Journal of Integer Sequences, Vol. 17 (2014),
Article 14.4.3

Combinatorial Expressions Involving Fibonacci, Hyperfibonacci, and Incomplete Fibonacci Numbers

Hacène Belbachir and Amine Belkhir
USTHB, Faculty of Mathematics
RECITS Laboratory, DG-RSDT
BP 32, El Alia
16111 Bab Ezzouar, Algiers
Algeria
hbelbachir@usthb.dz
hacenebelbachir@gmail.com
abelkhir@usthb.dz
ambelkhir@gmail.com

Abstract

We give a combinatorial interpretation, an explicit formula and some other properties of hyperfibonacci numbers. Further, we deduce relationships between Fibonacci, hyperfibonacci, and incomplete Fibonacci numbers.

1 Introduction

The hyperfibonacci numbers $F_n^{(r)}$ introduced recently by Dil and Mező [5]. There are defined by the relation

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \text{ with } F_n^{(0)} = F_n \text{ and } F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \tag{1}$$

where r is a positive integer and F_n is the n-th Fibonacci number defined recursively by

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2, \text{ and } F_0 = 0, F_1 = 1.$$

The double recurrence relation for the hyperfibonacci numbers is given by

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}. \tag{2}$$

The Fibonacci number F_{n+1} counts the number of tilings of a $(1 \times n)$-board with cells labeled $1, 2, \ldots, n$ using (1×1)-squares and (1×2)-dominoes. We follow the notation introduced by Benjamin and Quinn [4] and define $f_n = F_{n+1}$ and get $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$.

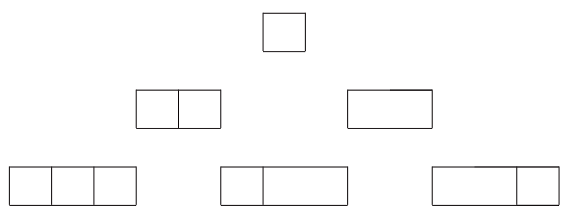

Figure 1: Tilings of length $1, 2$ and 3 using squares and dominoes.

The following lemma will be used to establish our results.

Lemma 1. *[4] The number of n-tilings using exactly k dominoes is*

$$\binom{n-k}{k}, \quad (k = 0, 1, \ldots, \lfloor n/2 \rfloor), \tag{3}$$

where $\lfloor n \rfloor$ is the integer part of n.

From Lemma 1, Benjamin and Quinn [4] gave a closed form for f_n by summing over all values of k, the number of ways to tile an n-board with squares and dominoes is

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}. \tag{4}$$

Our aim is to investigate, as the authors do for generalized Fibonacci and Lucas sequences [1, 2], the tilings approach to give a combinatorial interpretation for hyperfibonacci numbers. More precisely, in Section 2, a combinatorial interpretation of hyperfibonacci numbers is presented. In Section 3, we give a closed form for hyperfibonacci numbers. Finally, in Section 4, we provide a combinatorial interpretation for incomplete Fibonacci numbers and we establish a relation between incomplete Fibonacci numbers and hyperfibonacci numbers.

2 Combinatorial interpretation

In this section, we present combinatorial interpretation for hyperfibonacci numbers. Later, we derive a relation involving hyperfibonacci and Fibonacci numbers.

Theorem 2. *Let* $f_n^{(r)}$ *counts the number of ways to tile an* $(n+2r)$*-board with at least* r *dominoes. Then* $f_0^{(r)} = 1$, $f_n^{(0)} = f_n$, *and for* $n \geq 2$,

$$f_n^{(r)} = f_{n-1}^{(r)} + f_n^{(r-1)}. \tag{5}$$

Proof. We start by verifying the initial conditions. For $n = 0$, there is one $2r$ -tiling with at least r dominoes, and for $r = 0$, there are f_n n-tilings with at least 0 dominoes (there is no restriction on the number of dominoes). Now, if $n \geq 2$, an $(n+2r)$-board can either end with a square or with a domino. If it ends with a square, then the remaining $(n+2r-1)$-board can be tiled with at least r dominoes in $f_{n-1}^{(r)}$ ways. If it ends with a domino, then the remaining $(n+2r-2)$-board can be tiled with at least $r-1$ dominoes in $f_n^{(r-1)}$ ways. □

As $f_0^{(r)} = F_1^{(r)} = 1$ and $f_n^{(0)} = F_{n+1}^{(0)} = F_{n+1}$, it seen that for $n \geq 0$, we have $f_n^{(r)} = F_{n+1}^{(r)}$. Letting $f_{-1}^{(r)} = 0$, because there is not $(2r-1)$-tiling with at least r dominoes. Now, we have a combinatorial interpretation for the hyperfibonacci numbers.

Theorem 3. *For* $n,r \geq 0$, $F_{n+1}^{(r)} = f_n^{(r)}$ *counts the number of ways to tile an* $(n+2r)$*-board with at least* r *dominoes.*

The first few values of $f_n^{(r)}$ are as follows:

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$f_n^{(0)}$	1	1	2	3	5	8	13	21	34	55	89
$f_n^{(1)}$	1	2	4	7	12	20	33	54	88	143	232
$f_n^{(2)}$	1	3	7	14	26	46	79	133	221	364	594
$f_n^{(3)}$	1	4	11	25	51	97	179	309	530	894	1490

Table 1: Some values of $f_n^{(r)}$

Theorem 4. *For* $n \geq 0$, *and* $r \geq 1$, *we have*

$$f_n^{(r)} = \sum_{k=r-1}^{\lfloor n/2 \rfloor + r-1} (n+2r-k-1)\, f_n^{(r-1)}. \tag{6}$$

Proof. The number of ways to tile a board of length $n + 2r - 2$ with at least $r - 1$ dominoes is $f_n^{(r-1)}$. Now, to obtain an $(n + 2r)$-tilings with at least r dominoes from an $(n + 2r - 2)$-tilings with at least $r - 1$ dominoes, it suffices to add a domino. Let k $(r - 1 \leq k \leq \lfloor n/2 \rfloor + r - 1)$ be the number of dominos in an $(n + 2r - 2)$-tilings, then it contains $n+2r-2k-2$ squares, so there are $n+2r-k-2$ tiles in the $(n+2r-2)$-tilings. The number of ways to place a domino in an $(n+2r-2)$ -tiling with k $(r - 1 \leq k \leq \lfloor n/2 \rfloor + r - 1)$ dominoes is $n + 2r - k - 1$. □

The hyperfibonacci numbers $f_n^{(r)}$ can be expressed as a sum of a product of binomial coefficients and Fibonacci numbers.

Theorem 5. *For $n \geq 0$, and $r \geq 1$, we have*

$$f_n^{(r)} = \sum_{k=0}^{n} \binom{n + r - k - 1}{r - 1} f_k. \tag{7}$$

Proof. Let $k+1$,$k+2$ $(0 \leq k \leq n)$ be the position of the r-th (from the right) domino, then there are f_k ways to tile the first k cells, and there are $\binom{n+r-k-1}{r-1}$ ways to tile cells from $k+3$ to $n + 2r$ with exactly $r - 1$ dominoes. Thus, there are $\binom{n+r-k-1}{r-1}$ f_k $(n + 2r)$-tilings with the r-th domino covering cells $k + 1$,$k + 2$. Summing over k, we get relation (7). □

From the relation (7), the following convolution is derived, the hyperfibonacci numbers are obtained as a convolution between the anti-diagonal terms of Pascal's triangle and the Fibonacci numbers.

Corollary 6. *For $n \geq 0$, and $r \geq 1$, we have*

$$f_n^{(r)} = \sum_{k=0}^{n} \binom{r - 1 + k}{k} f_{n-k}. \tag{8}$$

3 Closed form for hyperfibonacci numbers

The following theorem gives an explicit expression of $f_n^{(r)}$ in terms of binomial coefficients.

Theorem 7. *For $n \geq 0$, and $r \geq 1$, we have*

$$f_n^{(r)} = \sum_{k=r}^{\lfloor n/2 \rfloor + r} \binom{n + 2r - k}{k}. \tag{9}$$

Proof. An $(n + 2r)$-tiling with at least r dominoes can contains k dominoes where $k = r, r+1, \ldots, \lfloor n/2 \rfloor + r$. Using Lemma 1, the number of $(n + 2r)$-tilings with exactly k dominoes is $\binom{n+2r-k}{k}$. Summing over k we get (9). □

The relation (9) is a truncated diagonal sum of Pascal's Triangle. This allow us to state the following:

Theorem 8. *For $n \geq 0$, and $r \geq 1$, we have*

$$f_n^{(r)} = f_{n+2r} - \sum_{k=0}^{r-1} \binom{n+2r-k}{k}. \tag{10}$$

Remark 9. For $n \geq 0$, we have some special cases

$$f_n^{(1)} = \sum_{k=0}^{n} f_k = f_{n+2} - 1. \tag{11}$$

$$f_n^{(2)} = \sum_{k=0}^{n} (k+1) f_{n-k} = f_{n+4} - n - 4. \tag{12}$$

4 Relationships between the hyperfibonacci and incomplete Fibonacci numbers

We give a combinatorial interpretation for the incomplete Fibonacci numbers. This allow us to obtain a relationship involving the Fibonacci, hyperfibonacci, and incomplete Fibonacci numbers.

Filipponi [6] defined the incomplete Fibonacci numbers $F_n(k)$ by the following relation for $n \geq 0$

$$F_{n+1}(k) = \sum_{j=0}^{k} \binom{n-j}{j} \quad \left(0 \leq k \leq \left\lfloor \tfrac{n+1}{2} \right\rfloor\right). \tag{13}$$

Theorem 10. *Let $f_n(k)$ counts the number of ways to tile an n-board with at most k dominoes. Then*

$$f_n(k) = \sum_{j=0}^{k} \binom{n-j}{j} \quad \left(0 \leq k \leq \left\lfloor \tfrac{n}{2} \right\rfloor\right). \tag{14}$$

Proof. It follows from Lemma 1, by summing over j. □

Note that, if we take $k = \left\lfloor \frac{n}{2} \right\rfloor$, then the $f_n(k)$ is reduced to the Fibonacci number f_n.

Theorem 11. *For $n \geq 0$, we have*

$$f_n(k) = f_{n-1}(k) + f_{n-2}(k-1), \tag{15}$$

with $f_n(0) = f_0(k) = 1$.

Proof. An n-tilings with at most k dominoes either ends with a square or a domino. If it ends with a square, there are $f_{n-1}(k)$ ways to tile the first $n-1$ cells with at most k dominoes and if it ends with a domino, there are $f_{n-2}(k-1)$ ways to tile the first $n-2$ cells with at most $k-1$ dominoes. □

The following theorem gives a combinatorial interpretation for incomplete Fibonacci numbers.

Theorem 12. *For $n,k \geq 0$ with $0 \leq k \leq \lfloor n/2 \rfloor$, we have $F_{n+1}(k) = f_n(k)$. That is, $F_{n+1}(k)$ counts the number of ways to tile an n-board with at most k dominoes.*

From relations (13) and (15), we obtain the following non-homogenous second order recurrence relation as stated by Filipponi [6].

For $n \geq 0$, we have

$$f_n(k) = f_{n-1}(k) + f_{n-2}(k) - \binom{n-k}{k}. \tag{16}$$

Using the approach of Benjamin et al., we recover Filipponi's formula [6].

Theorem 13. *For $n \geq 0$, we have*

$$f_{n+2h}(k+h) = \sum_{j=0}^{h} \binom{h}{j} f_{n+j}(k+j) \quad \left(0 \leq k \leq \frac{n-h}{2}\right). \tag{17}$$

Proof. The left hand side counts the number of ways to tile an $(n+2h)$-board with at most $k+h$ dominoes. Now, we show that the right hand side counts the same tilings by conditioning on the number of dominoes that appear among the first h tiles. There are $\binom{h}{j}$ ways to select j positions for the dominoes among the first h tiles and $f_{n+h-j}(k+h-j)$ ways to tile remaining $n+h-j$ cells with at most $k+h-j$ dominoes. □

Using (10) and (14), we give a relation between Fibonacci numbers, incomplete Fibonacci numbers and hyperfibonacci numbers.

Corollary 14. *For integers $n,r \geq 0$, we have*

$$f_{n+2r} = f_n^{(r)} + f_{n+2r}(r-1). \tag{18}$$

This states that, for given nonnegative integers n and r, every Fibonacci number can be written as a combination of an incomplete Fibonacci number and an hyperfibonacci number.

5 Acknowledgments

The authors thank the anonymous referee for the throughout reading of the manuscript and valuable comments.

References

[1] H. Belbachir and A. Belkhir. Identities related to generalized Fibonacci numbers via tiling approach, submitted.

[2] H. Belbachir and A. Belkhir. Tiling approach to obtain identities for generalized Fibonacci and Lucas numbers, *Ann. Math. Inform.*, **41** (2013), 13–17.

[3] A. T. Benjamin, J. J. Quinn, and F. E. Su. Phased tilings and generalized Fibonacci identities. *Fibonacci Quart.* **38** (2000), 282–288.

[4] A. T. Benjamin and J. J. Quinn. *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003.

[5] A. Dil and I. Mező. A symmetric algorithm for hyperharmonic and Fibonacci numbers, *Appl. Math. Comput.* **206** (2008), 942–951.

[6] P. Filipponi. Incomplete Fibonacci and Lucas numbers, *Rend. Circ. Mat. Palermo* **45** (1996) 37–56.

2000 *Mathematics Subject Classification*: Primary 11B39; Secondary 05B45; 05A19; 11B37.
Keywords: Fibonacci number, hyperfibonacci number, tiling, bijective proofs, incomplete Fibonacci number.

(Concerned with sequences A000045, A000071, and A136431.)

Received October 2 2013; revised version received January 22 2014. Published in *Journal of Integer Sequences*, February 16 2014.

Return to Journal of Integer Sequences home page.