},$$

and

$$\frac{z^{p-4}}{\prod_{j=0}^{p-1}\left(z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s\right)}$$

$$= (-1)^{s+1}\sum_{j=0}^{p-1}\frac{1}{\prod_{i=0,i\neq j}^{p-1}\left((-1)^{sj}L_{s(2p-1-2j)}-(-1)^{si}L_{s(2p-1-2i)}\right)}\times \times\frac{\left(L_{2s(2p-1-2j)}+(-1)^s\right)z-(-1)^{sj}\left(L_{3s(2p-1-2j)}+(-1)^sL_{s(2p-1-2j)}\right)}{z^2-(-1)^{sj}L_{s(2p-1-2j)}z+(-1)^s}.$$

∎

Some examples are the following:

$$\binom{n+2}{4}_{F_s} = \frac{1}{L_{4s}+(-1)^{s+1}L_{2s}}\sum_{t=0}^{n}\left(\frac{F_{4st}}{F_{4s}}-(-1)^{s(t+1)}\frac{F_{2st}}{F_{2s}}\right). \tag{93}$$

$$\binom{n+1}{4}_{F_s} = \frac{1}{(-1)^sL_{2s}-L_{4s}}\sum_{t=0}^{n}\left(\begin{array}{c} F_{4st+1}+\frac{F_{4s-1}-L_{4s}}{F_{4s}}F_{4st} \\ +(-1)^{st+1}\left(F_{2st+1}+\frac{F_{2s-1}-L_{2s}}{F_{2s}}F_{2st}\right)\end{array}\right). \tag{94}$$

$$\binom{n}{3}_{F_s} = \frac{(-1)^{s+1}}{L_{3s} - (-1)^s L_s} \left(F_{3sn+1} + \frac{F_{3s-1} - L_{3s}}{F_{3s}} F_{3sn} + (-1)^{sn+1} \left(F_{sn+1} + \frac{F_{s-1} - L_s}{F_s} F_{sn} \right) \right). \tag{95}$$

$$\begin{aligned} \binom{n}{5}_{F_s} &= \frac{-L_{5s}}{(L_{5s} - (-1)^s L_{3s})(L_{5s} - L_s)} \left(F_{5sn+1} + \frac{F_{5s-1} - \frac{L_{10s}+(-1)^s}{L_{5s}}}{F_{5s}} F_{5sn} \right) \\ &+ \frac{(-1)^{s(n+1)+1} L_{3s}}{((-1)^s L_{3s} - L_{5s})((-1)^s L_{3s} - L_s)} \left(F_{3sn+1} + \frac{F_{3s-1} - \frac{L_{6s}+(-1)^s}{L_{3s}}}{F_{3s}} F_{3sn} \right) \\ &+ \frac{-L_s}{(L_s - L_{5s})(L_s - (-1)^s L_{3s})} \left(F_{sn+1} + \frac{F_{s-1} - \frac{L_{2s}+(-1)^s}{L_s}}{F_s} F_{sn} \right). \end{aligned} \tag{96}$$

$$\binom{n+1}{6}_{F_s} = \sum_{t=0}^{n} \left(\begin{array}{c} \frac{(-1)^{s(n-t)+1} L_{6s}}{(L_{6s} - (-1)^s L_{4s})(L_{6s} - L_{2s})} \left(F_{6st+1} + \frac{F_{6s-1} - \frac{L_{12s}+1}{L_{6s}}}{F_{6s}} F_{6st} \right) \\ + \frac{(-1)^{s(n+1)+1} L_{4s}}{((-1)^s L_{4s} - L_{6s})((-1)^s L_{4s} - L_{2s})} \left(F_{4st+1} + \frac{F_{4s-1} - \frac{L_{8s}+1}{L_{4s}}}{F_{4s}} F_{4st} \right) \\ + \frac{(-1)^{s(n-t)+1} L_{2s}}{(L_{2s} - L_{6s})(L_{2s} - (-1)^s L_{4s})} \left(F_{2st+1} + \frac{F_{2s-1} - \frac{L_{4s}+1}{L_{2s}}}{F_{2s}} F_{2st} \right) \end{array} \right). \tag{97}$$

5 Final remarks

We can write (53) (with $G = F$ and $m_1 = \cdots = m_l = 0$) as

$$(-1)^{s+1} \sum_{i=0}^{p} \sum_{j=0}^{i} (-1)^{\frac{(sj+2(s+1))(j+1)}{2}} \binom{p+1}{j}_{F_s} F_{t_1 s(i-j)}^{k_1} \cdots F_{t_l s(i-j)}^{k_l} \binom{n+p-i}{p}_{F_s} = F_{t_1 sn}^{k_1} \cdots F_{t_l sn}^{k_l}, \tag{98}$$

where $p = t_1k_1 + \cdots + t_lk_l$. It is possible to see (98) as a linear system in the p indeterminates $\binom{n+p-i}{p}_{F_s}$, $i = 1, 2, \ldots, p$, with m equations, where m is the number of terms $F_{t_1 sn}^{k_1} \cdots F_{t_l sn}^{k_l}$ we can form such that $t_1k_1 + \cdots + t_lk_l = p$ (of course we refer to non-trivial terms and up to natural equivalences).

Conjecture 19. *Any s-Fibonomial* $\binom{n+k}{p}_{F_s}$, $k = 0, 1, \ldots, p-1$, *can be written as a linear combination of homogeneous terms* $F_{t_1 sn}^{k_1} F_{t_2 sn}^{k_2} \cdots F_{t_l sn}^{k_l}$ *where* $t_1k_1 + t_2k_2 + \cdots + t_lk_l = p$.

In the simplest case $p = 2$ we have two homogeneous terms, namely F_{sn}^2 and F_{2sn}, and (98) can be solved for $\binom{n}{2}_{F_s}$ and $\binom{n+1}{2}_{F_s}$ to obtain

$$\binom{n}{2}_{F_s} = \frac{(-1)^s}{2}\left(\frac{F_{sn}^2}{F_s^2} - \frac{F_{2sn}}{F_{2s}}\right). \tag{99}$$

$$\binom{n+1}{2}_{F_s} = \frac{1}{2}\left(\frac{F_{sn}^2}{F_s^2} + \frac{F_{2sn}}{F_{2s}}\right). \tag{100}$$

In the case $p = 3$ we have three homogeneous terms F_{3sn}, $F_{2sn}F_{sn}$ and F_{sn}^3, and (98) can be solved for $\binom{n}{3}_{F_s}$, $\binom{n+1}{3}_{F_s}$ and $\binom{n+2}{3}_{F_s}$ to obtain

$$\binom{n}{3}_{F_s} = \frac{1}{2}(-1)^{s+1}\frac{F_{2sn}F_{sn}}{F_{2s}F_s} + \frac{1}{3}(-1)^s\frac{F_{3sn}}{F_{3s}} + \frac{1}{6}(-1)^s\frac{F_{sn}^3}{F_s^3}. \tag{101}$$

$$\binom{n+1}{3}_{F_s} = \frac{(-1)^{s+1}}{3L_s}\frac{F_{3sn}}{F_{3s}} + \frac{(-1)^s}{3L_s}\frac{F_{sn}^3}{F_s^3}. \tag{102}$$

$$\binom{n+2}{3}_{F_s} = \frac{1}{2}\frac{F_{2sn}F_{sn}}{F_{2s}F_s} + \frac{1}{3}\frac{F_{3sn}}{F_{3s}} + \frac{1}{6}\frac{F_{sn}^3}{F_s^3}. \tag{103}$$

In the case $p = 4$ we have 5 homogeneous terms

$$F_{4sn}\ ,\ F_{3sn}F_{sn}\ ,\ F_{sn}^4\ ,\ F_{2sn}^2\ ,\ F_{2sn}F_{sn}^2. \tag{104}$$

The corresponding system (98) can be solved for the s-Fibonomials $\binom{n-i}{4}_{F_s}$, $i = 0, 1, 2, 3$, to obtain

$$\binom{n}{4}_{F_s} = -\frac{1}{4}\frac{F_{4sn}}{F_{4s}} + \frac{L_{3s}}{10L_sF_s^2}\frac{F_{3sn}F_{sn}}{F_{3s}F_s} + \frac{(-1)^{s+1}}{10F_s^2}\frac{F_{2sn}^2}{F_{2s}^2} - \frac{1}{4}\frac{F_{2sn}F_{sn}^2}{F_s^2F_{2s}}. \tag{105}$$

$$\binom{n+1}{4}_{F_s} = \frac{(-1)^s F_s}{4F_{3s}}\frac{F_{4sn}}{F_{4s}} + \frac{(-1)^s}{10F_s^2}\frac{F_{3sn}F_{sn}}{F_{3s}F_s} + \frac{(-1)^{s+1}}{10F_s^2}\frac{F_{2sn}^2}{F_{2s}^2} + \frac{(-1)^{s+1}F_s}{4F_{3s}}\frac{F_{2sn}F_{sn}^2}{F_s^2F_{2s}}. \tag{106}$$

$$\binom{n+2}{4}_{F_s} = \frac{(-1)^{s+1} F_s}{4F_{3s}}\frac{F_{4sn}}{F_{4s}} + \frac{(-1)^s}{10F_s^2}\frac{F_{3sn}F_{sn}}{F_{3s}F_s} + \frac{(-1)^{s+1}}{10F_s^2}\frac{F_{2sn}^2}{F_{2s}^2} + \frac{(-1)^{s}F_s}{4F_{3s}}\frac{F_{2sn}F_{sn}^2}{F_s^2F_{2s}}. \tag{107}$$

$$\binom{n+3}{4}_{F_s} = \frac{1}{4}\frac{F_{4sn}}{F_{4s}} + \frac{L_{3s}}{10L_sF_s^2}\frac{F_{3sn}F_{sn}}{F_{3s}F_s} + \frac{(-1)^{s+1}}{10F_s^2}\frac{F_{2sn}^2}{F_{2s}^2} + \frac{1}{4}\frac{F_{2sn}F_{sn}^2}{F_s^2F_{2s}}. \tag{108}$$

However, in this case a linear dependence relation appears, namely:

$$5F_{sn}^4 + 3F_{2sn}^2 - 4F_{3sn}F_{sn} = 0, \tag{109}$$

so the way of representing the s-Fibonomials as linear combinations of homogeneous terms stated in our conjecture should be not unique.

It remains (for a future work) to have a proof of the conjecture above and to identify what kind of linear dependencies exist among the homogeneous terms.

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