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On the Lcm-Sum Function

Soichi Ikeda and Kaneaki Matsuoka
Graduate School of Mathematics
Nagoya University
Furocho Chikusaku Nagoya 464-8602
Japan
m10004u@math.nagoya-u.ac.jp
m10041v@math.nagoya-u.ac.jp

Abstract

We consider a generalization of the lcm-sum function, and we give two kinds of asymptotic formulas for the sum of that function. Our results include a generalization of Bordellès's results and a refinement of the error estimate of Alladi's result. We prove these results by the method similar to those of Bordellès.

1 Introduction

Pillai [6] first defined the gcd-sum function

$$g(n) = \sum_{j=1}^{n} \gcd(j, n)$$

and studied this function. The function $g(n)$ was defined again by Broughan [3]. Broughan considered

$$G_\alpha(x) = \sum_{n \le x} n^{-\alpha} g(n)$$

for $\alpha \in \mathbb{R}$ and $x \ge 1$, and obtained some asymptotic formulas for $G_\alpha(x)$. The function $G_\alpha(x)$ was studied by some authors (see, for example, [4, 8]). Some generalizations of the function $g(n)$ was considered (see, for example, [2, 10]).

On the other hand, the lcm-sum function

$$l(n) := \sum_{j=1}^{n} \operatorname{lcm}(j,n)$$

was considered by some authors. Alladi [1] studied the sum

$$\sum_{j=1}^{n} (\operatorname{lcm}(j,n))^r \qquad (r \in \mathbb{R},\ r \geq 1)$$

and obtained

$$\sum_{n\leq x} \sum_{j=1}^{n} (\operatorname{lcm}(j,n))^r = \frac{\zeta(r+2)}{2(r+1)^2\zeta(2)} x^{2r+2} + O(x^{2r+1+\epsilon}). \tag{1}$$

We define the functions

$$L_a(n) := \sum_{j=1}^{n} (\operatorname{lcm}(j,n))^a$$
$$T_a(x) := \sum_{n\leq x} L_a(n)$$

for $a \in \mathbb{Z}$ and $x \geq 1$.

Bordellès studied the sums $T_1(x)$ and $T_{-1}(x)$ and obtained

$$l(n) = \frac{1}{2}((\mathrm{Id}^2 \cdot (\varphi + \tau_0)) * \mathrm{Id})(n),$$
$$\sum_{n\leq x} \sum_{j=1}^{n} \operatorname{lcm}(j,n) = \frac{\zeta(3)}{8\zeta(2)} x^4 + O(x^3 (\log x)^{2/3} (\log\log x)^{4/3}) \quad (x > e),$$
$$\sum_{n\leq x} \sum_{j=1}^{n} \frac{1}{\operatorname{lcm}(j,n)} = \frac{(\log x)^3}{6\zeta(2)} + \frac{(\log x)^2}{2\zeta(2)} \left(\gamma + \log\left(\frac{\mathcal{A}^{12}}{2\pi}\right)\right) + O(\log x),$$

where $\mathrm{Id}^a(n) = n^a$ $(a \in \mathbb{Z})$,

$$\tau_0(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{otherwise,} \end{cases}$$

$F * G$ is the usual Dirichlet convolution product, and $\mathcal{A}$ is the Glaisher-Kinkelin constant [7, p. 25]). Gould and Shonhiwa [5] stated that the log-factors in the error term in the second formula can be removed.

In this paper we study $T_a(x)$ for $a \geq 2$ and $a \leq -2$. The following theorems are our main results. These results are proved by the methods similar to those of Bordellès [2, Section 6].

We write $f(x) = O(g(x))$, or equivalently $f(x) \ll g(x)$, where there is a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all values of x under consideration.

Theorem 1. *Let* B_n *be Bernoulli numbers defined by*

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}.$$

If we define

$$\varphi_k(n):=\sum_{d|n}\mu(d)\Big(\frac{d}{n}\Big)^k$$

and

$$M_a(n):=\left(\mathrm{Id}^{2a}\cdot\left(\frac{1}{a+1}\varphi+\frac{1}{2}\tau_0+\frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\varphi_k\right)\right)(n),$$

then for $a\in\mathbb{Z}$ *we have*

$$L_a(n)=(M_a*\mathrm{Id}^a)(n).$$

Theorem 2. *Let* $x>e$ *and* $a\in\mathbb{N}$*. Then we have*

$$\sum_{n\le x}L_a(n)=\frac{\zeta(a+2)}{2(a+1)^2\zeta(2)}x^{2a+2}+O(x^{2a+1}(\log x)^{2/3}(\log\log x)^{4/3})\quad(\textit{as } x\to\infty),$$

where the implied constant depends on a*.*

Theorem 3. *Let* $x\ge 1$ *and* $k\in\mathbb{N}$ *with* $k\ge 2$*. Then we have*

$$\sum_{n=1}^{\infty}L_{-k}(n)=\frac{\zeta(k)}{2}\left(1+\frac{\zeta(k)^2}{\zeta(2k)}\right)\tag{2}$$

and

$$\sum_{n\le x}L_{-k}(n)=\frac{\zeta(k)}{2}\left(1+\frac{\zeta(k)^2}{\zeta(2k)}\right)-\frac{\zeta(k)x^{-k+1}\log x}{(k-1)\zeta(k+1)}+O(x^{-k+1})\quad(\textit{as } x\to\infty),$$

where the implied constant depends on k*.*

We note that the function $L_a(n)$ is not multiplicative for all $a\in\mathbb{Z}\setminus\{0\}$, but we can write $L_a(n)$ $(a\ge 1)$ explicitly by Dirichlet convolution. In the proof of Theorem 2 we use this fact. The error estimates in Theorem 2 are better than (1). Since we have

$$g_r(n):=\sum_{j=1}^{n}(\gcd(j,n))^r>\varphi(n)$$

for all $r\in\mathbb{R}$, the sum

$$\sum_{n=1}^{\infty}g_r(n)$$

is divergent for all r. Therefore the behavior of the sum $T_a(x)$ ($a\in\mathbb{Z}$ and $a\le -2$) is completely different from that of the sum

$$\sum_{n\le x}g_a(n).$$

2 Lemmas for the proof of theorems

In this section, we collect some auxiliary results and definitions.

Let $B_n(x)$ be Bernoulli polynomials defined by

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}.$$

The following relations are well-known [7, p. 59].

$$B_n(x+1)-B_n(x)=nx^{n-1},$$
$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k},$$
$$B_n(0)=B_n(1)=B_n \quad (n>1).$$

Lemma 4. *Let $m,n\in\mathbb{N}$ and*

$$S_n(m):=\sum_{l=1}^{m}l^n.$$

Then we have

$$S_n(m)=\frac{m^{n+1}}{n+1}+\frac{1}{2}m^n+\frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k+1}B_{k+1}m^{n-k}.$$

Proof. We have

$$\begin{aligned}S_n(m)&=\frac{1}{n+1}(B_{n+1}(m+1)-B_{n+1}(1))\\&=\frac{1}{n+1}(B_{n+1}(m)+(n+1)m^n-B_{n+1})\\&=\frac{1}{n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}B_km^{n+1-k}+(n+1)m^n-B_{n+1}\right)\\&=\frac{m^{n+1}}{n+1}+\frac{1}{2}m^n+\frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k+1}B_{k+1}m^{n-k}.\end{aligned}$$

□

We use the following lemmas in the proof of Theorem 2.

Lemma 5. *Let $r,k\in\mathbb{N}$ with $r>k$ and $x\geq 1$. We have*

$$\sum_{n\leq x}n^r\varphi_k(n)\leq x^{r+1}.$$

Proof. We have

$$\begin{aligned}\sum_{n\le x} n^r \varphi_k(n) &= \sum_{n\le x} n^{r-k} \sum_{d\mid n} \mu(d) d^k \\ &= \sum_{d\le x} \mu(d) d^k \big(d^{r-k} + (2d)^{r-k} + \cdots + (d\lfloor x/d \rfloor)^{r-k}\big) \\ &= \sum_{d\le x} \mu(d) d^r \sum_{j\le x/d} j^{r-k} \\ &\le x^{r+1}.\end{aligned}$$

□

Lemma 6. *Let $r \in \mathbb{N}$ and $x > e$. We have*

$$\sum_{n\le x} n^r \varphi(n) = \frac{x^{r+2}}{(r+2)\zeta(2)} + O(x^{r+1}(\log x)^{2/3}(\log\log x)^{4/3}) \quad (\text{as } x \to \infty),$$

where the implied constant depends on r.

Proof. We can obtain the lemma by the estimate [11]

$$\sum_{n\le x} \varphi(n) = \frac{x^2}{2\zeta(2)} + O(x(\log x)^{2/3}(\log\log x)^{4/3})$$

and the partial summation formula. □

3 Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. We have

$$\begin{aligned}\sum_{j=1}^{n} \left(\frac{j}{\gcd(n,j)}\right)^a &= \sum_{d\mid n} \frac{1}{d^a} \sum_{\substack{j=1 \\ \gcd(j,n)=d}}^{n} j^a \\ &= \sum_{d\mid n} \frac{1}{d^a} \sum_{\substack{k\le n/d \\ \gcd(k,n/d)=1}} (kd)^a = \sum_{d\mid n} \sum_{\substack{k\le n/d \\ \gcd(k,n/d)=1}} k^a.\end{aligned}$$

By Lemma 4 we have

$$
\begin{aligned}
\sum_{\substack{k\le N\\ \gcd(k,N)=1}} k^a &= \sum_{k\le N} k^a \sum_{d|\gcd(k,N)} \mu(d) = \sum_{d|N} d^a\mu(d) \sum_{m\le N/d} m^a \\
&= \sum_{d|N} d^a \mu(d)\Bigg(\frac{1}{a+1}\left(\frac{N}{d}\right)^{a+1} + \frac{1}{2}\left(\frac{N}{d}\right)^{a} + \\
&\qquad + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\left(\frac{N}{d}\right)^{a-k}\Bigg) \\
&= \frac{N^a}{a+1}\sum_{d|N}\mu(d)\frac{N}{d} + \frac{N^a}{2}\sum_{d|N}\mu(d) + \\
&\qquad + \frac{N^a}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\sum_{d|N}\mu(d)\left(\frac{d}{N}\right)^{k} \\
&= N^a\left(\frac{1}{a+1}\varphi + \frac{1}{2}\tau_0 + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\varphi_k\right)(N).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
L_a(n) &= n^a \sum_{j=1}^{n}\left(\frac{j}{\gcd(n,j)}\right)^{a} \\
&= \sum_{d|n}\left(\frac{n}{d}\right)^{2a}\cdot\left(\frac{1}{a+1}\varphi + \frac{1}{2}\tau_0 + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1}B_{k+1}\varphi_k\right)(n/d)\cdot d^a \\
&= (M_a * \mathrm{Id}^a)(n).
\end{aligned}
$$

$\square$

Proof of Theorem 2. By Lemma 5, Lemma 6 and Theorem 1, we have

$$\begin{aligned}
\sum_{n\le x} L_a(n) &= \sum_{n\le x}(M_a * \mathrm{Id}^a)(n) = \sum_{d\le x} d^a \sum_{m\le x/d} M_a(m) \\
&= \sum_{d\le x} d^a \sum_{m\le x/d} m^{2a}\left(\frac{1}{a+1}\varphi + \frac{1}{2}\tau_0 + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1} B_{k+1}\varphi_k\right)(m) \\
&= \frac{1}{a+1}\sum_{d\le x} d^a \sum_{m\le x/d} m^{2a}\varphi(m) + O(x^{a+1}) + \\
&\quad + \frac{1}{a+1}\sum_{k=1}^{a-1}\binom{a+1}{k+1} B_{k+1}\sum_{d\le x} d^a \sum_{m\le x/d} m^{2a}\varphi_k(m) \\
&= \frac{1}{a+1}\sum_{d\le x} d^a\left(\frac{1}{(2a+2)\zeta(2)}\left(\frac{x}{d}\right)^{2a+2} + \right. \\
&\quad \left. + O\left(\left(\frac{x}{d}\right)^{2a+1}(\log(x/d))^{2/3}(\log\log(x/d))^{4/3}\right)\right) + \\
&\quad + O(x^{a+1}) + O\left(\sum_{d\le x} d^a (x/d)^{2a+1}\right) \\
&= \frac{x^{2a+2}}{(a+1)(2a+2)\zeta(2)}\sum_{d\le x}\frac{1}{d^{a+2}} + O(x^{2a+1}(\log x)^{2/3}(\log\log x)^{4/3}) + \\
&\quad + O(x^{2a+1}).
\end{aligned}$$

This implies the theorem. □

4 Proof of Theorem 3

Proof of Theorem 3. Since we have

$$\begin{aligned}
L_{-k}(n) &= \sum_{j=1}^{n}\frac{1}{(\mathrm{lcm}(n,j))^k} = \frac{1}{n^k}\sum_{j=1}^{n}\frac{(\gcd(n,j))^k}{j^k} = \frac{1}{n^k}\sum_{d\mid n} d^k \sum_{\substack{j=1 \\ \gcd(j,n)=d}}^{n}\frac{1}{j^k} \\
&= \frac{1}{n^k}\sum_{d\mid n} d^k \sum_{\substack{i\le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}}\frac{1}{i^k d^k} \\
&= \frac{1}{n^k}\sum_{d\mid n}\sum_{\substack{i\le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}}\frac{1}{i^k},
\end{aligned}$$

we obtain

$$\sum_{n=1}^{\infty} L_{-k}(n) = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{d|n} \sum_{\substack{i \le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}} \frac{1}{i^k} = \sum_{d=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j^k d^k} \sum_{\substack{i \le j \\ \gcd(i,j)=1}} \frac{1}{i^k}$$

$$= \zeta(k) \sum_{j=1}^{\infty} \frac{1}{j^k} \sum_{\substack{i \le j \\ \gcd(i,j)=1}} \frac{1}{i^k}$$

$$= \zeta(k) \sum_{n=1}^{\infty} \frac{1}{n^k} (\sum_{\substack{i \le j \\ \gcd(i,j)=1 \\ ij=n}} 1).$$

Also we have

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\sum_{\substack{i \le j \\ \gcd(i,j)=1 \\ ij=n}} 1 \right) = 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^k} \left(\sum_{\substack{\gcd(i,j)=1 \\ ij=n}} 1 \right)$$

and the relation [9, 1.2.8]

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\sum_{\substack{\gcd(i,j)=1 \\ ij=n}} 1 \right) = \frac{\zeta(k)^2}{\zeta(2k)}.$$

Therefore we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \left(\sum_{\substack{i \le j \\ \gcd(i,j)=1 \\ ij=n}} 1 \right) = 1 + \frac{1}{2} \left(\frac{\zeta(k)^2}{\zeta(2k)} - 1 \right) = \frac{1}{2} \left(1 + \frac{\zeta(k)^2}{\zeta(2k)} \right).$$

This implies (2).

By the relation

$$\sum_{n \le x} L_{-k}(n) = \frac{\zeta(k)}{2} \left(1 + \frac{\zeta(k)^2}{\zeta(2k)} \right) - \sum_{n > x} L_{-k}(n),$$

the remaining task is to estimate the sum $\sum_{n>x} L_{-k}(n)$. We have

$$\begin{aligned}
\sum_{n>x} L_{-k}(n) &= \sum_{n>x} \frac{1}{n^k} \sum_{d|n} \sum_{\substack{i\le \frac{n}{d} \\ \gcd(i,\frac{n}{d})=1}} \frac{1}{i^k} = \sum_{d=1}^{\infty} \sum_{h>\frac{x}{d}} \frac{1}{(hd)^k} \sum_{\substack{j\le h \\ \gcd(j,h)=1}} \frac{1}{j^k} \\
&= \sum_{d=1}^{\infty} \sum_{h>\frac{x}{d}} \frac{1}{(hd)^k} \sum_{j\le h} \frac{1}{j^k} \sum_{\delta|\gcd(j,h)} \mu(\delta) \\
&= \sum_{d=1}^{\infty} \sum_{h>\frac{x}{d}} \frac{1}{(hd)^k} \sum_{\delta|h} \sum_{m\le \frac{h}{\delta}} \frac{\mu(\delta)}{m^k \delta^k} \\
&= \sum_{d=1}^{\infty} \frac{1}{d^k} \sum_{\delta=1}^{\infty} \sum_{l>\frac{x}{d\delta}} \frac{\mu(\delta)}{l^k \delta^{2k}} \sum_{m\le l} \frac{1}{m^k} \\
&= \sum_{q=1}^{\infty} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k} \sum_{m\le l} \frac{1}{m^k} \\
&= \sum_{q=1}^{\infty} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k} \left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k}) \right) \\
&= \sum_{q<x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k} \left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k}) \right) + \\
&\qquad + \sum_{q\ge x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k} \left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k}) \right) \\
&=: S_1 + S_2,
\end{aligned}$$

say. We have

$$
\begin{aligned}
S_1 &= \sum_{q<x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k}\left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k})\right) \\
&= \sum_{q<x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta)\left(\frac{\zeta(k)}{k-1}(x/q)^{-k+1} + O((x/q)^{-k}) - \right. \\
&\qquad \left. - \frac{(\frac{x}{q})^{-2k+2}}{(k-1)(2k-2)} + O((x/q)^{-2k+1})\right) \\
&= \sum_{q<x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta)\left(\frac{\zeta(k)}{k-1}(x/q)^{-k+1} + O((x/q)^{-k})\right) \\
&= \sum_{q<x} \frac{1}{q^k} \sum_{d|q} \frac{\mu(d)}{d^k}\left(\frac{\zeta(k)}{k-1}(x/q)^{-k+1} + O((x/q)^{-k})\right) \\
&= \frac{\zeta(k)x^{-k+1}}{k-1} \sum_{q<x} q^{-1} \sum_{d|q} \frac{\mu(d)}{d^k} + O\left(x^{-k} \sum_{q<x} \sum_{d|q} \frac{|\mu(d)|}{d^k}\right) \\
&= \frac{\zeta(k)x^{-k+1}}{k-1} \sum_{d<x} \sum_{j<\frac{x}{d}} j^{-1} \frac{\mu(d)}{d^{k+1}} + O(x^{-k+1}) \\
&= \frac{\zeta(k)x^{-k+1}}{k-1} \sum_{d<x} \log\left(\frac{x}{d}\right) \frac{\mu(d)}{d^{k+1}} + O(x^{-k+1}) \\
&= \frac{\zeta(k)x^{-k+1}\log x}{(k-1)\zeta(k+1)} + O(x^{-k+1})
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{q\geq x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l>\frac{x}{q}} \frac{1}{l^k}\left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k})\right) \\
&\ll \sum_{q\geq x} \frac{1}{q^{2k}} \sum_{d\delta=q} d^k \mu(\delta) \sum_{l=1}^{\infty} \frac{1}{l^k}\left(\zeta(k) - \frac{l^{1-k}}{k-1} + O(l^{-k})\right) \\
&\ll \sum_{q\geq x} q^{-k} \sum_{d|q} \frac{|\mu(d)|}{d^k} \\
&\ll x^{-k+1}.
\end{aligned}
$$

Therefore we obtain

$$
\sum_{n>x} L_{-k}(n) = \frac{\zeta(k)x^{-k+1}\log x}{(k-1)\zeta(k+1)} + O(x^{-k+1}).
$$

This completes the proof. □

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