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Quadrant Marked Mesh Patterns

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Abstract

In this paper we begin the first systematic study of distributions of quadrant marked mesh patterns. Mesh patterns were introduced recently by Brändén and Claesson in connection with permutation statistics. Quadrant marked mesh patterns are based on how many elements lie in various quadrants of the graph of a permutation relative to the coordinate system centered at one of the points in the graph of the permutation. We study the distribution of several quadrant marked mesh patterns in a symmetric group and in certain subsets of the symmetric group. We find explicit formulas for the generating function of such distributions in several general cases and develop recursions to compute the numbers in question in other cases. In addition, certain q-analogues of our results are discussed.

1 Introduction

The notion of mesh patterns was introduced by Brändén and Claesson [1] to provide explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns. This notion was further studied in [2, 10]. In particular, the notion of a mesh pattern was extended to that of a *marked mesh pattern* by Úlfarsson in [10].

In this paper, we study the number of occurrences of what we call *quadrant marked mesh patterns*. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers and S_n denote the symmetric group of permutations of $1, \ldots, n$. If $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, then we will consider the graph of σ, $G(\sigma)$, to be the set of points (i, σ_i) for $i = 1, \ldots, n$. For example, the graph of the permutation $\sigma = 471569283$ is pictured in Figure 1. Then if we draw a coordinate system centered at a point (i, σ_i), we will be interested in the points that lie in the four quadrants I, II, III, and IV of that coordinate system as pictured in Figure 1. For any $a, b, c, d \in \mathbb{N}$ and any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we say that σ_i matches the quadrant marked mesh pattern $MMP(a, b, c, d)$ in σ if in $G(\sigma)$ relative to the coordinate system which has the point (i, σ_i) as its origin, there are $\geq a$ points in quadrant I, $\geq b$ points in quadrant II, $\geq c$ points in quadrant III, and $\geq d$ points in quadrant IV. For example, if $\sigma = 471569283$, the point $\sigma_4 = 5$ matches the quadrant marked mesh pattern $MMP(2, 1, 2, 1)$ since relative to the coordinate system with origin $(4, 5)$, there are 3 points in $G(\sigma)$ in quadrant I, there is 1 point in $G(\sigma)$ in quadrant II, there are 2 points in $G(\sigma)$ in quadrant III, and there are 2 points in $G(\sigma)$ in quadrant IV. Note that if a coordinate in $MMP(a, b, c, d)$ is 0, then there is no condition imposed on the points in the corresponding quadrant. In addition, we shall consider patterns $MMP(a, b, c, d)$ where $a, b, c, d \in \mathbb{N} \cup \{\emptyset\}$. Here when one the parameters a, b, c, or d in $MMP(a, b, c, d)$ is the empty set, then for σ_i to match $MMP(a, b, c, d)$ in $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, it must be the case that there are no points in $G(\sigma)$ relative to the coordinate system with origin (i, σ_i) in the corresponding quadrant. For example, if $\sigma = 471569283$, the point $\sigma_3 = 1$ matches the marked mesh pattern $MMP(4, 2, \emptyset, \emptyset)$ since relative to the coordinate system with origin $(3, 1)$, there are 6 points in $G(\sigma)$ in quadrant I, 2 points in $G(\sigma)$ in quadrant II, no points in $G(\sigma)$ in quadrant III, and no points in $G(\sigma)$ in quadrant IV. We let $\mathrm{mmp}^{(a,b,c,d)}(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a, b, c, d)$ in σ.

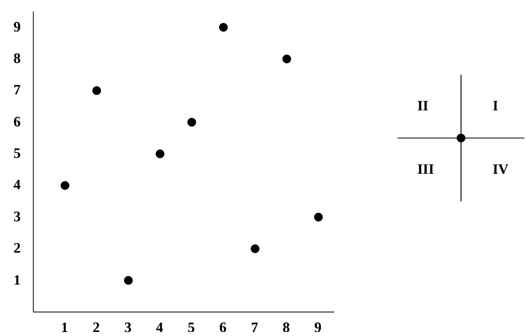
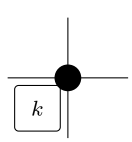
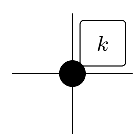


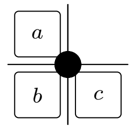
Figure 1: The graph of $\sigma = 471569283$.

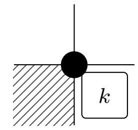
Note how the (two-dimensional) notation of Úlfarsson [10] for marked mesh patterns corresponds to our (one-line) notation for quadrant marked mesh patterns. For example,

$MMP(0, 0, k, 0) =$ 

, $MMP(k, 0, 0, 0) =$ 

,

$$MMP(0,a,b,c) =$$ 

and $$MMP(0,0,\emptyset,k) =$$ 

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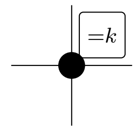
In Section 2 we will consider $MMP(=k,0,0,0)$, another type of quadrant marked mesh patterns, which requires presence of exactly k elements in quadrant I. This type of patterns is expressed in the terminology of Úlfarsson [10] as follows:

$$MMP(=k,0,0,0) =$$ 

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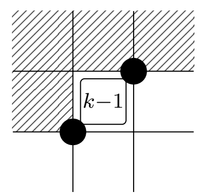
Also, in Section 5 we define and study $MMP(k{\leq}\max,\emptyset,0,0)$, yet another type of quadrant marked mesh patterns, which is equivalent to the following pattern in the terminology of Úlfarsson [10]:

$$MMP(k{\leq}\max,\emptyset,0,0) =$$ 

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Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i-th smallest integer that appears in σ by i. For example, if $\sigma = 2754$, then $\mathrm{red}(\sigma) = 1432$. Given a permutation $\tau = \tau_1 \cdots \tau_j \in S_j$, we say that the pattern τ *occurs* in $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ provided there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that a permutation σ *avoids* the pattern τ if τ does not occur in σ. Let $S_n(\tau)$ denote the set of permutations in S_n which avoid τ. In the theory of permutation patterns, τ is called a *classical pattern*. See [3] for a comprehensive introduction to permutation patterns.

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we say that σ_j is a *right-to-left maximum* of σ if $\sigma_j > \sigma_i$ for all $i > j$. We let $\mathrm{RLmax}(\sigma)$ denote the number of right-to-left maxima of σ.

The main goal of this paper is to study the generating functions

$$R^{(a,b,c,d)}(t,x) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

For any $a,b,c,d \in \{\emptyset\} \cup \mathbb{N}$, let

$$R_n^{(a,b,c,d)}(x) = \sum_{\sigma \in S_n} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

Note that there is a natural action of the symmetries of the square on the graphs of permutations. That is, one can identify each permutation $\sigma \in S_n$ with a permutation matrix $P(\sigma)$ and it is clear that rotating such matrices counter-clockwise by 90 degrees counterclockwise or reflecting permutation matrices about any central axis or any diagonal axis preserves the property of being a permutation matrix. For any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$\sigma^r = \sigma_n \cdots \sigma_1$ be the *reverse* of σ and $\sigma^c = (n+1-\sigma_1)\cdots(n+1-\sigma_n)$ be the *complement* of σ. Then it is easy to see that the map $\sigma \to \sigma^r$ corresponds to reflecting a permutation matrix about its central vertical axis and the map $\sigma \to \sigma^c$ corresponds to reflecting a permutation matrix about its central horizontal axis. It is also easy to see that rotating a permutation matrix by 90 degrees corresponds to replacing σ by $(\sigma^{-1})^r$. It follows that the map $\sigma \to (\sigma^{-1})^r$ shows that $R_n^{(a,b,c,d)}(x) = R_n^{(d,a,b,c)}(x)$, the map $\sigma \to \sigma^r$ shows that for $R_n^{(a,b,c,d)}(x) = R_n^{(b,a,d,c)}(x)$ and the map $\sigma \to \sigma^c$ shows that for $R_n^{(a,b,c,d)}(x) = R_n^{(d,c,b,a)}(x)$. Moreover, taking the inverse of permutations shows that $R_n^{(a,b,c,d)}(x) = R_n^{(a,d,c,b)}(x)$, applying the composition of reverse and complement shows that $R_n^{(a,b,c,d)}(x) = R_n^{(c,d,a,b)}(x)$, applying the composition of reverse and inverse shows that $R_n^{(a,b,c,d)}(x) = R_n^{(b,c,d,a)}(x)$, and finally, applying the composition of reverse, complement and inverse, that is, the map $\sigma \to (\sigma^{rc})^{-1}$, shows that $R_n^{(a,b,c,d)}(x) = R_n^{(c,b,a,d)}(x)$. Any other composition of the three trivial bijections on S_n will not give us any new symmetries. Thus we have the following lemma:

Lemma 1. *For any $a,b,c,d \in \{\emptyset\} \cup \mathbb{N}$,*

$$\begin{aligned} R_n^{(a,b,c,d)}(x) &= R_n^{(d,a,b,c)}(x) = R_n^{(c,b,a,d)}(x) = \\ R_n^{(b,a,d,c)}(x) &= R_n^{(d,c,b,a)}(x) = R_n^{(a,d,c,b)}(x) = R_n^{(c,d,a,b)}(x) = R_n^{(b,c,d,a)}(x). \end{aligned}$$

2 $R_n^{(k,0,0,0)}(x) = R_n^{(0,k,0,0)}(x) = R_n^{(0,0,k,0)}(x) = R_n^{(0,0,0,k)}(x)$ and $R_n^{(=k,0,0,0)}(x)$

The equalities in the section title are true by Lemma 1. Thus we only need to consider the quadrant marked mesh pattern $MMP(k,0,0,0)$.

First assume that $k \geq 1$. It is easy to see that $R_n^{(k,0,0,0)}(x) = n!$ if $n \leq k$. For $n > k$, suppose that we start with a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and then we let $\sigma^{(i)}$ denote the permutation of S_{n+1} that is produced by adding 1 to each element of σ and inserting 1 before the element $\sigma_i + 1$ if $1 \leq i \leq n$ and inserting 1 at the end if $i = n+1$. Then it is easy to see that

$$\mathrm{mmp}^{(k,0,0,0)}(\sigma^{(i)}) = \begin{cases} \mathrm{mmp}^{(k,0,0,0)}(\sigma), & \text{if } i > n-k+1; \\ 1 + \mathrm{mmp}^{(k,0,0,0)}(\sigma), & \text{if } i \leq n-k+1. \end{cases}$$

That is, clearly the placement of 1 in position i in $\sigma^{(i)}$ cannot effect the elements in quadrant I relative to any pair $(j, \sigma_j + 1)$ for any $j < i$ or relative to any pair $(j+1, \sigma_j + 1)$ for any $j \geq i$. In addition, the 1 in position i can contribute to $\mathrm{mmp}^{(k,0,0,0)}(\sigma^{(i)})$ if and only if $i \leq n+1-k$. It then follows that for $n \geq k$,

$$R_{n+1}^{(k,0,0,0)}(x) = (k + x(n+1-k))R_n^{(k,0,0,0)}(x). \tag{1}$$

Iterating this recursion, we see that

$$R_{k+s}^{(k,0,0,0)}(x) = k! \prod_{i=1}^{s} (k + ix) \tag{2}$$

for all $s \geq 1$. In this case, we can form a simple generating function. That is, let

$$P^{(k,0,0,0)}(t,x) = \sum_{n\geq k} \frac{t^{n-k}}{(n-k)!} \sum_{\sigma\in S_n} x^{\mathrm{mmp}^{(k,0,0,0)}(\sigma)}.$$

Then we know that

$$\begin{aligned}
P^{(k,0,0,0)}(t,x) &= k! + \sum_{n>k} R_n^{(k,0,0,0)}(x) \frac{t^{n-k}}{(n-k)!} \\
&= k! + \sum_{n>k} \left(kR_{n-1}^{(k,0,0,0)}(x) + (n-k)xR_{n-1}^{(k,0,0,0)}(x) \right) \frac{t^{n-k}}{(n-k)!} \\
&= k! + tx \sum_{n>k} R_{n-1}^{(k,0,0,0)}(x) \frac{t^{n-k-1}}{(n-k-1)!} + k \sum_{n>k} R_{n-1}^{(k,0,0,0)}(x) \frac{t^{n-k}}{(n-k)!}.
\end{aligned}$$

Hence

$$(1-tx)P^{(k,0,0,0)}(t,x) = k! + k \sum_{n>k} R_{n-1}^{(k,0,0,0)}(x) \frac{t^{n-k}}{(n-k)!}.$$

Taking the derivative of both sides with respect to t, we see that

$$\frac{\partial}{\partial t}((1-tx)P^{(k,0,0,0)}(t,x)) = kP^{(k,0,0,0)}(t,x) = \frac{k}{1-tx}(1-tx)P^{(k,0,0,0)}(t,x).$$

Thus

$$\frac{\frac{\partial}{\partial t}((1-tx)P^{(k,0,0,0)}(t,x))}{(1-tx))P^{(k,0,0,0)}(t,x)} = \frac{k}{1-tx}.$$

It then follows that

$$\ln((1-tx)P^{(k,0,0,0)}(t,x)) = -\frac{k}{x}\ln(1-tx) + c.$$

Hence, using the fact that $P^{(k,0,0,0)}(0,x) = k!$, we obtain the following theorem.

Theorem 2. *For all $k \geq 1$,*

$$P^{(k,0,0,0)}(t,x) = \sum_{n\geq k} \frac{t^{n-k}}{(n-k)!} \sum_{\sigma\in S_n} x^{\mathrm{mmp}^{(k,0,0,0)}(\sigma)} = k! \left(\frac{1}{1-tx}\right)^{\frac{k}{x}+1}.$$

Next, we consider another type of quadrant marked mesh pattern. That is, given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we say that σ_i matches the marked mesh pattern $MMP(=k,0,0,0)$ in σ if and only if, relative to the coordinate system with origin at (i,σ_i), there are exactly k points in $G(\sigma)$ in quadrant I. Let $\mathrm{mmp}^{(=k,0,0,0)}(\sigma)$ denote the number of i such that σ_i matches $MMP(=k,0,0,0)$ and

$$R_n^{(=k,0,0,0)}(x) = \sum_{\sigma\in S_n} x^{\mathrm{mmp}^{(=k,0,0,0)}(\sigma)}.$$

Then we can use the same reasoning to show that

$$R_n^{(=k,0,0,0)}(x) = n!$$

if $n \leq k$, and, for $n \geq k$,

$$R_{n+1}^{(=k,0,0,0)}(x) = (n+x)R_n^{(=k,0,0,0)}(x).$$

The difference in this case is that insertion of 1 will contribute to $\text{mmp}^{(=k,0,0,0)}(\sigma^{(i)})$ if and only if $i = n+1-k$. Iterating this recursion, we see that

$$R_{k+s}^{(=k,0,0,0)}(x) = k! \prod_{i=1}^{s} (k+i-1+x)$$

for all $s \geq 1$. It is then easy to see using the same reasoning as above that if we let

$$P^{(=k,0,0,0)}(t,x) = \sum_{n \geq k} \frac{t^{n-k}}{(n-k)!} \sum_{\sigma \in S_n} x^{\text{mmp}^{(=k,0,0,0)}(\sigma)},$$

then

$$P^{(=k,0,0,0)}(t,x) = k! \left(\frac{1}{1-t}\right)^{x+k}.$$

It is also known [1] and is easy to see that a point σ_i matches the pattern $MMP(\emptyset,0,0,0)$ in $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ if and only if σ_i is a right-to-left maximum of σ. It is well-known that the number of permutations $\sigma \in S_n$ such that $\text{RLmax}(\sigma) = k$ is equal to the signless Stirling number of the first kind $c(n,k)$ which is the number of permutations $\tau \in S_n$ such that τ has k cycles. Thus

$$\begin{aligned} R_n^{(\emptyset,0,0,0)}(x) &= \sum_{\sigma \in S_n} x^{\text{RLmax}(\sigma)} = \sum_{k=1}^{n} c(n,k)x^k \\ &= x(x+1)\cdots(x+n-1). \end{aligned}$$

It is well-known that $\sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=1}^n c(n,k)x^k = \left(\frac{1}{1-t}\right)^x$. Thus

$$R^{(\emptyset,0,0,0)}(t,x) = 1 + \sum_{n\geq 1} R_n^{(\emptyset,0,0,0)}(x)\frac{t^n}{n!} = \left(\frac{1}{1-t}\right)^x.$$

3 $R_n^{(a,b,0,0)}(x)$ and some other cases of two non-zero parameters

Note that it follows from Lemma 1 that $R_n^{(a,b,0,0)}(x)$ equals

$$R_n^{(b,0,0,a)}(x) = R_n^{(0,0,a,b)}(x) = R_n^{(0,a,b,0)}(x) = R_n^{(b,a,0,0)}(x) =$$

$$R_n^{(0,b,a,0)}(x) = R_n^{(0,0,b,a)}(x) = R_n^{(a,0,0,b)}(x).$$

First, we consider the case where $a, b \geq 1$. It is easy to see that $R_n^{(a,b,0,0)}(x) = n!$ if $n \leq a+b$. For $n \geq a+b$, it is easy to see that if $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, then

$$\mathrm{mmp}^{(a,b,0,0)}(\sigma^{(i)}) = \begin{cases} \mathrm{mmp}^{(a,b,0,0)}(\sigma), & \text{if } i \leq b \text{ or } i > n-a+1; \\ 1 + \mathrm{mmp}^{(a,b,0,0)}(\sigma), & \text{if } b+1 \leq i \leq n-a+1. \end{cases} \tag{3}$$

That is, clearly the placement of 1 in position i in $\sigma^{(i)}$ cannot effect the elements in quadrant I or II relative to any pair $(j, \sigma_j + 1)$ for any $j < i$ or relative to any pair $(j+1, \sigma_j + 1)$ for any $j \geq i$, and the 1 in position i can contribute to $\mathrm{mmp}^{(k,0,0,0)}(\sigma^{(i)})$ if and only if $b+1 \leq i \leq n-a+1$. It then follows that for $n \geq a+b$,

$$R_{n+1}^{(a,b,0,0)}(x) = (a+b)R_n^{(a,b,0,0)}(x) + (n+1-(a+b))xR_n^{(a,b,0,0)}(x). \tag{4}$$

Note this is the same recursion as the recursion for $R_{n+1}^{(k,0,0,0)}(x)$ if $a+b=k$. Thus it follows that

$$R_{a+b+s}^{(a,b,0,0)}(x) = (a+b)! \prod_{i=1}^{s} ((a+b)+ix) \tag{5}$$

for all $s \geq 1$ and that

$$P^{(a,b,0,0)}(t,x) = \sum_{n \geq a+b} \frac{t^{n-a-b}}{(n-a-b)!} R_n^{(a,b,0,0)}(x) = (a+b)! \left(\frac{1}{1-tx}\right)^{\frac{a+b}{x}+1} \tag{6}$$

for all $a, b \geq 1$.

Next we consider the case $R_n^{(1,\emptyset,0,0)}(x)$. Clearly, $R_1^{(1,\emptyset,0,0)}(x) = 1$. For $n \geq 1$ and any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we again consider the permutations $\sigma^{(i)}$ in S_{n+1}. For $i = 1$, $\mathrm{mmp}^{(1,\emptyset,0,0)}(\sigma^{(1)}) = 1 + \mathrm{mmp}^{(1,\emptyset,0,0)}(\sigma)$ and, for $i > 1$, $\mathrm{mmp}^{(1,\emptyset,0,0)}(\sigma^{(i)}) = \mathrm{mmp}^{(1,\emptyset,0,0)}(\sigma)$. It follows that for all $n \geq 1$,

$$R_{n+1}^{(1,\emptyset,0,0)}(x) = (x+n)R_n^{(1,\emptyset,0,0)}(x)$$

so that for $n > 1$,

$$R_n^{(1,\emptyset,0,0)}(x) = (x+1)\cdots(x+n-1).$$

It then easily follows that

$$R^{(1,\emptyset,0,0)}(t,x) = 1 + \frac{(1-t)^{-x} - 1}{x}. \tag{7}$$

Next fix $k \geq 2$. Clearly, $R_n^{(k,\emptyset,0,0)}(x) = n!$ for $n \leq k$. For $n \geq k$ and $\sigma \in S_n$, it is easy to see that for $i = 1$, $\mathrm{mmp}^{(k,\emptyset,0,0)}(\sigma^{(1)}) = 1 + \mathrm{mmp}^{(k,\emptyset,0,0)}(\sigma)$ and that for $i > 1$, $\mathrm{mmp}^{(k,\emptyset,0,0)}(\sigma^{(i)}) = \mathrm{mmp}^{(k,\emptyset,0,0)}(\sigma)$. Hence, for all $n \geq k$,

$$R_{n+1}^{(k,\emptyset,0,0)}(x) = (x+n)R_n^{(k,\emptyset,0,0)}(x).$$

Thus for $n > k$,

$$R_n^{(k,\emptyset,0,0)}(x) = k!(x+k)\cdots(x+n-1).$$

It follows that for $k \geq 2$,

$$P^{(k,\emptyset,0,0)}(t,x) = \sum_{n \geq k} R_n^{(k,\emptyset,0,0)}(x) \frac{t^{n-k}}{(n-k)!} = k! \left(\frac{1}{1-t}\right)^{x+k}$$

and that

$$\begin{aligned} R^{(k,\emptyset,0,0)}(t,x) &= 1 + \sum_{n \geq 1} R_n^{(k,\emptyset,0,0)}(x) \frac{t^n}{n!} \\ &= \sum_{j=0}^{k} t^j + \frac{k!}{\prod_{i=1}^{k-1}(x+i)} \left(R^{(1,\emptyset,0,0)}(t,x) - 1 - t - \sum_{j=2}^{k} \frac{t^j}{j!} \prod_{i=1}^{j-1}(x+i) \right). \quad (8) \end{aligned}$$

4 q-analogues to marked mesh patterns considered above

We let

$$\begin{aligned} [n]_q &= 1 + q + \cdots + +q^{n-1} = \frac{1-q^n}{1-q}, \\ [n]_q! &= [1]_q[2]_q \cdots [n]_q, \text{ and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!} \end{aligned}$$

denote the usual q-analogues of n, $n!$, and $\binom{n}{k}$. We shall use the standard conventions that $[0]_q = 0$ and $[0]_q! = 1$. For any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we let $\text{coinv}(\sigma)$ equal the number of *co-inversions* in σ, that is, the number of $1 \leq i < j \leq n$ such that $\sigma_i < \sigma_j$.

Let

$$R_n^{(a,b,c,d)}(x,q) = \sum_{\sigma \in S_n} x^{\text{mmp}^{(a,b,c,d)}(\sigma)} q^{\text{coinv}(\sigma)}.$$

It turns out that we can easily obtain q-analogues of the recursions (1) and (4). That is, for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, it is easy to see that

$$q^{\text{coinv}(\sigma^{(i)})} x^{\text{mmp}^{(k,0,0,0)}(\sigma^{(i)})} = \begin{cases} q^{n+1-i} q^{\text{coinv}(\sigma)} x^{\text{mmp}^{(k,0,0,0)}(\sigma)}, & \text{if } i > n-k; \\ q^{n+1-i} q^{\text{coinv}(\sigma)} x^{1+\text{mmp}^{(k,0,0,0)}(\sigma)}, & \text{if } i \leq n-k. \end{cases}$$

It then follows that for $n \geq k$,

$$R_{n+1}^{(k,0,0,0)}(x,q) = [k]_q R_n^{(k,0,0,0)}(x,q) + xq^k [n+1-k]_q R_n^{(k,0,0,0)}(x,q).$$

Thus for $n \leq k$,

$$R_{n+1}^{(k,0,0,0)}(x,q) = [n]_q!$$

and for $s \geq 1$,

$$R_{k+s}^{(k,0,0,0)}(x,q) = [k]_q! \prod_{i=1}^{s} ([k]_q + xq^k [i]_q).$$

Similarly, we can find a q-analogue of the recursion (3). That is, for any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, it is easy to see that

$$q^{\mathrm{coinv}(\sigma^{(i)})} x^{\mathrm{mmp}^{(a,b,0,0)}(\sigma^{(i)})} = \begin{cases} q^{n+1-i} q^{\mathrm{coinv}(\sigma)} x^{\mathrm{mmp}^{(a,b,0,0)}(\sigma)}, & \text{if } i \leq b \text{ or } i > n-a+1; \\ q^{n+1-i} q^{\mathrm{coinv}(\sigma)} x^{1+\mathrm{mmp}^{(a,b,0,0)}(\sigma)}, & \text{if } b+1 \leq i \leq n-a+1. \end{cases}$$

It then follows that for $n \geq a+b$,

$$R_{n+1}^{(a,b,0,0)}(x,q) = ([a]_q + q^{n-b}[b]_q) R_n^{(k,0,0,0)}(x,q) + q^a [n+1-(a+b)]_q x R_n^{(k,0,0,0)}(x,q).$$

Thus for $n \leq a+b$,

$$R_{n+1}^{(a,b,0,0)}(x,q) = [n]_q!$$

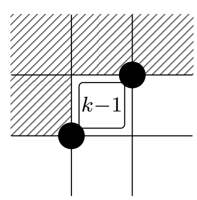
and for $s \geq 1$,

$$R_{a+b+s}^{(a,b,0,0)}(x,q) = [k]_q! \prod_{i=1}^{s} ([a]_q + q^{a+i}[b]_q + q^a x [i]_q).$$

Note that if $a+b=k$ where $a,b \geq 1$ and $n>k$, then $R_n^{(a,b,0,0)}(x,q) \neq R_n^{(k,0,0,0)}(x,q)$.

5 $R_n^{(k\leq \max,\emptyset,0,0)}(x)$ – another type of quadrant marked mesh patterns

We say that σ_i matches the quadrant marked mesh pattern $MMP(k\leq \max, \emptyset, 0, 0)$ in $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ if in $G(\sigma)$ relative to the coordinate system with origin (i, σ_i), there are no points in quadrant II in $G(\sigma)$ and there are at least $k-1$ points that lie in quadrant I to the left of the largest value occurring in quadrant I. Said another way, σ_i matches the pattern $MMP(k\leq \max, \emptyset, 0, 0)$ in σ if none of $\sigma_1, \ldots, \sigma_{i-1}$ is greater than σ_i and if $\sigma_j > \sigma_i$ is the maximum of $\sigma_{i+1}, \ldots, \sigma_n$, then there are at least k points among $\sigma_{i+1}, \ldots, \sigma_j$ which are greater than σ_i. It is easy to see that $MMP(k\leq \max, \emptyset, 0, 0)$ is equivalent to the following (ordinary) marked mesh pattern in the terminology of Úlfarsson [10]:



.

For $\sigma \in S_n$, we let $\mathrm{mmp}^{(k\leq \max,\emptyset,0,0)}(\sigma)$ be the number of i such that σ_i matches

$$MMP(k\leq \max, \emptyset, 0, 0)$$

in σ and we let

$$R_n^{(k\leq \max,\emptyset,0,0)}(x) = \sum_{\sigma \in S_n} x^{\mathrm{mmp}^{(k\leq \max,\emptyset,0,0)}(\sigma)}.$$

Clearly, $R_n^{(1\leq \max,\emptyset,0,0)}(x) = R_n^{(1,\emptyset,0,0)}(x)$. We can compute $R_n^{(k\leq \max,\emptyset,0,0)}(x)$ for $k \geq 2$ as follows. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $\sigma^{[i]}$ denote the permutation of S_{n+1}

that results by inserting $n+1$ immediately before σ_i if $1 \leq i \leq n$ and inserting $n+1$ at the end of σ if $i = n+1$. It is easy to see that the set of all $\sigma^{[i]}$ for $\sigma \in S_n$ contributes $\binom{n}{i-1} R_{i-1}^{(k-1,\emptyset,0,0)}(x)(n+1-i)!$ to $R_{n+1}^{(k\leq \max,\emptyset,0,0)}(x)$. That is, the presence of $n+1$ in the i-th position of $\sigma^{[i]}$ means that none of the elements to the right of $n+1$ in $\sigma^{[i]}$ can contribute to $\mathrm{mmp}^{(k\leq \max,\emptyset,0,0)}(\sigma^{[i]})$ while an element to the left of $n+1$ in $\sigma^{[i]}$ will contribute 1 to $\mathrm{mmp}^{(k\leq \max,\emptyset,0,0)}(\sigma^{[i]})$ if and only if it contributes 1 to $\mathrm{mmp}^{(k-1,\emptyset,0,0)}(\sigma_1 \cdots \sigma_{i-1})$. It follows that for all $n \geq 0$,

$$R_{n+1}^{(k\leq \max,\emptyset,0,0)}(x) = \sum_{i=1}^{n+1} (n+1-i)! \binom{n}{i-1} R_{i-1}^{(k-1,\emptyset,0,0)}(x)$$

or, equivalently,

$$\frac{R_{n+1}^{(k\leq \max,\emptyset,0,0)}(x)}{n!} = \sum_{i=1}^{n+1} \frac{R_{i-1}^{(k-1,\emptyset,0,0)}(x)}{(i-1)!}. \tag{9}$$

Thus multiplying (9) by t^n and summing it over all $n \geq 0$, we obtain that

$$\begin{aligned} \frac{\partial}{\partial t} R^{(k\leq \max,\emptyset,0,0)}(t,x) &= \sum_{n\geq 0} \frac{R_{n+1}^{(k\leq \max,\emptyset,0,0)}(x) t^n}{n!} \\ &= \sum_{n\geq 0} t^n \sum_{i=1}^{n+1} \frac{R_{i-1}^{(k-1,\emptyset,0,0)}(x)}{(i-1)!} \\ &= \frac{1}{1-t} R^{(k-1,\emptyset,0,0)}(t,x). \end{aligned}$$

Hence we obtain the recursion

$$R^{(k\leq \max,\emptyset,0,0)}(t,x) = 1 + \int_0^t \frac{1}{1-z} R^{(k-1,\emptyset,0,0)}(z,x)dz. \tag{10}$$

Note that one can find an explicit formula for $R^{(k-1,\emptyset,0,0)}(z,x)$ by using formulas (7) and (8). For example, one can use Mathematica to compute that

$$\begin{aligned} &R^{(2\leq \max,\emptyset,0,0)}(t,x) = 1 + t + t^2 + \frac{1}{6}(5+x)t^3 + \\ &\frac{1}{24}\left(17+6x+x^2\right)t^4 + \frac{1}{120}\left(74+35x+10x^2+x^3\right)t^5 + \\ &\frac{1}{720}\left(394+225x+85x^2+15x^3+x^4\right)t^6 + \\ &\frac{1}{5040}\left(2484+1624x+735x^2+175x^3+21x^4+x^5\right)t^7 + \\ &\frac{1}{40320}\left(18108+13132x+6769x^2+1960x^3+322x^4+28x^5+x^6\right)t^8 + \\ &\frac{1}{362880}\left(149904+118124x+67284x^2+22449x^3+4536x^4+546x^5+36x^6+x^7\right)t^9 + \cdots \end{aligned}$$

and

$$R^{(3\leq\max,\emptyset,0,0)}(t,x) = 1+t+t^2+t^3+\frac{1}{24}2(11+x)t^4+\frac{1}{120}2\left(50+9x+x^2\right)t^5+$$
$$\frac{1}{720}2\left(274+71x+14x^2+x^3\right)t^6+\frac{1}{5040}2(1764+580x+155x^2+20x^3+x^4)t^7+$$
$$\frac{1}{40320}2(13068+5104x+1665x^2+295x^3+27x^4+x^5)t^8+$$
$$\frac{1}{362880}2(100584+48860x+18424x^2+4025x^3+511x^4+35x^5+x^6)t^9+\cdots.$$

and

$$R^{(4\leq\max,\emptyset,0,0)}(t,x) = 1+t+t^2+t^3+t^4+\frac{1}{120}6\left(19+x\right)t^5+\frac{1}{720}6\left(107+12x+x^2\right)t^6+$$
$$\frac{1}{5040}6(702+119x+18x^2+x^3)t^7+\frac{1}{40320}6(5274+1175x+245x^2+25x^3+x^4)t^8+$$
$$\frac{1}{362880}6(44712+12154x+3135x^2+445x^3+33x^4+x^5)t^9+\cdots.$$

There are several of the coefficients of $R_n^{(k\leq\max,\emptyset,0,0)}(x)$ that we can explain. First, we claim that

$$R_n^{(k\leq\max,\emptyset,0,0)}(x)|_{x^{n-k}} = (k-1)! \text{ for } n\geq k+1. \tag{11}$$

That is, it is easy to see that for $n \geq k+1$, the only permutations $\sigma = \sigma_1\cdots\sigma_n \in S_n$ such that $\mathrm{mmp}^{(k\leq\max,\emptyset,0,0)}(\sigma) = n-k$ must have $\sigma_i = i$ for $i = 1,\ldots,n-k$, $\sigma_n = n$ and $\sigma_{n-k+1}\cdots\sigma_{n-1}$ be some permutation of $(n-k+1),(n-k+2),\ldots,(n-1)$.

Next, we claim that

$$R_n^{(k\leq\max,\emptyset,0,0)}(x)|_{x^{n-k-1}} = (k-1)!\left(\binom{n}{2}-\binom{k-1}{2}\right) \text{ for } n\geq k+2. \tag{12}$$

First observe that any permutation $\sigma = \sigma_1\cdots\sigma_n \in S_n$ such that $n \in \{\sigma_1,\ldots,\sigma_{n-2}\}$ cannot have $\mathrm{mmp}^{(k\leq\max,\emptyset,0,0)}(\sigma) = n-k-1$. Now assume that $n \geq k+2$ and $\sigma = \sigma_1\cdots\sigma_n \in S_n$ is such that $\mathrm{mmp}^{(k\leq\max,\emptyset,0,0)}(\sigma) = n-k-1$. Thus it must be that case that $n = \sigma_{n-1}$ or $n = \sigma_n$. If $\sigma_{n-1} = n$, then it must be the case that $\mathrm{mmp}^{(k\leq\max,\emptyset,0,0)}(\mathrm{red}(\sigma_1\cdots\sigma_{n-1})) = n-k-1$. In this case, we have $n-1$ choices for σ_n and $(k-1)!$ choices for $\mathrm{red}(\sigma_1\cdots\sigma_{n-1})$ by our argument above. Thus there are $(n-1)((k-1)!)$ such elements. Next assume that $n = \sigma_n$. Then in this case, the only way to produce a $\sigma = \sigma_1\cdots\sigma_n \in S_n$ such that $\mathrm{mmp}^{(k\leq\max,\emptyset,0,0)}(\sigma) = n-k-1$ is to start with a permutation

$$\tau = \tau_1\cdots\tau_n = 1\ 2\ \cdots(n-k)\tau_{n-k+1}\cdots\tau_{n-1}\ n$$

where $\tau_{n-k+1}\cdots\tau_{n-1}$ is some permutation of $(n-k+1),(n-k+2),\ldots,(n-1)$ so that $\mathrm{mmp}^{(k\leq\max,\emptyset,0,0)}(\tau) = n-k$. Now if $k \geq 2$, then take some $i \in \{1,\ldots,n-k\}$ in τ and move i immediately after one of $\tau_{i+1},\ldots,\tau_{n-1}$. This will ensure that i does not match the pattern $MMP(k\leq\max,\emptyset,0,0)$ in the resulting permutation because i will have a larger element to its left. Each such τ gives rise to $(n-2)+(n-3)+\cdots+(k-1) = \binom{n-1}{2}-\binom{k-1}{2}$ such permutations. If $k=1$, then take some $i \in \{1,\ldots,n-2\}$ in τ and move i immediately after one of $\tau_{i+1},\ldots,\tau_{n-1}$. In this case, each such τ gives rise to $(n-2)+(n-3)+\cdots+1 = \binom{n-1}{2} = \binom{n-1}{2} - \binom{0}{2}$ such permutations. Thus in either case, the number of $\sigma \in S_n$ with $\sigma_n = n$ and $\mathrm{mmp}^{(k\leq \max,\emptyset,0,0)}(\sigma) = n-k-1$ is $(k-1)!\left(\binom{n-1}{2} - \binom{k-1}{2}\right)$. It thus follows that the number of $\sigma \in S_n$ with $\mathrm{mmp}^{(k\leq \max,\emptyset,0,0)}(\sigma) = n-k-1$ is $(k-1)!\left(\binom{n}{2} - \binom{k-1}{2}\right)$.

Next we observe that the sequence $(R_n^{(2\leq \max,\emptyset,0,0)}(0))_{n\geq 1}$ is

$$1,2,5,17,74,394,2484,18108,149904,\ldots.$$

This is sequence A000774 from the OEIS where the n-th term in the sequence is

$$n!(1+\sum_{i=1}^{n}\frac{1}{i}).$$

The sequence $(R_n^{(2\leq \max,\emptyset,0,0)}(x)|_x)_{n\geq 3}$ is

$$1,6,35,225,1624,13,132,118124,\ldots$$

which is sequence A000399 in the OEIS. The n-th term of this sequence is $c(n,3)$ which is the number of permutations $\sigma \in S_n$ which have three cycles. We can give a direct proof of this result. That is, we have the following proposition.

Proposition 3. *For all $n \geq 3$,*

$$R_n^{(2\leq \max,\emptyset,0,0)}(x)|_x = c(n,3).$$

Proof. Suppose that $n \geq 3$ and σ is a permutation of S_n with 3 cycles C_1, C_2, C_3 where have arranged the cycles so that the largest element in each cycle is on the left and we order the cycles by increasing largest elements. Then we let $\overline{\sigma}$ be the result of erasing the parentheses and commas in $C_1C_2C_3$. For example, if $\sigma = (5,3,4,1)\ (8,2,6)\ (9,7)$, then $\overline{\sigma} = 5\ 3\ 4\ 1\ 8\ 2\ 6\ 9\ 7$. It is easy to see that under this map only the first element of $\overline{\sigma}$ matches the pattern $MMP(2\leq \max,\emptyset,0,0)$ in $\overline{\sigma}$.

We claim that every $\tau \in S_n$ such that $\mathrm{mmp}^{(2\leq \max,\emptyset,0,0)}(\tau) = 1$ is equal to $\overline{\sigma}$ for some $\sigma \in S_n$ that has two three cycles. That is, suppose that $\mathrm{mmp}^{(2\leq \max,\emptyset,0,0)}(\tau) = 1$ where $\tau = \tau_1\cdots\tau_n \in S_n$ and τ_i matches the pattern $MMP(2\leq \max,\emptyset,0,0)$ in τ. First we claim that $i=1$. It cannot be that any of $\tau_1,\ldots,\tau_{i-1}$ are greater than τ_i since otherwise τ_i would not match the pattern $MMP(2\leq \max,\emptyset,0,0)$ in τ. But if $i \neq 1$, then $\tau_1 < \tau_i$ and τ_1 would match the pattern $MMP(2\leq \max,\emptyset,0,0)$ in τ which would mean that $\mathrm{mmp}^{(2\leq \max,\emptyset,0,0)}(\tau) \geq 2$. Next suppose that $\tau_k = n$ and τ_j is the maximum element of $\{\tau_2,\ldots,\tau_{k-1}\}$. It must be that case that $\tau_1 < \tau_j$ since otherwise τ_1 would not match the pattern $MMP(2\leq \max,\emptyset,0,0)$ in τ. We claim that $\tau_2,\ldots,\tau_{j-1}$ must all be less than τ_j. That is, if τ_r is the maximum of $\{\tau_2,\ldots,\tau_{j-1}\}$ and $\tau_r > \tau_1$, then we would have $\tau_r < \tau_j < \tau_k = n$ so that τ_r would match the pattern $MMP(2\leq \max,\emptyset,0,0)$ in τ which would imply that $\mathrm{mmp}^{(2\leq \max,\emptyset,0,0)}(\tau) \geq 2$. It then follows that if $\sigma = (\tau_1,\ldots,\tau_{j-1})\ (\tau_j,\ldots,\tau_{j-1})\ (\tau_k,\ldots,\tau_n)$, then $\overline{\sigma} = \tau$. Thus we have proved that for $n \geq 3$, $R_n^{(2\leq \max,\emptyset,0,0)}(x)|_x = c(n,3)$. $\square$

The sequence $(\frac{1}{2}R_n^{(3\leq \max,\emptyset,0,0)}(0))_{n\geq 2}$ is

$$1, 3, 11, 50, 274, 1764, 13068, 109584, 1026576, \ldots$$

which is sequence A000254 in the OEIS whose n-th term is $c(n+1,2)$ which is the number of permutations of S_{n+1} with 2 cycles. We shall give a direct proof of this fact.

Proposition 4. *For all $n \geq 2$,*

$$R_n^{(3\leq \max,\emptyset,0,0)}(0) = 2c(n,2).$$

Proof. Suppose that $\sigma = \sigma_1 \cdots \sigma_n$ is a permutation in S_n such that $\mathrm{mmp}^{(3\leq \max,\emptyset,0,0)}(\sigma) = 0$. Let $a(\sigma)$ be the i such that $\sigma_i = n$. If $a(\sigma) \neq 1$, then let $b(\sigma) = j$ where $\sigma_j = \max(\{\sigma_1, \ldots, \sigma_{a(\sigma)-1}\})$. If $b(\sigma) \neq 1$, then let $c(\sigma) = i$ where $\sigma_i = \max(\{\sigma_1, \ldots, \sigma_{b(\sigma)-1}\})$. First we claim that if $c(\sigma)$ is defined, then $c(\sigma) = 1$. That is, if $c(\sigma) \neq 1$, then $\sigma_1 < \sigma_{c(\sigma)} < \sigma_{b(\sigma)} < \sigma_{a(\sigma)} = n$ so that σ_1 would match the pattern $MMP(3 \leq \max, \emptyset, 0, 0)$. We also claim that if $c(\sigma)$ is defined, then σ_1 must be the second largest element among $\{\sigma_1, \ldots, \sigma_{a(\sigma)-1}\}$ otherwise σ_1 would match the pattern $MMP(3 \leq \max, \emptyset, 0, 0)$ in σ. Thus we have three possible cases for a $\sigma \in S_n$ such that $\mathrm{mmp}^{(3\leq \max,\emptyset,0,0)}(\sigma) = 0$.

Case 1. $a(\sigma) = 1$.
There are clearly $(n-1)!$ such permutations as $\sigma_2, \ldots, \sigma_n$ can be any arrangement of $1, \ldots, n-1$.

Case 2. $b(\sigma) = 1$.

Case 3. $c(\sigma) = 1$ and σ_1 is the second largest element in $\{\sigma_1, \ldots, \sigma_{a(\sigma)-1}\}$.

Next we define a map θ which takes the permutations in Cases 2 and 3 into the set of permutations τ of S_n which have 2 cycles C_1C_2 where we have arranged the cycles so that the largest element in each cycle is on the left and we have $\max(C_1) < \max(C_2) = n$. In Case 2, we let

$$\theta(\sigma) = (\sigma_{b(\sigma)}, \ldots, \sigma_{a(\sigma)-1})\ (\sigma_{a(\sigma)}, \ldots, \sigma_n).$$

In Case 3, we let

$$\theta(\sigma) = (\sigma_{b(\sigma)}, \ldots, \sigma_{a(\sigma)-1}, \sigma_{c(\sigma)}, \ldots, \sigma_{b(\sigma)-1})\ (\sigma_{a(\sigma)}, \ldots, \sigma_n).$$

Now suppose that τ is a permutation with two cycles C_1C_2, $C_1 = (\alpha_1, \ldots, \alpha_k)$ and $C_2 = (\beta_1, \ldots, \beta_{n-k})$, where $|C_1| = k > 1$. Again we assume that we have arranged the cycles so that the largest element in the cycle is on the left and $\max(C_1) < \max(C_2) = n$. Then there exists a σ in Case 2 such that $\theta(\sigma) = \tau$ and a γ in Case 3 such that $\theta(\gamma) = \tau$. That is, if α_j is the second largest element in $\{\alpha_1, \ldots, \alpha_k\}$, then

$$\sigma = \alpha_1, \ldots, \alpha_k\beta_1 \cdots \beta_{n-k}$$

is a permutation in Case 2 such that $\theta(\sigma) = \tau$ and

$$\gamma = \alpha_j \cdots \alpha_k\alpha_1, \ldots, \alpha_{j-1}\beta_1 \cdots \beta_{n-k}$$

is a permutation in Case 3 such that $\theta(\gamma) = \tau$. The only permutations with 2 cycles C_1C_2 that we have not accounted for are the permutations τ where $|C_1| = 1$. But if $|C_1| = 1$, then $\sigma = \alpha_1\beta_1 \cdots \beta_{n-1}$ is a permutation in Case 2 such that $\theta(\sigma) = \tau$. θ shows that the number of permutations in Cases 2 and 3 is equal to $2c(n, 2)$ minus the number of τ such that $|C_1| = 1$. However, it is easy to see that the number of τ with two cycles C_1C_2 such that $|C_1| = 1$ is $(n - 1)!$ which is equal to the number of permutations in Case 1. Thus we have shown that $R_n^{(3\leq \max,\emptyset,0,0)}(0) = 2c(n, 2)$. □

The sequence $(\frac{1}{6}R_n^{(\leq 4\max,\emptyset,0,0)}(0))_{n\geq 3}$ which is

$$1, 4, 19, 107, 702, 5274, 44712, 422, 568, \ldots$$

does not appear in the OEIS. However, the sequence $(\frac{1}{6}R_n^{(4\leq \max,\emptyset,0,0)}(x)|_x)_{n\geq 5}$ which is

$$1, 12, 119, 1175, 12154, 133938, \ldots$$

seems to be A001712 in the OEIS whose n-th term is $\sum_{k=0}^{n}(-1)^{n+k}\binom{k+2}{2}3^k s(n + 2, k + 2)$ where $s(n, k)$ is the Stirling number of the first kind.

Problem 5. Can we prove this formula (directly)?

6 $R_n^{(a,0,b,0)}(x) = R_n^{(b,0,a,0)}(x) = R_n^{(0,a,0,b)}(x) = R_n^{(0,b,0,a)}(x)$

The equalities in the section title are true by Lemma 1. We will consider $R_n^{(a,0,b,0)}(x)$.

In this case, we do not know how to compute the generating function $R^{(1,0,1,0)}(t, x)$. However, we can develop a recursion to compute $R_n^{(1,0,1,0)}(x)$. That is, for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$\begin{aligned} A_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(1, 0, 1, 0) \text{ in } \sigma) \text{ and} \\ B_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(\emptyset, 0, 1, 0) \text{ in } \sigma) \end{aligned}$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Note that at most one of $A_i(\sigma)$ and $B_i(\sigma)$ can be equal to 1. Then let

$$F_n^{(1,0,1,0)}(x_2, \ldots, x_n; y_2, \ldots, y_n) = \sum_{\sigma\in S_n}\prod_{i=2}^{n} x_i^{A_i(\sigma)} y_i^{B_i(\sigma)}. \tag{13}$$

Note that $(1, \sigma_1)$ never contributes to $\mathrm{mmp}^{(1,0,1,0)}(\sigma)$ or $\mathrm{mmp}^{(\emptyset,0,1,0)}(\sigma)$ so that $F_n^{(1,0,1,0)}(x_2, \ldots, x_n; y_2, \ldots, y_n)$ records all the information about the contributions of $\sigma_2, \ldots, \sigma_n$ to $\mathrm{mmp}^{(1,0,1,0)}(\sigma)$ or $\mathrm{mmp}^{(\emptyset,0,1,0)}(\sigma)$ as σ ranges over S_n.

Recall that given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, $\sigma^{[i]}$ denotes the permutation of S_{n+1} that results by inserting $n + 1$ immediately before σ_i if $1 \leq i \leq n$ and inserting $n + 1$ at the end of σ if $i = n + 1$. First consider all the permutations $\sigma^{[1]}$ as σ ranges over S_n. Clearly, since each of these permutations start with $n + 1$, the first element of $\sigma^{[1]}$, which is $n + 1$, does not contribute to either $\mathrm{mmp}^{(1,0,1,0)}(\sigma^{[1]})$ or $\mathrm{mmp}^{(\emptyset,0,1,0)}(\sigma^{[1]})$. Moreover $n + 1$ does not effect whether any other σ_i contributes to $\mathrm{mmp}^{(1,0,1,0)}(\sigma^{[1]})$ or $\mathrm{mmp}^{(\emptyset,0,1,0)}(\sigma^{[1]})$ but it does shift the corresponding indices by 1. Thus the second element of $\sigma^{[1]}$ which is σ_1 does not contribute to either $\mathrm{mmp}^{(1,0,1,0)}(\sigma^{[1]})$ or $\mathrm{mmp}^{(\emptyset,0,1,0)}(\sigma^{[1]})$. However, for the $(i+1)$-st element $\sigma^{[1]}$ which is the old σ_i, we have

1. $A_{i+1}(\sigma^{[1]}) = 1$ if and only if $A_i(\sigma) = 1$ and
2. $B_{i+1}(\sigma^{[1]}) = 1$ if and only if $B_i(\sigma) = 1$.

Thus it follows that the contribution of the permutations of the form $\sigma^{[1]}$ to $F_{n+1}^{(1,0,1,0)}(x_2, \ldots, x_{n+1}; y_2, \ldots, y_{n+1})$ is just $F_n^{(1,0,1,0)}(x_3, \ldots, x_{n+1}; y_3, \ldots, y_{n+1})$. For $i \geq 2$, again consider all the permutations $\sigma^{[i]}$ as σ ranges over S_n. For $j < i$, if $A_j(\sigma) = 1$, then $A_j(\sigma^{[i]}) = 1$ and if $B_j(\sigma) = 1$, then $A_j(\sigma^{[i]}) = 1$. For $j \geq i$, if $A_j(\sigma) = 1$, then $A_{j+1}(\sigma^{[i]}) = 1$ and if $B_j(\sigma) = 1$, then $B_{j+1}(\sigma^{[i]}) = 1$. Moreover the fact that $n+1$ is in position i in $\sigma^{[i]}$ means that $B_i(\sigma^{[i]}) = 1$. Since at most one of $A_i(\sigma)$ and $B_i(\sigma)$ can equal 1, it follows that the contribution of the permutations of the form $\sigma^{[i]}$ to $F_{n+1}(x_2, \ldots, x_{n+1}; y_2, \ldots, y_{n+1})$ is just

$$y_i F_n^{(1,0,1,0)}(x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}; x_2, \ldots, x_{i-1}, y_{i+1}, \ldots, y_{n+1}).$$

Thus

$$\begin{aligned} &F_{n+1}^{(1,0,1,0)}(x_2, \ldots, x_{n+1}; y_2, \ldots, y_{n+1}) = F_n^{(1,0,1,0)}(x_3, \ldots, x_{n+1}; y_3, \ldots, y_{n+1}) + \\ &\sum_{i=2}^{n+1} y_i F_n^{(1,0,1,0)}(x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}; x_2, \ldots, x_{i-1}, y_{i+1}, \ldots, y_{n+1}). \end{aligned} \tag{14}$$

It is then easy to see that

$$R_n^{(1,0,1,0)}(x) = F_n^{(1,0,1,0)}(x, \ldots, x; 1, \ldots, 1). \tag{15}$$

Using (14) and (15), one can compute that

$$\begin{aligned} &R_1^{(1,0,1,0)}(x) = 1, \\ &R_2^{(1,0,1,0)}(x) = 2, \\ &R_3^{(1,0,1,0)}(x) = 5 + x, \\ &R_4^{(1,0,1,0)}(x) = 14 + 8x + 2x^2, \\ &R_5^{(1,0,1,0)}(x) = 42 + 46x + 26x^2 + 6x^3, \\ &R_6^{(1,0,1,0)}(x) = 132 + 232x + 220x^2 + 112x^3 + 24x^5, \\ &R_7^{(1,0,1,0)}(x) = 429 + 1093x + 1527x^2 + 1275x^3 + 596x^4 + 120x^5, \text{ and} \\ &R_8^{(1,0,1,0)}(x) = 1430 + 4944x + 9436x^2 + 11384x^3 + 8638x^4 + 3768x^5 + 720x^6. \end{aligned}$$

It is easy to see that a permutation σ avoids $MMP(1,0,1,0)$ if and only if it avoids the pattern 123, that is, if $\sigma \in S_n(123)$. From a well-known fact (see, e.g. [3]), it follows that the sequence $(R_n^{(1,0,1,0)}(0))_{n\geq 1}$ is the Catalan numbers. Thus it is not surprising that it is difficult to find a simple expression for $R^{(1,0,1,0)}(t,x)$ since then $R^{(1,0,1,0)}(t,0)$ would give us an exponential generating function for the Catalan numbers which is not known. It is also easy to see that for any $n > 2$, the most occurrences of the pattern $MMP(1,0,1,0)$ occurs when $\sigma_1 = 1$ and $\sigma_n = n$. Clearly, there are $(n-2)!$ such permutations which explains the coefficients of the largest power of x appearing in $(R_n^{(1,0,1,0)}(x))_{n\geq 1}$. We note that neither the sequence $(R_n^{(1,0,1,0)}(x)|_x)_{n\geq 3}$ nor the sequence $(R_n^{(1,0,1,0)}(x)|_{x^{n-3}})_{n\geq 3}$ appear in the OEIS.

We can use similar reasoning to develop a recursion to compute $R_n^{(1,0,a,0)}(x)$. That is, for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$\begin{aligned} A_i^a(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(1,0,a,0) \text{ in } \sigma) \text{ and} \\ B_i^a(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(\emptyset,0,a,0) \text{ in } \sigma). \end{aligned}$$

Then let

$$F_n^{(1,0,a,0)}(x_2,\ldots,x_n;y_2,\ldots,y_n) = \sum_{\sigma\in S_n}\prod_{i=2}^{n} x_i^{A_i^a(\sigma)} y_i^{B_i^a(\sigma)}. \tag{16}$$

We can now use the same analysis that we did to prove (14) to prove that

$$\begin{aligned} &F_{n+1}^{(1,0,a,0)}(x_2,\ldots,x_{n+1};y_2,\ldots,y_{n+1}) = F_n^{(1,0,a,0)}(x_3,\ldots,x_{n+1};y_3,\ldots,y_{n+1}) + \\ &\sum_{i=2}^{a} F_n^{(1,0,a,0)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1};x_2,\ldots,x_{i-1},y_{i+1},\ldots,y_{n+1}) + \\ &\sum_{i=a+1}^{n} y_i F_n^{(1,0,a,0)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1};x_2,\ldots,x_{i-1},y_{i+1},\ldots,y_{n+1}). \end{aligned} \tag{17}$$

The difference in this case is that if $n \geq a$, then inserting $n+1$ immediately before $\sigma_1,\ldots,\sigma_a$ does not give rise to an extra factor of y_i, while inserting $n+1$ immediately before $\sigma_{a+1},\ldots,\sigma_n$ or at the end does give rise to a factor of y_i since then $n+1$ matches the pattern $MMP(\emptyset,0,a,0)$. It is then easy to see that

$$R_n^{(1,0,a,0)}(x) = F_n^{(1,0,a,0)}(x,\ldots,x;1,\ldots,1). \tag{18}$$

Using (17) and (18) one can compute that for $a = 2$,

$$\begin{aligned} &R_1^{(1,0,2,0)}(x) = 1, \\ &R_2^{(1,0,2,0)}(x) = 2, \\ &R_3^{(1,0,2,0)}(x) = 6, \\ &R_4^{(1,0,2,0)}(x) = 22 + 2x, \\ &R_5^{(1,0,2,0)}(x) = 90 + 26x + 4x^2, \\ &R_6^{(1,0,2,0)}(x) = 394 + 232x + 82x^2 + 12x^3, \\ &R_7^{(1,0,2,0)}(x) = 1806 + 1776x + 1062x^2 + 348x^3 + 48x^4, \text{ and} \\ &R_8^{(1,0,2,0)}(x) = 8558 + 12546x + 11118x^2 + 6022x^3 + 1836x^4 + 240x^5. \end{aligned}$$

Again, it is easy to see that one obtains the maximum number of $MMP(1,0,2,0)$-matches in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ if either $\sigma_1 = 1, \sigma_2 = 2$, and $\sigma_n = n$ or $\sigma_1 = 2, \sigma_2 = 1$, and $\sigma_n = n$. It follows that for $n \geq 4$, the coefficient of the highest power of x in $R_n^{(1,0,2,0)}(x)$ is $2(n-3)!$. The sequence $(R_n^{(1,0,2,0)}(0))_{n\geq 1}$ whose initial terms are

$$1, 2, 6, 22, 90, 394, 1806, 8558, \ldots$$

is the so-called *large Schröder numbers*, A006318 in the OEIS. This follows from the fact that avoiding $MMP(1,0,2,0)$ is the same as avoiding simultaneously the patterns 1234 and 2134, a known case (see [3, Table 2.2]). The same numbers count so-called *separable permutations* (those avoiding simultaneously the patterns 2413 and 3142) and other eight (non-equivalent modulo trivial bijections) classes of avoidance of two (classical) patterns of length 4 (see [3, Table 2.2]). We note that neither the sequence $(R_n^{(1,0,2,0)}(x)|_x)_{n\geq 4}$ nor the sequence $(R_n^{(1,0,2,0)}(x)|_{x^{n-4}})_{n\geq 4}$ appear in the OEIS.

One can also develop a recursion to compute $R_n^{(b,0,a,0)}(x)$ when $b \geq 2$ but it is more complicated. That is, in such a case, we have to keep track of the patterns $MMP(\emptyset,0,a,0)$, $MMP(1,0,a,0), \ldots, MMP(b,0,a,0)$. We will show how this works in the case when $a = b = 2$. That is, for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$\begin{aligned}
A_i^{2,2}(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(2,0,2,0) \text{ in } \sigma),\\
B_i^{2,2}(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(1,0,2,0) \text{ in } \sigma, \text{ but does not match the pattern}\\
&\qquad MMP(2,0,2,0) \text{ in } \sigma) \text{ and}\\
C_i^{2,2}(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(\emptyset,0,2,0) \text{ in } \sigma).
\end{aligned}$$

Then let

$$G_n^{(2,0,2,0)}(x_3,\ldots,x_n;y_3,\ldots,y_n;z_3,\ldots,z_n) = \sum_{\sigma\in S_n}\prod_{i=3}^{n} x_i^{A_i^{2,2}(\sigma)} y_i^{B_i^{2,2}(\sigma)} z_i^{C_i^{2,2}(\sigma)}. \tag{19}$$

Note that for any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we cannot have either σ_1 or σ_2 match any of the patterns $MMP(\emptyset,0,2,0)$, $MMP(1,0,2,0)$, or $MMP(2,0,2,0)$ in σ so that is why we do not need variables with subindices 1 or 2. Clearly, for $n < 3$,

$$G_n^{(2,0,2,0)}(x_3,\ldots,x_n;y_3,\ldots,y_n;z_3,\ldots,z_n) = n!.$$

For $n = 3$, we can never have a pattern match for $MMP(2,0,2,0)$ or $MMP(1,0,2,0)$. We can have a pattern match for $MMP(\emptyset,0,2,0)$ only if $\sigma = 123$ or $\sigma = 213$. Thus $G_3^{2,2}(x_3,y_3,z_3) = 4 + 2z_3$.

Consider all the permutations $\sigma^{[i]}$ where $i \in \{1,2\}$ as σ ranges over S_n. Clearly, since each of these permutations start with $n+1$ or has its second element equal to $n+1$, $n+1$ does not contribute to either $\mathrm{mmp}^{(2,0,2,0)}(\sigma^{[i]})$, $\mathrm{mmp}^{(1,0,2,0)}(\sigma^{[i]})$, or $\mathrm{mmp}^{(\emptyset,0,2,0)}(\sigma^{[i]})$. Moreover $n+1$ does not effect whether any other σ_i contributes to $\mathrm{mmp}^{(2,0,2,0)}(\sigma^{[i]})$, $\mathrm{mmp}^{(1,0,2,0)}(\sigma^{[i]})$, or $\mathrm{mmp}^{(\emptyset,0,2,0)}(\sigma^{[i]})$, but it does shift the index over by 1. Thus the contribution of the permutations of the form $\sigma^{[1]}$ and $\sigma^{[2]}$ to $G_{n+1}^{(2,0,2,0)}(x_3,\ldots,x_{n+1};y_3,\ldots,y_{n+1};z_3,\ldots,z_{n+1})$ is just $2G_n^{(2,0,2,0)}(x_4,\ldots,x_{n+1};y_4,\ldots,y_{n+1},z_4,\ldots,z_{n+1})$. Fix $i \geq 3$ and consider all the permutations $\sigma^{[i]}$ as σ ranges over S_n. For $j < i$, if $A_j(\sigma) = 1$, then $A_j(\sigma^{[i]}) = 1$, if $B_j(\sigma) = 1$, then $A_j(\sigma^{[i]}) = 1$ and if $C_j(\sigma) = 1$, then $B_j(\sigma^{[i]}) = 1$. For $j \geq i$, if $A_j(\sigma) = 1$, then

$A_{j+1}(\sigma^{[i]}) = 1$, if $B_j(\sigma) = 1$, then $B_{j+1}(\sigma^{[i]}) = 1$ and if $C_j(\sigma) = 1$, then $C_{j+1}(\sigma^{[i]}) = 1$. Moreover, the fact that $n+1$ is in position i in $\sigma^{[i]}$ means that $C_i(\sigma^{[i]}) = 1$. Since at most one of $A_i(\sigma) = 1$, $B_i(\sigma) = 1$, and $C_i(\sigma) = 1$ can hold for any i, it follows that the contribution of the permutations of the form $\sigma^{[i]}$ to $G_{n+1}^{(2,0,2,0)}(x_3, \ldots, x_{n+1}; y_3, \ldots, y_{n+1}; z_3, \ldots, z_{n+1})$ is just $z_i F_n(x_3, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}; x_3, \ldots, x_{i-1}, y_{i+1}, \ldots, y_{n+1}; y_3, \ldots, y_{i-1}, z_{i+1}, \ldots, z_{n+1})$. Thus

$$\begin{aligned} G_{n+1}^{(2,0,2,0)}&(x_3, \ldots, x_{n+1}; y_3, \ldots, y_{n+1}; z_3, \ldots, z_{n+1}) = \\ &2G_{n+1}^{(2,0,2,0)}(x_4, \ldots, x_{n+1}; y_4, \ldots, y_{n+1}; z_4, \ldots, z_{n+1}) + \\ &\sum_{i=3}^{n+1} z_i G_n^{(2,2)}(x_3, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}; x_2, \ldots, x_{i-1}, \\ &\qquad y_{i+1}, \ldots, y_{n+1}; y_3, \ldots, y_{i-1}, z_{i+1}, \ldots, z_{n+1}). \end{aligned} \tag{20}$$

It is then easy to see that

$$R_n^{(2,0,2,0)}(x) = G_n^{(2,0,2,0)}(x, \ldots, x; 1, \ldots, 1; 1, \ldots, 1). \tag{21}$$

Using (20) and (21) one can compute that
$R_1^{(2,0,2,0)}(x) = 1$,
$R_2^{(2,0,2,0)}(x) = 2$,
$R_3^{(2,0,2,0)}(x) = 6$,
$R_4^{(2,0,2,0)}(x) = 24$,
$R_5^{(2,0,2,0)}(x) = 116 + 4x$,
$R_6^{(2,0,2,0)}(x) = 632 + 80x + 8x^2$,
$R_7^{(2,0,2,0)}(x) = 3720 + 1056x + 240x^2 + 24x^3$, and
$R_8^{(2,0,2,0)}(x) = 23072 + 11680x + 4480x^2 + 992x^3 + 96x^4$.

Again, it is easy to see that one obtains the maximum number of $MMP(2,0,2,0)$-matches in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ if $\{\sigma_1, \sigma_2\} = \{1, 2\}$ and $\{\sigma_{n-1}, \sigma_n\} = \{n-1, n\}$. It follows that for $n \geq 5$, the coefficient of the highest power of x in $R_n^{(1,0,2,0)}(x)$ is $4(n-4)!$. In this case, the sequence $(R^{(2,0,2,0)}(0))_{n\geq 1}$ does not appear in the OEIS, but it clearly counts the number of permutations which avoid the permutations 12345, 21345, 12354, and 21354.

While we cannot find a formula for $R^{(1,0,1,0)}(t,x)$, we can find a restricted version of this generating function. That is, let $S_n(1 \to n)$ denote the set of permutations $\sigma = \sigma_1 \cdots \sigma_n$ such that 1 appears to the left of n in σ. Then let

$$B^{(1,0,1,0)}(t,x) = \sum_{n\geq 2} B_n^{(1,0,1,0)}(x) \frac{t^{n-2}}{(n-2)!} \tag{22}$$

where

$$B_n^{(1,0,1,0)}(x) = \sum_{\sigma \in S_n(1\to n)} x^{mpp^{(1,0,1,0)}(\sigma)}.$$

Let $S_n(i,j)$ denote the set of all permutations $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ such that $\sigma_i = 1$ and $\sigma_j = n$ where $1 \leq i < j \leq n$. It is easy to see that for $s < i$, σ_s matches the pattern $MMP(1,0,1,0)$

in σ if and only if σ_s matches the pattern $MMP(0,0,1,0)$ in $\sigma_1\cdots\sigma_{i-1}$. Clearly if $i<s<j$, then σ_s matches the pattern $MMP(1,0,1,0)$ in σ. Finally, if $j<s\leq n$, then σ_s matches the pattern $MMP(1,0,1,0)$ in σ if and only if σ_s matches the pattern $MMP(1,0,0,0)$ in $\sigma_{j+1}\cdots\sigma_n$. It follows that

$$B_n^{(1,0,1,0)}(x)=\sum_{1\leq i<j\leq n}\binom{n-2}{i-1,j-i-1,n-j}(j-i-1)!x^{j-i-1}R_{i-1}^{(0,0,1,0)}(x)R_{n-j}^{(1,0,0,0)}(x)$$

or, equivalently,

$$\begin{aligned}\frac{B_n^{(1,0,1,0)}(x)}{(n-2)!} &= \sum_{1\leq i<j\leq n}x^{j-i-1}\frac{R_{i-1}^{(0,0,1,0)}(x)}{(i-1)!}\frac{R_{n-j}^{(1,0,0,0)}(x)}{(n-j)!}\\ &= \sum_{a,b,c\geq 0,a+b+c=n-2}x^a\frac{R_b^{(0,0,1,0)}(x)}{b!}\frac{R_c^{(1,0,0,0)}(x)}{c!}.\end{aligned}\tag{23}$$

It is easy to see from (2) that $R^{(1,0,0,0)}(t,x)=R^{(0,0,1,0)}(t,x)=(1-tx)^{-1/x}$. Multiplying (23) by $\frac{t^{n-2}}{(n-2)!}$ and summing for $n\geq 2$, we obtain

$$B^{(1,0,1,0)}(t,x)=\frac{1}{(1-tx)}R^{(0,0,1,0)}(t,x)R^{(0,0,1,0)}(t,x)=(1-tx)^{-1-\frac{2}{x}}.\tag{24}$$

One can use (24) to show that for $n\geq 1$,

$$B_{2n}^{(1,0,1,0)}(x)=2^{n-1}\prod_{i=1}^{n-1}(1+ix)\prod_{i=1}^{n-1}(2+(2i-1)x)\tag{25}$$

and

$$B_{2n-1}^{(1,0,1,0)}(x)=2^{n-1}\prod_{i=1}^{n-2}(1+ix)\prod_{i=1}^{n-1}(2+(2i-1)x).\tag{26}$$

Problem 6. Can one find a direct explanation of formulas (25) and (26)?

For example, it follows that the number of permutations $\sigma\in S_n(1\rightarrow n)$ which avoid $MMP(1,0,1,0)$ is 2^{n-2}. This is easy to see since to avoid $MMP(1,0,1,0)$ one must have 1 and n adjacent and there must be decreasing sequences on either side of 1 and n. Thus such a permutation is determined by choosing for each $i\in\{2,\ldots,n-1\}$ whether it is placed on the left or the right of the adjacent pair $1n$.

There is a second class of permutations for which it is easy to compute the generating function for the distribution of matches of the pattern $MMP(1,0,1,0)$. That is, suppose that $k,\ell>0$ and $n>k+\ell$, Let $\beta=\beta_1\cdots\beta_k$ be any permutation of $n-k+1,\ldots,n$ and $\alpha=\alpha_1\ldots\alpha_\ell$ be any permutation of $1,\ldots,\ell$. Then we let $S_n^{\overline{\beta\alpha}}$ denote the set of permutations $\sigma\in S_n$ such that $\beta\alpha$ is a consecutive sequence in σ. Now suppose that $a\leq k$, $b\leq\ell$, $\mathrm{mmp}^{(a,0,0,0)}(\mathrm{red}(\beta))=0$, $\mathrm{mmp}^{(0,0,b,0)}(\mathrm{red}(\alpha))=0$, and

$$R_n^{(a,0,b,0),\overline{\beta\alpha}}(x)=\sum_{\sigma\in S_n^{\overline{\beta\alpha}}}x^{\mathrm{mmp}^{(a,0,b,0)}(\sigma)}.$$

Then we claim that for $n > k+\ell$,

$$R_n^{(a,0,b,0),\overline{\beta\alpha}}(x) = \sum_{i=0}^{n-k-\ell} \binom{n-k-\ell}{i} R_i^{(0,0,b,0)}(x) R_{n-k-\ell-i}^{(a,0,0,0)}(x).$$

That is, suppose that $\sigma \in S_n^{\overline{\beta\alpha}}$ is a permutation where there are i elements to the left of the occurrence of $\beta\alpha$. Then a σ_j with $j \leq i$ matches the pattern $MMP(a,0,b,0)$ in σ if and only if σ_j matches the pattern $MMP(0,0,b,0)$ in $\sigma_1 \cdots \sigma_i$. Similarly for $j > i+k+\ell$, σ_j matches the pattern $MMP(a,0,b,0)$ in σ if and only if σ_j matches the pattern $MMP(a,0,0,0)$ in $\sigma_{k+\ell+1} \cdots \sigma_n$. Finally our assumptions ensure that none of the elements in $\overline{\beta\alpha}$ matches the pattern $MMP(a,0,b,0)$ in σ. It then follows that

$$R^{(a,0,b,0),\overline{\beta\alpha}}(t,x) = \sum_{n \geq k+\ell} \frac{t^{n-k-\ell}}{(n-k-\ell)!} R_n^{(a,0,b,0),\overline{\beta\alpha}}(x) = R^{(a,0,0,0)}(t,x) R^{(0,0,b,0)}(t,x).$$

For example, since $R^{(0,0,1,0)}(t,x) = R^{(1,0,0,0)}(t,x) = (1-tx)^{-1/x}$, it follows that

$$\begin{aligned} R^{(1,0,1,0),\overline{n1}}(t,x) &= \sum_{n \geq 2} \frac{t^{n-2}}{(n-2)!} R_n^{(1,0,1,0),\overline{n1}}(x) \\ &= (1-tx)^{-2/x} = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \prod_{i=0}^{n-1} (2+ix). \end{aligned}$$

7 $R_n^{(1,0,1,1)}(x) = R_n^{(0,1,1,1)}(x) = R_n^{(1,1,0,1)}(x) = R_n^{(1,1,1,0)}(x)$

The equalities in the section title are true by Lemma 1. Thus we shall only consider $R_n^{(1,0,1,1)}(x)$.

In this case, we can develop a recursion to compute $R_n^{(1,0,1,1)}(x)$ which is very similar to the recursion that we developed to compute $R_n^{(1,0,1,0)}(x)$. That is, for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$\begin{aligned} C_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(1,0,1,1) \text{ in } \sigma) \text{ and} \\ D_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(\emptyset,0,1,1) \text{ in } \sigma). \end{aligned}$$

Then let

$$F_n^{(1,0,1,1)}(x_2,\ldots,x_{n-1};y_2,\ldots,y_{n-1}) = \sum_{\sigma \in S_n} \prod_{i=2}^{n-1} x_i^{C_i(\sigma)} y_i^{D_i(\sigma)}.$$

Note that σ_1 and σ_n never contribute to $\mathrm{mmp}^{(1,0,1,1)}(\sigma)$ or $\mathrm{mmp}^{(\emptyset,0,1,1)}(\sigma)$ so that $F_n^{(1,0,1,1)}(x_2,\ldots,x_{n-1};y_2,\ldots,y_{n-1})$ records all the information about the contributions of $2,\ldots,n-1$ to $\mathrm{mmp}^{(1,0,1,1)}(\sigma)$ or $\mathrm{mmp}^{(\emptyset,0,1,1)}(\sigma)$ as σ ranges over S_n.

First consider all the permutations $\sigma^{[1]}$ as σ ranges over S_n. Clearly, since each of these permutations starts with $n+1$, $n+1$ does not contribute to either $\mathrm{mmp}^{(1,0,1,1)}(\sigma^{[1]})$ or

$\mathrm{mmp}^{(\emptyset,0,1,1)}(\sigma^{[1]})$. Moreover, $n+1$ does not effect whether any other σ_i contributes to $\mathrm{mmp}^{(1,0,1,1)}(\sigma^{[1]})$ or $\mathrm{mmp}^{(\emptyset,0,1,1)}(\sigma^{[1]})$, but it does shift the index over by 1. Thus the contribution of the permutations of the form $\sigma^{[1]}$ to $F_{n+1}^{(1,0,1,1)}(x_2,\ldots,x_n;y_2,\ldots,y_n)$ is just $F_n^{(1,0,1,1)}(x_3,\ldots,x_n;y_3,\ldots,y_n)$. Next consider all the permutations $\sigma^{[n+1]}$ as σ ranges over S_n. Clearly, since each of these permutations ends with $n+1$, $n+1$ does not contribute to either $\mathrm{mmp}^{(1,0,1,1)}(\sigma)$ or $\mathrm{mmp}^{(\emptyset,0,1,1)}(\sigma)$. However, while $n+1$ does not effect whether any other i contributes to $\mathrm{mmp}^{(1,0,1,1)}(\sigma)$, it does ensure that each (i,σ_i) that contributed to $\mathrm{mmp}^{(\emptyset,0,1,1)}(\sigma)$ will now contribute to $\mathrm{mmp}^{(1,0,1,1)}(\sigma^{[n+1]})$. Thus it is easy to see that the contribution of the permutations of the form $\sigma^{[n+1]}$ to $F_{n+1}^{(1,0,1,1)}(x_2,\ldots,x_n;y_2,\ldots,y_n)$ is just $F_n^{(1,0,1,1)}(x_2,\ldots,x_{n-1};x_2,\ldots,x_{n-1})$. For $2\leq i\leq n$, again consider all the permutations $\sigma^{[i]}$ as σ ranges over S_n. For $j<i$, if $C_j(\sigma)=1$, then $C_j(\sigma^{[i]})=1$ and if $D_j(\sigma)=1$, then $C_j(\sigma^{[i]})=1$. For $j\geq i$, if $C_j(\sigma)=1$, then $C_{j+1}(\sigma^{[i]})=1$ and if $D_j(\sigma)=1$, then $D_{j+1}(\sigma^{[i]})=1$. Moreover, the fact that $n+1$ is in position i in $\sigma^{[i]}$ means that $D_i(\sigma^{[i]})=1$. Since at most one of $C_i(\sigma)=1$ and $D_i(\sigma)=1$ holds, it follows that the contribution of the permutations of the form $\sigma^{[i]}$ to $F_{n+1}^{(1,0,1,1)}(x_2,\ldots,x_n;y_2,\ldots,y_n)$ is just

$$y_iF_n^{(1,0,1,1)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n;x_2,\ldots,x_{i-1},y_{i+1},\ldots,y_n).$$

Thus

$$\begin{aligned}&F_{n+1}^{(1,0,1,1)}(x_2,\ldots,x_n;y_2,\ldots,y_n)=F_n^{(1,0,1,1)}(x_3,\ldots,x_n;y_3,\ldots,y_n)+\\&F_n^{(1,0,1,1)}(x_2,\ldots,x_{n-1};x_2,\ldots,x_{n-1})+\\&\sum_{i=2}^{n-1}y_iF_n^{(1,0,1,1)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n;x_2,\ldots,x_{i-1},y_{i+1},\ldots,y_n).\end{aligned}\tag{27}$$

It is then easy to see that

$$R_n^{(1,0,1,1)}(x)=F_n^{(1,0,1,1)}(x,\ldots,x;1,\ldots,1).\tag{28}$$

Using (27) and (28) one can compute that

$$\begin{aligned}&R_1^{(1,0,1,1)}(x)=1,\\&R_2^{(1,0,1,1)}(x)=2,\\&R_3^{(1,0,1,1)}(x)=6,\\&R_4^{(1,0,1,1)}(x)=20+4x,\\&R_5^{(1,0,1,1)}(x)=70+42x+8x^2,\\&R_6^{(1,0,1,1)}(x)=252+300x+144x^2+24x^3,\\&R_7^{(1,0,1,1)}(x)=924+1812x+1572x^2+636x^3+96x^4,\text{ and}\\&R_8^{(1,0,1,1)}(x)=3432+9960x+13440x^2+9576x^3+3432x^4+480x^5.\end{aligned}$$

In this case, the sequence $(R_n^{(1,0,1,1)}(0))$ is A000984 in the OEIS which is the sequence whose n-th term is the central binomial coefficient $\binom{2n}{n}$ which has lots of combinatorial interpretations. That is, we claim that $R_n^{(1,0,1,1)}(0)=\binom{2n-2}{n-1}$. Note that avoiding $MMP(1,0,1,1)$

is equivalent to simultaneously avoiding the patterns 1324, 1342, 2314 and 2341, which, in turn, by applying the complement is equivalent to simultaneous avoidance of the patterns 4231, 4213, 3241 and 3214. It is a known fact (see [3, Subsection 6.1.1]) that the number of permutations in S_n avoiding simultaneously the patterns 4132, 4123, 3124 and 3142 is given by $\binom{2n-2}{n-1}$. Mark Tiefenbruck [8] has constructed a bijection between the set of permutations of S_n which simultaneously avoid the patterns 4132, 4123, 3124 and 3142 and the set of permutations of S_n which simultaneously avoid the patterns 4231, 4213, 3241 and 3214. Indeed, this is a special case of a more general bijection which will appear in a forthcoming paper by Remmel and Tiefenbruck [9] so we will not give the details here.

In this case, it is also easy to understand the coefficient of the highest power of x in the polynomial $R_n^{(1,0,1,1)}(x)$. That is, one obtains the maximum number of occurrences of the pattern $MMP(1,0,1,1)$ when the permutation σ either starts with 1 and ends with either 2 n or n 2 or it starts with 2 and ends with either 1 n or n 1. Thus it is easy to see that the coefficient of the highest power of x occurring in $R_n^{(1,0,1,1)}(x)$ is $4((n-3)!)x^{n-3}$ for $n \geq 4$.

While we cannot find a formula for $R^{(1,0,1,1)}(t,x)$, we can find generating functions for the distribution of $MMP(1,0,1,1)$-matches in certain restricted classes of permutations. That is, let

$$B^{(1,0,1,1)}(t,x) = \sum_{n\geq 2} B_n^{(1,0,1,1)}(x)\frac{t^{n-2}}{(n-2)!}$$

where

$$B_n^{(1,0,1,1)}(x) = \sum_{\sigma\in S_n(1\to n)} x^{mpp^{(1,0,1,1)}(\sigma)}.$$

Consider the set $S_n^{(k+1)}$ of all permutations $\sigma = \sigma_1\cdots\sigma_n \in S_n$ such that $\sigma_{k+1} = 1$ and $\sigma_j = n$ for some $k+1 < j \leq n$. It is easy to see that for $s \leq k$, σ_s matches the pattern $MMP(1,0,1,1)$ in σ if and only if σ_s matches the pattern $MMP(0,0,1,0)$ in $\sigma_1\cdots\sigma_k$. Clearly $\sigma_{k+1} = 1$ does not match the pattern $MMP(1,0,1,1)$ in σ. If $k+1 < s \leq n$, then σ_s matches the pattern $MMP(1,0,1,1)$ in σ if and only if σ_s matches the pattern $MMP(1,0,0,1)$ in $\sigma_{k+2}\cdots\sigma_n$. It follows that

$$B_n^{(1,0,1,1)}(x) = \sum_{k=0}^{n-2}\binom{n-2}{k} R_k^{(0,0,1,0)}(x)R_{n-k-1}^{(1,0,0,1)}(x)$$

or, equivalently,

$$\frac{B_n^{(1,0,1,1)}(x)}{(n-2)!} = \sum_{k=0}^{n-2}\frac{R_k^{(0,0,1,0)}(x)}{k!}\frac{R_{n-k-1}^{(1,0,0,1)}(x)}{(n-2-k)!} \tag{29}$$

Multiplying (29) by $\frac{t^{n-2}}{(n-2)!}$ and summing for $n \geq 2$, we obtain

$$B^{(1,0,1,1)}(t,x) = R^{(0,0,1,0)}(t,x)\sum_{n\geq 0}\frac{t^n}{n!}R_{n+1}^{(1,0,0,1)}(x).$$

By (6), we have that

$$\sum_{n\geq 0}\frac{t^n}{n!}R_{n+2}^{(1,0,0,1)}(x) = 2(1-tx)^{-\frac{2}{x}-1}.$$

Hence

$$\sum_{n\geq 0}\frac{t^n}{n!}R_{n+1}^{(1,0,0,1)}(x) = 1+\int_0^t 2(1-zx)^{-\frac{2}{x}-1}dz = (1-tx)^{-2/x}.$$

Since $R^{(0,0,1,0)}(t,x) = (1-tx)^{-1/x}$, we obtain

$$B^{(1,0,1,1)}(t,x) = (1-tx)^{-3/x} = 1+3t+\sum_{n\geq 2}\frac{t^n}{n!}\prod_{i=0}^{n-1}(3+ix).$$

For example, it follows that the number of permutations $\sigma \in S_n(1 \to n)$ which avoid $MMP(1,0,1,1)$ is 3^n.

Next suppose that $k,\ell > 0$ and $n > k+\ell$, and let $\gamma_{k,\ell} = n(n-1)\cdots(n-k+1)\ell(\ell-1)\cdots 1$. Then we let $S_n^{\overline{\gamma_{k,\ell}}}$ denote the set of permutations $\sigma \in S_n$ such that $\gamma_{k,\ell}$ is a consecutive sequence in σ and

$$R_n^{(1,0,1,1),\overline{\gamma_{k,\ell}}}(x) = \sum_{\sigma\in S_n^{\overline{\gamma_{k,\ell}}}} x^{\mathrm{mmp}^{(1,0,1,1)}(\sigma)}.$$

Then it is easy to see that for $n > k+\ell$,

$$R_n^{(1,0,1,1),\overline{\gamma_{k,\ell}}}(x) = \sum_{i=0}^{n-k-\ell}\binom{n-k-\ell}{i}R_i^{(0,0,1,0)}(x)R_{n-k-\ell-i}^{(1,0,0,1)}(x). \tag{30}$$

That is, if $\sigma \in S_n^{\overline{\gamma_{k,\ell}}}$ is such that there are i elements to the left of the occurrence of $\gamma_{k,\ell}$, then a σ_j with $j \leq i$ matches the pattern $MMP(1,0,1,1)$ in σ if and only if σ_j matches the pattern $MMP(0,0,1,0)$ in $\sigma_1\cdots\sigma_i$. Similarly for $j > i+k+\ell$, σ_j matches the pattern $MMP(1,0,1,1)$ in σ if and only if σ_j matches the pattern $MMP(1,0,0,1)$ in $\sigma_{k+\ell+1}\cdots\sigma_n$. Clearly no element that is part of $\gamma_{k,\ell}$ can match $MMP(1,0,1,1)$ in σ.

It then follows that

$$R^{(1,0,1,1),\overline{\gamma_{k,\ell}}}(t,x) = \sum_{n\geq k+\ell}\frac{t^{n-k-\ell}}{(n-k-\ell)!}R_n^{(1,0,1,1),\overline{\gamma_{k,\ell}}}(x) = R^{(1,0,0,0)}(t,x)R^{(1,0,0,1)}(t,x).$$

Since

$$\sum_{n\geq 1}\frac{t^{n-1}}{(n-1)!}R_n^{(1,0,0,1)}(x) = (1-tx)^{-2/x},$$

we have that

$$\sum_{n\geq 0}\frac{t^n}{n!}R_n^{(1,0,0,1)}(x) = 1+\int_0^t (1-zx)^{-2/x}dz = \frac{1-x+(1-tx)^{1-\frac{2}{x}}}{2-x}.$$

Hence,

$$\begin{aligned} R^{(1,0,1,1),\overline{n1}}(t,x) &= \sum_{n\geq 2}\frac{t^{n-2}}{(n-2)!}R_n^{(1,0,1,1),\overline{n1}}(x) \\ &= (1-tx)^{-1/x}\frac{1-x+(1-tx)^{1-\frac{2}{x}}}{2-x}. \end{aligned}$$

One can use Mathematica to show that

$$R^{(1,0,1,1),\overline{n1}}(t,x) = 1 + 2t + (5+x)\frac{t^3}{3!} + (14+8x+2x^2)\frac{t^4}{4!} +$$
$$(41+50x+23x^2+6x^3)\frac{t^5}{5!} + (122+268x+214x^2+92x^3_2 4x^4)\frac{t^6}{6!} +$$
$$(365+1283x+1689x^2+1117x^3+466x^4+120x^5)\frac{t^7}{7!} +$$
$$(1094+5660x+11412+11656x^3+6934x^4+2844x^5+720x^6)\frac{t^8}{8!} +$$
$$(3281+23524x+68042x^2+102880x^3+89849x^4+$$
$$49996x^5+20268x^6+5040x^7)\frac{t^9}{9!} + \cdots$$

The sequence $(R_n^{(1,0,1,1),\overline{n1}}(0))_{n\geq 0}$ is

$$1, 2, 5, 14, 41, 122, 365, 1094, 3281, \ldots$$

which is sequence A007051 in the OEIS whose n-th term is $\frac{1+3^n}{2}$. This is easily explained from the recursion (30) which shows that for $n \geq 2$,

$$R_n^{(1,0,1,1),\overline{n1}} = \sum_{k=0}^{n-2} \binom{n-2}{k} R_k^{(0,0,1,0)}(0) R_{n-k-1}^{(1,0,0,1)}(0).$$

Now $R_k^{(0,0,1,0)}(0) = 1$ since only the decreasing permutation $\sigma = n(n-1)\cdots 21$ has no $MMP(0,0,1,0)$ matches and $R_{n-k-1}^{(1,0,0,1)}(0) = 2^{n-k-1}$ by (5). Thus

$$\begin{aligned}
R_n^{(1,0,1,1),\overline{n1}}(0) &= \sum_{k=0}^{n-2} \binom{n-2}{k} 2^{n-k-1} \\
&= \frac{1}{2}(1 + \sum_{n=0}^{n-2} \binom{n-2}{k} 2^{n-k-2}) \\
&= \frac{1}{2}(1 + (1+2)^{n-2}) = \frac{1}{2}(1+3^{n-2}).
\end{aligned}$$

Also the coefficient of the highest power of x occurring in $R_n^{(1,0,1,1),\overline{n1}}(x)$ is $(n-3)!$ for $n \geq 4$ which counts all the permutations that start with 2 and end with $n1$.

8 The function $R_n^{(1,1,1,1)}(x)$

We can develop a recursion to compute $R_n^{(1,1,1,1)}(x)$ but our recursion will require 4 sets of variables. That is, for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$\begin{aligned}
K_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(1,1,1,1) \text{ in } \sigma), \\
L_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(\emptyset,1,1,1) \text{ in } \sigma), \\
M_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(1,\emptyset,1,1) \text{ in } \sigma), \text{ and} \\
N_i(\sigma) &= \chi(\sigma_i \text{ matches the pattern } MMP(\emptyset,\emptyset,1,1) \text{ in } \sigma).
\end{aligned}$$

Then let

$$H_n^{(1,1,1,1)}(x_2,\dots,x_{n-1};y_2,\dots,y_{n-1};z_2,\dots,z_{n-1};w_2,\dots,w_{n-1}) = \sum_{\sigma\in S_n}\prod_{i=2}^{n} x_i^{K_i(\sigma)} y_i^{L_i(\sigma)} z_i^{M_i(\sigma)} w_i^{N_i(\sigma)}.$$

First consider all the permutations $\sigma^{[1]}$ as σ ranges over S_n. The point $(1, n+1)$ in $\sigma^{[1]}$ does not contribute to either $\mathrm{mmp}^{(1,1,1,1)}(\sigma^{[1]})$, $\mathrm{mmp}^{(\emptyset,1,1,1)}(\sigma^{[1]})$, $\mathrm{mmp}^{(1,\emptyset,1,1)}(\sigma^{[1]})$, or $\mathrm{mmp}^{(\emptyset,\emptyset,1,1)}(\sigma^{[1]})$. However,

1. if $K_i(\sigma) = 1$, then $K_{i+1}(\sigma^{[1]}) = 1$,
2. if $L_i(\sigma) = 1$, then $L_{i+1}(\sigma^{[1]}) = 1$,
3. if $M_i(\sigma) = 1$, then $K_{i+1}(\sigma^{[1]}) = 1$, and
4. if $N_i(\sigma) = 1$, then $L_{i+1}(\sigma^{[1]}) = 1$.

It follows that the contribution of the permutations of the form $\sigma^{[1]}$ to $H_{n+1}^{(1,1,1,1)}(x_2,\dots,x_n;y_2,\dots,y_n;z_2,\dots,z_n;w_2,\dots,w_n)$ is

$$H_n^{(1,1,1,1)}(x_3,\dots,x_n;y_3,\dots,y_n;x_3,\dots,x_n;y_3,\dots,y_n).$$

Next consider all the permutations $\sigma^{[n+1]}$ as σ ranges over S_n. Clearly, since each of these permutations ends with $n+1$, $n+1$ does not contribute to $\mathrm{mmp}^{(1,1,1,1)}(\sigma^{[n+1]})$, $\mathrm{mmp}^{(\emptyset,1,1,1)}(\sigma^{[n+1]})$, $\mathrm{mmp}^{(1,\emptyset,1,1)}(\sigma^{[n+1]})$,or $\mathrm{mmp}^{(\emptyset,\emptyset,1,1)}(\sigma^{[n+1]})$. However,

1. if $K_i(\sigma) = 1$, then $K_i(\sigma^{[1]}) = 1$,
2. if $L_i(\sigma) = 1$, then $K_i(\sigma^{[1]}) = 1$,
3. if $M_i(\sigma) = 1$, then $M_i(\sigma^{[1]}) = 1$, and
4. if $N_i(\sigma) = 1$, then $M_i(\sigma^{[1]}) = 1$.

It follows that the contribution of the permutations of the form $\sigma^{[n+1]}$ to $H_{n+1}^{(1,1,1,1)}(x_2,\dots,x_n;y_2,\dots,y_n;z_2,\dots,z_n;w_2,\dots,w_n)$ is

$$H_n^{(1,1,1,1)}(x_2,\dots,x_{n-1};x_2,\dots,x_{n-1};z_2,\dots,z_{n-1};z_2,\dots,z_{n-1}).$$

For $2 \le i \le n$, again consider all the permutations $\sigma^{[i]}$ as σ ranges over S_n. First note that the fact that $n+1$ is in position i means that $N_i(\sigma^{[i]}) = 1$. Then for $j < i$,

1. if $K_j(\sigma) = 1$, then $K_j(\sigma^{[i]}) = 1$,
2. if $L_j(\sigma) = 1$, then $K_j(\sigma^{[i]}) = 1$,
3. if $M_j(\sigma) = 1$, then $M_j(\sigma^{[i]}) = 1$, and

4. if $N_j(\sigma) = 1$, then $M_j(\sigma^{[i]}) = 1$.

Similarly, for $j \geq i$,

1. if $K_j(\sigma) = 1$, then $K_{j+1}(\sigma^{[i]}) = 1$,
2. if $L_j(\sigma) = 1$, then $L_{j+1}(\sigma^{[i]}) = 1$,
3. if $M_j(\sigma) = 1$, then $K_{j+1}(\sigma^{[i]}) = 1$, and
4. if $N_j(\sigma) = 1$, then $L_{j+1}(\sigma^{[i]}) = 1$.

It follows that for $2 \leq i \leq n$, the contribution of the permutations of the form $\sigma^{[i]}$ to $H_{n+1}^{(1,1,1,1)}(x_2,\ldots,x_n;y_2,\ldots,y_n;z_2,\ldots,z_n;w_2,\ldots,w_n)$ is

$$w_i H_n^{(1,1,1,1)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n;x_2,\ldots,x_{i-1},y_{i+1},\ldots,y_n;\\ z_2,\ldots,z_{i-1},x_{i+1},\ldots,x_n;z_2,\ldots,z_{i-1},y_{i+1},\ldots,y_n)$$

Thus

$$\begin{aligned}
&H_{n+1}^{(1,1,1,1)}(x_2,\ldots,x_n;y_2,\ldots,y_n;z_2,\ldots,z_{n+1};w_2,\ldots,w_n) = \\
&H_n^{(1,1,1,1)}(x_3,\ldots,x_n;y_3,\ldots,y_n;x_3,\ldots,x_n;y_3,\ldots,y_n) + \\
&H_n^{(1,1,1,1)}(x_2,\ldots,x_{n-1};x_2,\ldots,x_{n-1};z_2,\ldots,z_{n-1};z_2,\ldots,z_{n-1}) + \\
&\sum_{i=2}^{n} w_i H_n^{(1,1,1,1)}(x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n;x_2,\ldots,x_{i-1},y_{i+1},\ldots,y_n; \\
&\qquad\qquad z_2,\ldots,z_{i-1},x_{i+1},\ldots,x_n;z_2,\ldots,z_{i-1},y_{i+1},\ldots,y_n).
\end{aligned} \tag{31}$$

It is then easy to see that

$$R_n^{(1,1,1,1)}(x) = H_n^{(1,1,1,1)}(x,\ldots,x;1,\ldots,1;1,\ldots,1;1,\ldots,1). \tag{32}$$

Note that for all $\sigma \in S_3$

$$\mathrm{mmp}^{(1,1,1,1)}(\sigma) = \mathrm{mmp}^{(\emptyset,1,1,1)}(\sigma) = \mathrm{mmp}^{(1,\emptyset,1,1)}(\sigma) = 0.$$

However, for $\mathrm{mmp}^{(\emptyset,\emptyset,1,1)}(\sigma) = 1$ if σ equals 132 or 231. Thus

$$H_3^{(1,1,1,1)}(x_2,y_2,z_2,w_2) = 4 + 2w_2.$$

Using (31) and (32), one can compute that

$$\begin{aligned}
&R_1^{(1,1,1,1)}(x) = 1, \\
&R_2^{(1,1,1,1)}(x) = 2, \\
&R_3^{(1,1,1,1)}(x) = 6, \\
&R_4^{(1,1,1,1)}(x) = 24, \\
&R_5^{(1,1,1,1)}(x) = 104 + 16x, \\
&R_6^{(1,1,1,1)}(x) = 464 + 224x + 32x^2, \\
&R_7^{(1,1,1,1)}(x) = 2088 + 2088x + 768x^2 + 96x^3, \text{ and} \\
&R_8^{(1,1,1,1)}(x) = 9392 + 16096x + 11056x^2 + 3392x^3 + 384x^4.
\end{aligned}$$

In this case, the sequence $(R_n^{(1,1,1,1)}(0))_{n\geq 1}$ seems to be A128652 in the OEIS, which is the number of square permutations of length n. There is a formula for the numbers, namely,

$$a(n) = 2(n+2)4^{n-3} - 4^{2n-5}\binom{2n-6}{n-3}.$$

Problem 7. Can we prove this formula (directly)?

In this case, it is again easy to understand the coefficient of the highest power of x occurring in the polynomial $R_n^{(1,1,1,1)}(x)$. That is, one obtains the maximum number of occurrences of the pattern $MMP(1,1,1,1)$ when the permutation σ either
(i) starts with 1 $(n-1)$ or $(n-1)$ 1 and ends with either 2 n or n 2,
(ii) starts with 2 $(n-1)$ or $(n-1)$ 2 and ends with either 1 n or n 1,
(iii) starts with 1 n or n 1 and ends with either 2 $(n-1)$ or $(n-1)$ 2, or
(iv) starts with 2 n or n 2 and ends with either 1 $(n-1)$ or $(n-1)$ 1.
Thus it is easy to see that the coefficient of the highest power of x occurring in $R_n^{(1,1,1,1)}(x)$ is $16((n-4)!)x^{n-4}$ for $n \geq 5$.

9 Conclusion

In this paper, we have began the study of the distributions of quadrant marked mesh patterns of the form $MMP(a,b,c,d)$ in S_n and certain subsets of S_n. As is evidenced by several of our examples, avoiding a quadrant mesh pattern $MMP(a,b,c,d)$ is equivalent to simultaneously avoiding certain classes of classical patterns. For example, avoiding the quadrant mesh pattern $MMP(1,1,1,1)$ is equivalent to simultaneously avoiding all classical patterns of the form $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ where $\sigma_3 = 3$ and $\{\sigma_1,\sigma_2\}$ is either $\{1,4\}$, $\{2,4\}$, $\{1,5\}$, or $\{2,5\}$. Thus our results also give results on classical pattern avoidance. Note, however, that our distribution results are not necessarily equivalent to studying the occurrences of such patterns. For example, avoiding $MMP(1,0,0,0)$ is equivalent to avoiding the classical pattern 12. The number of occurrences of the pattern 12 in a permutation σ is the number of coinversions in σ. However, $\mathrm{mmp}^{(1,0,0,0)}(132) = 1$ while $\mathrm{coinv}(\sigma)(132) = 2$ since 1 participates in 2 coinversions in 132 but only contributes 1 to $\mathrm{mmp}^{(1,0,0,0)}(132)$.

There are several other special subclasses of permutations where we have studied the distribution of the quadrant mesh patterns $MMP(a,b,c,d)$. For example, with Mark Tiefenbruck, we have studied the distribution of quadrant mesh patterns in 132-avoiding permutations where we can find explicit formulas for generating functions of the form

$$\sum_{n\geq 0} t^n \sum_{\sigma\in S_n(132)} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}.$$

See [5, 6]. Similarly in [4], we have found generating functions of the form

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma\in UD_{2n}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}$$

and

$$\sum_{n\geq 1} \frac{t^{2n-1}}{(2n-1)!} \sum_{\sigma \in UD_{2n-1}} x^{\mathrm{mmp}^{(a,b,c,d)}(\sigma)}$$

where UD_n is the set of up-down permutations of S_n and (a,b,c,d) is any four-tuple which has one 1 and three 0s. For example, we have shown that

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\sigma \in UD_{2n}} x^{\mathrm{mmp}^{(1,0,0,0)}(\sigma)} = \sec(xt)^{1/x}.$$

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