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On an Open Problem of Tóth

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Abstract

In a recent paper, Tóth mentioned that it is an open problem to give the asymptotic formula for $\sum_{n\le x} P^k(n)$, where $P(n)$ is the well-known gcd-sum function and $k \ge 2$ is a fixed integer. In this paper, we use the analytic properties of the Dirichlet divisor function to obtain the asymptotic formula for it.

1 Introduction

In 1933, Pillai [8] introduced the gcd-sum function

$$P(n) = \sum_{k=1}^{n} \gcd(k, n). \tag{1}$$

By grouping the terms according to the values of $\gcd(k, n)$ we have

$$P(n) = \sum_{d|n} d\phi(n/d) = n \sum_{d|n} \frac{\phi(d)}{d}, \tag{2}$$

where ϕ is Euler's function. Many authors have studied the properties of $P(n)$, see [1, 2, 3, 4, 5, 8, 9, 10]; it is Sloane's sequence A018804. Chidambaraswamy and Sitaramachandrarao [5] showed that, given an arbitrary $\epsilon > 0$,

$$\sum_{n \le x} P(n) = e_1 x^2 \log x + e_2 x^2 + O(x^{1+\theta+\epsilon}),$$

where θ is the constant appearing in the error term of the Dirichlet divisor problem, e_1, e_2 are certain constants.

It follows from (2) that the arithmetic mean of $\gcd(1, n), \ldots, \gcd(n, n)$ is given by

$$A(n) = \frac{P(n)}{n} = \sum_{d|n} \frac{\phi(d)}{d}. \tag{3}$$

Tóth [9] showed that

$$\sum_{n \le x} A^2(n) = x(C_1 \log^3 x + C_2 \log^2 x + C_3 \log x + C_4) + O(x^{1/2+\epsilon}), \tag{4}$$

where C_1, C_2, C_3, C_4 are computable constants. Furthermore, he listed some open problems concerning the gcd-sum function, one of which is to derive the asymptotic formula for $\sum_{n \le x} P^k(n)$, where $k \ge 2$ is a fixed integer.

In this paper, we use the analytic method to get the asymptotic formula for $\sum_{n \le x} A^k(n)$.

Theorem 1. *Let $k \ge 2$ be a fixed integer. Then*

$$\sum_{n \le x} A^k(n) = xQ_{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}), \tag{5}$$

where $Q_{2^k-1}(t)$ is a polynomial of degree $2^k - 1$ in t and

$$\beta_2 = \frac{1}{2}, \quad \beta_3 = \frac{5}{8}, \quad \beta_4 = \frac{7}{9}, \quad \beta_5 = \frac{31}{36}, \quad \beta_6 = \frac{207}{224}, \quad \beta_k = 1 - 2^{-\frac{2}{3}k}/50, \quad k \ge 7.$$

Corollary 2. *Let $k \ge 2$ be a fixed integer. Then*

$$\sum_{n \le x} P^k(n) = x^2 Q'_{2^k-1}(\log x) + O(x^{1+\beta_k+\epsilon}), \tag{6}$$

where $Q'_{2^k-1}(t)$ is a polynomial of degree $2^k - 1$ in t.

Theorem 3. *Let $k \geq 2$ be a fixed integer and*

$$E_k(x) = \sum_{n \leq x} A^k(n) - xQ_{2^k-1}(\log x).$$

Then for $k = 3, 4, 5,$ we have

$$\int_1^U E_k(x)dx \ll U^{1+\delta_k+\epsilon},$$

where

$$\delta_3 = 1/2, \quad \delta_4 = 0.6030739, \quad \delta_5 = 0.773114.$$

2 Preliminary Lemmas

Lemma 4. *Let s be a complex number with $Re(s) > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \zeta^{2^k}(s) G_k(s),$$

where $G_k(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$ is a Dirichlet series which is absolutely convergent for $Re(s) > 1/2$.

Proof. Recall that $A(n)$ is multiplicative function. Then it follows from the Euler product representation that for $\mathrm{Re}(s) > 1$,

$$F(s) := \sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{A^k(p^\alpha)}{p^{\alpha s}}\right). \tag{7}$$

where

$$A(p^\alpha) = 1 + \frac{\phi(p)}{p} + \cdots + \frac{\phi(p^\alpha)}{p^\alpha} = 1 + \alpha - \frac{\alpha}{p}, \quad \alpha \geq 1.$$

Thus we have

$$\begin{aligned} 1 + \sum_{\alpha=1}^{\infty} \frac{A^k(p^\alpha)}{p^{\alpha s}} &= 1 + \sum_{\alpha=1}^{\infty} \frac{(1 + \alpha - \frac{\alpha}{p})^k}{p^{\alpha s}} \\ &= 1 + \frac{2^k}{p^s} - \frac{k \cdot 2^{k-1}}{p^{s+1}} + \ldots + (-1)^k \frac{1}{p^{s+k}} \\ &\quad + \frac{3^k}{p^{2s}} - \frac{2k \cdot 3^{k-1}}{p^{2s+1}} + \ldots + (-1)^k \frac{2^k}{p^{2s+k}} \\ &\quad + \frac{4^k}{p^{3s}} - \frac{3k \cdot 4^{k-1}}{p^{3s+1}} + \ldots + (-1)^k \frac{3^k}{p^{3s+k}} \\ &\quad + \ldots =: x_k(s). \end{aligned}$$

We substitute the above formula to the formula (7) to get

$$\sum_{n=1}^{\infty}\frac{A^k(n)}{n^s}=\prod_p x_k(s)=\zeta^{2^k}(s)\cdot\prod_p(1-\frac{1}{p^s})^{2^k}\cdot x_k(s)=:\zeta^{2^k}(s)G_k(s),$$

where

$$\begin{aligned}G_k(s)\ &=\prod_p(1-\frac{1}{p^s})^{2^k}\cdot x_k(s)\\&=\prod_p\Bigg(1-\frac{k\cdot 2^{k-1}}{p^{s+1}}+\ldots+(-1)^k\frac{1}{p^{s+k}}\\&+\frac{3^k-4^k+\binom{2^k}{2}}{p^{2s}}+\frac{k\cdot 2^{2k-1}-2k\cdot 3^{k-1}}{p^{2s+1}}+\ldots\\&+\frac{4^k-6^k+\binom{2^k}{2}2^k-\binom{2^k}{3}}{p^{3s}}+\frac{2k\cdot 2^k3^{k-1}-3k\cdot 4^{k-1}-k\cdot 2^{k-1}\cdot\binom{2^k}{2}}{p^{3s+1}}+\ldots\Bigg).\end{aligned}$$

From the above formula, it is easy to see that $G_k(s)$ can be expanded to a Dirichlet series $\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$, which is absolutely convergent for $\mathrm{Re}(s)>1/2$.

□

Lemma 5. *Suppose* $1/2\le\sigma\le 1$, *then*

$$\zeta(\sigma+it)\ll(|t|+2)^{\frac{1-\sigma}{3}}\log(|t|+2). \tag{8}$$

Proof. We define the function $\mu(\sigma)$ for each σ as the infimum of number $c\ge 0$ such that $\zeta(\sigma+it)\ll t^c$, or alternatively as

$$\mu(\sigma)=\limsup_{t\to\infty}\frac{\log|\zeta(\sigma+it)|}{\log t}.$$

Then $\mu(\sigma)$ is continuous, nonincreasing and for $\sigma_1\le\sigma\le\sigma_2$,

$$\mu(\sigma)\le\mu(\sigma_1)\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}+\mu(\sigma_2)\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}.$$

By the well-known estimates

$$\zeta(1/2+it)\ll t^{1/6},\quad \zeta(1+it)\ll\log t,$$

we can easily get the formula (8). □

Lemma 6. *If* $\zeta(s)=\chi(s)\zeta(1-s)$, *then the estimate*

$$\chi(s)\ll(|t|+2)^{1/2-\sigma}$$

holds uniformly for $0\le\sigma\le 1$.

Proof. Using standard properties of the gamma-function one may write the functional equation of $\zeta(s)$ as

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) = (2\pi)^s/(2\Gamma(s)\cos(\pi s/2)).$$

From Stirling's formula

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi|t|}{2}}\left(1+O(|t|^{-1})\right) \qquad (|t| \geq t_0),$$

it follows that

$$\begin{aligned}\chi(s) &= \left(\frac{2\pi}{|t|}\right)^{\sigma+i|t|-\frac{1}{2}} e^{i(|t|+\frac{\pi}{4})}\left(1+O(|t|^{-1})\right)\\ &\ll (|t|+2)^{1/2-\sigma}.\end{aligned}$$

□

3 Proofs of Theorem 1 and Corollary 2

Recall that the generalized divisor function

$$d_k(n) = \sum_{n=n_1\cdots n_k} 1,$$

and its Dirichlet series is

$$\zeta^k(s) = \sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}.$$

From [7, Theorem 13.2 and 13.3] it follows that

$$\sum_{n\leq x} d_{2^k}(n) = xP_{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}), \tag{9}$$

where $P_{2^k-1}(t)$ is a polynomial of degree 2^k-1 in t, and

$$\beta_2 = \frac{1}{2}, \quad \beta_3 = \frac{5}{8}, \quad \beta_4 = \frac{7}{9}, \quad \beta_5 = \frac{31}{36}, \quad \beta_6 = \frac{207}{224}, \quad \beta_k = 1 - 2^{-\frac{2}{3}k}/50, \qquad k \geq 7.$$

Then by Lemma 4, we have that

$$\sum_{n\leq x} A^k(n) = \sum_{m\ell\leq x} d_{2^k}(m)g(\ell) = \sum_{\ell\leq x} g(\ell)\sum_{m\leq x/\ell} d_{2^k}(m),$$

and formula (9) applied to the inner sum gives

$$\sum_{n\leq x} A^k(n) = \sum_{\ell\leq x} g(\ell)\left\{\frac{x}{\ell}P_{2^k-1}\left(\log(\frac{x}{\ell})\right) + O\left((\frac{x}{\ell})^{\beta_k+\epsilon}\right)\right\}$$

$$= xQ_{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}),$$

if we notice from Lemma 4 that the infinite series $\sum_{\ell=1}^{\infty} \frac{g(\ell)}{\ell}$ and $\sum_{\ell=1}^{\infty} \frac{g(\ell)\log^k \ell}{\ell}$ are absolutely convergent, and

$$\sum_{\ell \le x} |g(\ell)| \ll x^{1/2+\epsilon}.$$

From the definitions of $P(n)$ and Abel's summation formula, we can easily get

$$\sum_{n \le x} P^k(n) = x^2 Q'_{2^k-1}(\log x) + O(x^{1+\beta_k+\epsilon}),$$

where $Q'_{2^k-1}(t)$ is a polynomial of degree $2^k - 1$ in t.

4 Proof of Theorem 3

It suffices to prove that

$$\int_U^{2U} E_k(x)dx \ll U^{1+\delta_k+\epsilon}, \tag{10}$$

where

$$\delta_3 = 1/2, \quad \delta_4 = 0.6030739, \quad \delta_5 = 0.773114.$$

By Perron's formula (see for example, [6, Chapter 5]), we have for $T \le x \le 2T$ that

$$\sum_{n \le x} A^k(n) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta^{2^k}(s) G_k(s) \frac{x^s}{s} ds + O(T^{\varepsilon}).$$

Then we move the integration to the parallel segment with $\mathrm{Re}(s) = 1 - \varepsilon$ to get

$$E_k(x) = \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \zeta^{2^k}(s) G_k(s) \frac{x^s}{s} ds + O(T^{\varepsilon}).$$

So

$$\begin{aligned} \int_U^{2U} E_k(x)dx &= \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\zeta^{2^k}(s) G_k(s)}{s} \left(\int_U^{2U} x^s dx \right) ds + O(U^{1+\varepsilon}) \\ &= \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\zeta^{2^k}(s) G_k(s) (2^{s+1} - 1) U^{s+1}}{s(s+1)} ds + O(U^{1+\varepsilon}). \end{aligned} \tag{11}$$

Moving the integral line in the last integral of (11) to $\sigma = c$, where $\frac{1}{2} < c < 1$, we have

$$\int_U^{2U} E_k(x)dx \ll U^{1+c} \int_{c-iT}^{c+iT} \frac{|\zeta(s)|^{2^k}}{|s(s+1)|} ds \ll U^{1+c} \int_1^T \frac{|\zeta(c+it)|^{2^k}}{T^2} ds, \tag{12}$$

if we notice that $G_k(s)$ is absolutely convergent in $\mathrm{Re}(s) > \frac{1}{2}$.

For the case $k = 3$, it follows from [7, Theorem 8.3] that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^8 dt \ll T^{\frac{3}{2}}.$$

On taking $c = 1/2 + \varepsilon$ in (12), we have

$$\int_U^{2U} E_3(x)dx \ll U^{3/2}. \tag{13}$$

For the case $k = 4$, from [7, Theorem 8.3] and [7, Theorem 8.4], it follows that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^{16} dt \ll T^{1+\frac{350}{216}}$$

and

$$\int_1^T |\zeta(\sigma + it)|^{16} dt \ll T^{1+\varepsilon},$$

where σ satisfies

$$\frac{12408}{4537 - 4890\sigma} = 16,$$

which gives $\sigma = 0.7692229$. By [7, Lemma 8.3], we have

$$\int_1^T |\zeta(0.6030739 + it)|^{16} dt \ll T^2.$$

In the formula (12), we take $c = 0.6030739$ to get

$$\int_U^{2U} E_4(x)dx \ll U^{1+0.6030739}. \tag{14}$$

Similarly, we can get $\delta_k, k \geq 5$. For example, $\delta_5 = 0.773114$.

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