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More Determinant Representations for Sequences

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Abstract

In this paper, we will find some new families of infinite (integer) matrices whose entries satisfy a non-homogeneous recurrence relation and such that the sequence of their leading principal minors is a subsequence of the Fibonacci, Lucas, Jacobsthal, or Pell sequences.

1 Introduction

Throughout this paper, unless noted otherwise, we will use the following notation. Let $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ be two arbitrary sequences starting with a common first term $\alpha_0 = \beta_0$. We denote by $P_{\alpha,\beta}(n)$ the *generalized Pascal triangle* associated with the sequences α and β, which is introduced as follows. Actually, $P_{\alpha,\beta}(n) = [P_{i,j}]_{0\leqslant i,j\leqslant n}$ is a square matrix of order $n+1$ whose (i,j)-entry $P_{i,j}$ obeys the following rules:

$$P_{i,0} = \alpha_i, \quad P_{0,j} = \beta_j \ \text{ for } \ i,j = 0,1,2,\ldots,n, \ \text{ and } \ P_{i,j} = P_{i,j-1} + P_{i-1,j} \ \text{ for } \ 1 \leqslant i,j \leqslant n.$$

We also denote by $T_{\alpha,\beta}(n) = [T_{i,j}]_{0\leqslant i,j\leqslant n}$ the *Toeplitz matrix* of order $n+1$ whose (i,j)-entry $T_{i,j}$ obeys the following rules:

$$T_{i,0} = \alpha_i, \quad T_{0,j} = \beta_j \ \text{ for } \ i,j = 0,1,2,\ldots,n, \ \text{ and } \ T_{i,j} = T_{k,l} \ \text{ if } \ i-j = k-l.$$

The *unipotent lower triangular matrix* $L(n) = [L_{i,j}]_{0\leqslant i,j\leqslant n}$ is again a square matrix of order $n+1$ with entries:

$$L_{i,j} = \begin{cases} 0, & \text{if } \ 0 \leqslant i < j \leqslant n; \\ \binom{i}{j}, & \text{if } \ 0 \leqslant j \leqslant i \leqslant n. \end{cases}$$

We put $U(n) = L(n)^t$, where A^t signifies the transpose of matrix A. Moreover, a *lower Hessenberg matrix* $H(n) = [H_{i,j}]_{0\leqslant i,j\leqslant n}$ is a square matrix of order $n+1$, where $H_{i,j} = 0$ whenever $j > i+1$ and $H_{i,i+1} \neq 0$ for some i, $0 \leqslant i \leqslant n-1$.

Given a matrix A, we denote by $\mathrm{R}_i(A)$ (resp., $\mathrm{C}_j(A)$) the row i (resp., the column j) of A. We also denote by $A^{[1]}$ the submatrix obtained from A by deleting the first column of A.

Given a sequence $\varphi = (\varphi_i)_{i\geq 0}$, define the *binomial transform* of φ to be the sequence $\hat{\varphi} = (\hat{\varphi}_i)_{i\geqslant 0}$ with

$$\hat{\varphi}_i = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \varphi_k.$$

The *Fibonacci sequence* (A000045 in [3]) is defined by the recurrence relation:

$$F_0 = 0, \ F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \ \text{ for } \ n \geqslant 2.$$

The *Lucas sequence* (A000032 in [3]) is defined by the recurrence relation:

$$L_0 = 2, \ L_1 = 1, \quad L_n = L_{n-1} = L_{n-2} \ \text{ for } \ n \geqslant 2.$$

The *Jacobsthal sequence* (A001045 in [3]) is defined by the recurrence relation:

$$J_0 = 0, \ J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \ \text{ for } \ n \geqslant 2.$$

The *Pell sequence* (A000129 in [3]) is defined by the recurrence relation:

$$P_0 = 0, \ P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \ \text{ for } \ n \geqslant 2.$$

Let $A=[A_{i,j}]_{i,j\geqslant 0}$ be an arbitrary infinite matrix. We denote the elementary row operation of type *three* by $O_{r,s}(\lambda)$, where $r\neq s$ and λ a scalar, that is

$$\mathrm{R}_k(O_{r,s}(\lambda)A)=\begin{cases}\mathrm{R}_r(A)+\lambda\mathrm{R}_s(A), & \text{if} \quad k=r;\\ \mathrm{R}_k(A), & \text{if} \quad k\neq r.\end{cases}$$

The *nth leading principal minor* of A, denoted by $d_n(A)$, is defined as follows:

$$d_n(A)=\det[A_{i,j}]_{0\leqslant i,j\leqslant n}, \quad (n=0,1,2,3,\ldots).$$

We put $D(A)=(d_n(A))_{n\geqslant 0}$. Two infinite matrices A and B are said to be *equimodular* if $D(A)=D(B)$. Given a sequence $\omega=(\omega_n)_{n\geqslant 0}$, a family $\{A_t|\ t\in I\}$ of equimodular matrices are said to be ω*-equimodular* if $D(A_t)=\omega$ for all $t\in I$. We will denote the family of ω-equimodular matrices by $\mathcal{A}_\omega$. The infinite matrices in $\mathcal{A}_\omega$ are said to be *determinant representations of* ω. Note that for *any* sequence $\omega=(\omega_n)_{n\geqslant 0}$, there is a determinant representation of ω, in other words $\mathcal{A}_\omega\neq\emptyset$. Indeed, expanding along the last rows, it is easy to see that

$$\begin{pmatrix}\omega_0 & 1 & * & * & * & \cdots\\ -\omega_1 & 0 & 1 & * & * & \cdots\\ \omega_2 & 0 & 0 & 1 & * & \cdots\\ -\omega_3 & 0 & 0 & 0 & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\in\mathcal{A}_\omega,$$

(see also Theorem 3.2 and the Remark after this theorem in [4]). Especially, there are many different determinant representations of ω, when ω is a (sub-)sequence of Fibonacci, Lucas, Jacobsthal and Pell sequences. Some examples of such matrices can be found in [1, 2].

In this paper, we are going to find some determinant representations of the sequences:

$$\mathcal{F}=(F_{n+1})_{n\geqslant 0}, \quad \mathcal{L}=(L_{n+1})_{n\geqslant 0}, \quad \mathcal{J}=(J_{n+1})_{n\geqslant 0} \ \text{ and } \ \mathcal{P}=(P_{n+1})_{n\geqslant 0}.$$

It is worthwhile to point out that we will use *non-homogeneous recurrence relations* to construct these determinant representations.

In the sequel, we introduce a new family of (infinite) matrices $A(\infty)=[A_{i,j}]_{i,j\geqslant 0}$, whose entries obey a non-homogeneous recurrence relation. Actually, for two constants u and v, and arbitrary sequences $\lambda=(\lambda_i)_{i\geqslant 0}$ and $\mu=(\mu_i)_{i\geqslant 0}$ with $\mu_0=0$, the first column and row of matrix $A(\infty)$ are the sequences

$$(A_{i,0})_{i\geqslant 0}=(\lambda_0,\lambda_1,\lambda_2,\ldots,A_{i,0}=\lambda_i,\ldots),$$

and

$$(A_{0,j})_{j\geqslant 0}=(\lambda_0,\lambda_0+u,\lambda_0+2u,\ldots,A_{0,j}=\lambda_0+ju,\ldots),$$

respectively, while the remaining entries $A_{i,j}$ $(i,j\geqslant 1)$ are obtained from the following non-homogeneous recurrence relation:

$$A_{i,j}=A_{i,j-1}+A_{i-1,j}-\lambda_{i-1}+\mu_i-\mu_{i-1}+(j-1)(v-u), \qquad i,j\geqslant 1.$$

We denote by $A(n)$ the submatrix of $A(\infty)$ consisting of the entries in its first $n+1$ rows and columns. The matrix $A(3)$, for example, is then given by

$$A(3) = \begin{pmatrix} \lambda_0 & \lambda_0 + u & \lambda_0 + 2u & \lambda_0 + 3u \\ \lambda_1 & \lambda_1 + \mu_1 + u & \lambda_1 + 2\mu_1 + 2u + v & \lambda_1 + 3\mu_1 + 3u + 3v \\ \lambda_2 & \lambda_2 + \mu_2 + u & \lambda_2 + 2\mu_2 + \mu_1 + 2u + 2v & \lambda_2 + 3\mu_2 + 3\mu_1 + 3u + 7v \\ \lambda_3 & \lambda_3 + \mu_3 + u & \lambda_3 + 2\mu_3 + \mu_2 + \mu_1 + 2u + 3v & \lambda_3 + 3\mu_3 + 3\mu_2 + 4\mu_1 + 3u + 12v \end{pmatrix}.$$

Finally, the main result of this paper can be stated as follows:

Main Theorem. *The matrix $A(n)$, $n \geqslant 0$, defined as above, satisfies the following statements:*

(a) $A(n) = L(n) \cdot H(n) \cdot U(n)$, *where*

$$H(n) = \left(\begin{array}{c|cccc} \hat{\lambda}_0 & u & 0 & \cdots & 0 \\ \hline \hat{\lambda}_1 & & & & \\ \hat{\lambda}_2 & & & & \\ \vdots & & T_{(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\ldots),(\hat{\mu}_1,v,0,0,\ldots)}(n-1) & & \\ \hat{\lambda}_n & & & & \end{array}\right).$$

In particular, we have $\det(A(n)) = \det(H(n))$.

(b) *In the case when $u = v = 1$ and $\lambda_i = (2^i - 1)c + 1$, we have the following statements:*

(b. 1) *if* $\mu_i = \left(2^i + \frac{(i-2)(i+1)}{2}\right) c - \frac{i(i-3)}{2}$, *then* $\det(A(n)) = F_{n+1}$.

(b. 2) *if* $\mu_i = \left(\frac{5\cdot 3^i}{4} - 2^i - \frac{2i+1}{4}\right) c + \frac{5(3^i-1)}{4} + \frac{i}{2}$, *then* $\det(A(n)) = L_{n+1}$.

(b. 3) *if* $\mu_i = i^2 c - i^2 + 2i$, *then* $\det(A(n)) = J_{n+1}$.

(b. 4) *if* $\mu_i = \left(2^{i+1} + \frac{(i+1)(i-4)}{2}\right) c + \frac{(5-i)i}{2}$, *then* $\det(A(n)) = P_{n+1}$.

As mentioned previously, we have obtained some determinant representations of the sequences:

$$\mathcal{F} = (F_{n+1})_{n\geqslant 0}, \quad \mathcal{L} = (L_{n+1})_{n\geqslant 0}, \quad \mathcal{J} = (J_{n+1})_{n\geqslant 0} \text{ and } \mathcal{P} = (P_{n+1})_{n\geqslant 0},$$

which are presented in the following:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots \\ c+1 & 2c+3 & 3c+6 & \cdots \\ 3c+1 & 7c+3 & 12c+8 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}_{\mathcal{F}}, \qquad \begin{pmatrix} 1 & 2 & 3 & \cdots \\ c+1 & 2c+5 & 3c+10 & \cdots \\ 3c+1 & 9c+13 & 16c+30 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}_{\mathcal{L}},$$

$$\begin{pmatrix} 1 & 2 & 3 & \cdots \\ c+1 & 2c+3 & 3c+6 & \cdots \\ 3c+1 & 7c+2 & 12c+6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}_{\mathcal{J}} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & \cdots \\ c+1 & 2c+4 & 3c+8 & \cdots \\ 3c+1 & 8c+5 & 14c+13 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}_{\mathcal{P}}.$$

2 Main results

As the first result of this paper, we consider the following theorem.

Theorem 1. *For two arbitrary sequences* $(\lambda_i)_{i\geqslant 0}$ *and* $(\mu_i)_{i\geqslant 0}$, *with* $\mu_0 = 0$, *and some integers* u *and* v, *let* $A(\infty) = [A_{i,j}]_{i,j\geqslant 0}$ *be an infinite dimensional matrix whose entries are given by*

$$A_{i,j} = A_{i,j-1} + A_{i-1,j} - \lambda_{i-1} + \mu_i - \mu_{i-1} + (j-1)(v-u), \qquad i,j \geqslant 1 \tag{1}$$

and the initial conditions $A_{i,0} = \lambda_i$ *and* $A_{0,i} = \lambda_0 + iu$, $i \geqslant 0$. *If* $A(n) = [A_{i,j}]_{0\leqslant i,j\leqslant n}$, *then we have*

$$A(n) = L(n) \cdot H(n) \cdot U(n), \tag{2}$$

where

$$H(n) = \left(\begin{array}{c|cccc} \hat{\lambda}_0 & u & 0 & \cdots & 0 \\ \hline \hat{\lambda}_1 & & & & \\ \hat{\lambda}_2 & & & & \\ \vdots & & T_{(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\ldots),(\hat{\mu}_1,v,0,0,\ldots)}(n-1) & & \\ \hat{\lambda}_n & & & & \end{array}\right).$$

Proof. First of all, we recall that the entries of $L(n) = [L_{i,j}]_{0\leqslant i,j\leqslant n}$ satisfy the following recurrence

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad 1 \leqslant i,j \leqslant n. \tag{3}$$

Similarly, for the entries of $U(n) = [U_{i,j}]_{0\leqslant i,j\leqslant n}$ we have

$$U_{i,j} = U_{i-1,j-1} + U_{i,j-1}, \quad 1 \leqslant i,j \leqslant n. \tag{4}$$

In what follows, for convenience, we will let $A = A(n)$, $L = L(n)$, $H = H(n)$ and $U = U(n)$. Now, for the proof of the desired factorization we compute the (i,j)-entry of $L \cdot H \cdot U$, that is

$$(L \cdot H \cdot U)_{i,j} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i,r} H_{r,s} U_{s,j}. \tag{5}$$

In fact, we should establish

$$\begin{aligned} \mathrm{R}_0(L \cdot H \cdot U) &= \mathrm{R}_0(A) = && (\lambda_0, \lambda_0 + u, \ldots, \lambda_0 + nu), \\ \mathrm{C}_0(L \cdot H \cdot U) &= \mathrm{C}_0(A) = && (\lambda_0, \lambda_1, \ldots, \lambda_n), \end{aligned}$$

and finally, show that

$$(L \cdot H \cdot U)_{i,j} = (L \cdot H \cdot U)_{i-1,j-1} + (L \cdot H \cdot U)_{i-1,j} - \lambda_{i-1} + \mu_i - \mu_{i-1} + (j-1)(v-u), \tag{6}$$

for $1 \leqslant i, j \leqslant n$.

Let us do the required calculations. Assume first that $i = 0$. Then, we have

$$(L \cdot H \cdot U)_{0,j} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{0,r} H_{r,s} U_{s,j} = \sum_{s=0}^{n} H_{0,s} U_{s,j} = H_{0,0} U_{0,j} + H_{0,1} U_{1,j} = \lambda_0 + ju,$$

and so $\mathrm{R}_0(L \cdot H \cdot U) = \mathrm{R}_0(A) = (\lambda_0, \lambda_0 + u, \ldots, \lambda_0 + nu)$.

Assume next that $j = 0$. In this case, we obtain

$$(L \cdot H \cdot U)_{i,0} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i,r} H_{r,s} U_{s,0} = \sum_{r=0}^{n} L_{i,r} H_{r,0} = \sum_{r=0}^{n} \binom{i}{r} \hat{\lambda}_r = \lambda_i,$$

and hence we have $\mathrm{C}_0(L \cdot H \cdot U) = \mathrm{C}_0(A) = (\lambda_0, \lambda_1, \ldots, \lambda_n)$.

Finally, we must establish (6). Let us for the moment assume that $1 \leqslant i, j \leqslant n$. In this case, we have

$$(L \cdot H \cdot U)_{i,j} = \sum_{r=0}^{n} \sum_{s=0}^{n} L_{i,r} H_{r,s} U_{s,j} = \sum_{r=0}^{n} L_{i,r} H_{r,0} U_{0,j} + \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s,j}. \tag{7}$$

Let $\Omega(i,j) = \sum\limits_{r=0}^{n} \sum\limits_{s=1}^{n} L_{i,r} H_{r,s} U_{s,j}$. Then, using (4), we obtain

$$\begin{aligned} \Omega(i,j) &= \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} \big(U_{s-1,j-1} + U_{s,j-1} \big) = \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s-1,j-1} + \sum_{r=0}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s,j-1} \\ &= \sum_{r=1}^{n} \sum_{s=1}^{n} L_{i,r} H_{r,s} U_{s-1,j-1} + (L \cdot H \cdot U)_{i,j-1} + \sum_{s=1}^{n} L_{i,0} H_{0,s} U_{s-1,j-1} - \sum_{r=0}^{n} L_{i,r} H_{r,0} U_{0,j-1} \end{aligned} \tag{8}$$

For convenience, we write $\Theta(i,j) = \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i,r}H_{r,s}U_{s-1,j-1}$. Now, we apply (3), to get

$$
\begin{aligned}
\Theta(i,j) &= \sum_{r=1}^{n}\sum_{s=1}^{n} \big(L_{i-1,r-1} + L_{i-1,r}\big) H_{r,s}U_{s-1,j-1} \\
&= \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r-1}H_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}U_{s-1,j-1} \\
&= \sum_{r=2}^{n}\sum_{s=2}^{n} L_{i-1,r-1}H_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} \\
&\quad + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}U_{s-1,j-1} \\
&= \sum_{r=2}^{n}\sum_{s=2}^{n} L_{i-1,r-1}H_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} \\
&\quad + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}\big(U_{s,j} - U_{s,j-1}\big) \quad \big(\text{by } (4)\big) \\
&= \sum_{r=2}^{n}\sum_{s=2}^{n} L_{i-1,r-1}H_{r-1,s-1}U_{s-1,j-1} + \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} \\
&\quad + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}U_{s,j} \\
&\quad - \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}U_{s,j-1} \qquad \big(\text{by the structure of } H\big) \\
&= \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}U_{s,j-1} + \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=1}^{n}\sum_{s=0}^{n} L_{i-1,r}H_{r,s}U_{s,j} - \sum_{r=1}^{n} L_{i-1,r}H_{r,0}U_{0,j} - \sum_{r=1}^{n}\sum_{s=1}^{n} L_{i-1,r}H_{r,s}U_{s,j-1} \\
&\quad \big(\text{note that } L_{i-1,n-1} = U_{n-1,j-1} = 0\big) \\
&= \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=0}^{n}\sum_{s=0}^{n} L_{i-1,r}H_{r,s}U_{s,j} \\
&\quad - \sum_{s=0}^{n} L_{i-1,0}H_{0,s}U_{s,j} - \sum_{r=1}^{n} L_{i-1,r}H_{r,0}U_{0,j} \\
&= \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} + (L \cdot H \cdot U)_{i-1,j} \\
&\quad - \sum_{s=0}^{n} L_{i-1,0}H_{0,s}U_{s,j} - \sum_{r=1}^{n} L_{i-1,r}H_{r,0}U_{0,j} \quad \big(\text{by } (5)\big).
\end{aligned}
$$

By substituting this in (8), we obtain

$$\begin{aligned}\Omega(i,j) &= (L\cdot H\cdot U)_{i,j-1} + (L\cdot H\cdot U)_{i-1,j}\\ &\quad + \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1}\\ &\quad - \sum_{s=0}^{n} L_{i-1,0}H_{0,s}U_{s,j} - \sum_{r=1}^{n} L_{i-1,r}H_{r,0}U_{0,j}\\ &\quad + \sum_{s=1}^{n} L_{i,0}H_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n} L_{i,r}H_{r,0}U_{0,j-1}.\end{aligned}$$

Finally, if the above expression is substituted in (7) and the sums are put together, then we obtain

$$(L\cdot H\cdot U)_{i,j} = (L\cdot H\cdot U)_{i-1,j} + (L\cdot H\cdot U)_{i,j-1} + \Psi(i,j),$$

where

$$\begin{aligned}\Psi(i,j) &:= \sum_{r=0}^{n} L_{i,r}H_{r,0}U_{0,j} + \sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1}\\ &\quad - \sum_{s=0}^{n} L_{i-1,0}H_{0,s}U_{s,j} - \sum_{r=1}^{n} L_{i-1,r}H_{r,0}U_{0,j} + \sum_{s=1}^{n} L_{i,0}H_{0,s}U_{s-1,j-1}\\ &\quad - \sum_{r=0}^{n} L_{i,r}H_{r,0}U_{0,j-1}.\end{aligned}$$

However, by easy calculations one can show that

$$\begin{aligned}&\sum_{r=0}^{n} L_{i,r}H_{r,0}U_{0,j} - \sum_{r=0}^{n} L_{i,r}H_{r,0}U_{0,j-1} = 0,\\ &\sum_{r=1}^{n} L_{i-1,r-1}H_{r,1}U_{0,j-1} = \sum_{r=1}^{n} \tbinom{i-1}{r-1}\hat{\mu}_r = \sum_{r=1}^{n}\left(\tbinom{i}{r} - \tbinom{i-1}{r}\right)\hat{\mu}_r = \mu_i - \mu_{i-1},\\ &\sum_{r=1}^{n} L_{i-1,r}H_{r,0}U_{0,j} = \sum_{r=0}^{n} \hat{\lambda}_r - \lambda_0 = \lambda_{i-1} - \lambda_0,\\ &\sum_{s=2}^{n} L_{i-1,0}H_{1,s}U_{s-1,j-1} = (j-1)v,\\ &\sum_{s=0}^{n} L_{i-1,0}H_{0,s}U_{s,j} = \lambda_0 + ju,\\ &\sum_{s=1}^{n} L_{i,0}H_{0,s}U_{s-1,j-1} = u,\end{aligned}$$

and so

$$\Psi(i,j) = \mu_i - \mu_{i-1} - \lambda_{i-1} + (j-1)(v-u).$$

This completes the proof. □

Before stating the next result, we need to introduce some additional definitions. Let $\lambda = (\lambda_i)_{i\geqslant 0}$ and $\mu = (\mu_i)_{i\geqslant 0}$ be two arbitrary sequences. The *convolution* of λ and μ is the sequence $\nu = (\nu_i)_{i\geqslant 0}$, where

$$\nu_i = \sum_{k=0}^{i} \lambda_k \mu_{i-k}.$$

The *convolution matrix* associated with sequences λ and μ is the infinite matrix $A(\infty)$ whose first column $\mathrm{C}_0(A(\infty))$ is λ and whose jth column ($j = 1, 2, \ldots$) is the convolution of sequences $\mathrm{C}_{j-1}(A(\infty))$ and μ. We say that the convolution matrix of the sequences λ and λ is the convolution matrix of the sequence λ. There are many well-known integer matrices which can be written as convolution matrices of some sequences. For instance, $U(\infty)$ is the convolution matrix of the sequences $(1, 0, 0, \ldots)$ and $(1, 1, 0, 0, \ldots)$ and $P_{(1,1,\ldots),(1,1,\ldots)}(\infty)$ is the convolution matrix of the sequence $(1, 1, \ldots)$.

We will need the following technical result [4, Theorem 3.1].

Proposition 2. *Let*

$$A(x) = \sum_{n=1}^{\infty} a_n x^{n-1}, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n, \quad V(x) = \sum_{n=0}^{\infty} v_n x^n \quad \textit{and} \quad W(x) = \sum_{n=0}^{\infty} w_n x^n$$

be the generating functions for the sequences $(a_n)_{n\geqslant 1}$, $(b_n)_{n\geqslant 0}$, $(v_n)_{n\geqslant 0}$, and $(w_n)_{n\geqslant 0}$, respectively. Consider an infinite dimensional matrix of the following form:

$$M(\infty) = \left(\begin{array}{c|ccc} b_0 & v_0 & v_0 w_0 & \cdots \\ b_1 & v_1 & v_0 w_1 + v_1 w_0 & \cdots \\ b_2 & v_2 & v_0 w_2 + v_1 w_1 + v_2 w_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

where $\mathrm{C}_0(M(\infty)) = (b_0, b_1, \ldots)^t$ and $M(\infty)^{[1]}$ is the convolution matrix of the sequences $(v_i)_{i\geqslant 0}$ and $(w_j)_{j\geqslant 0}$. If

$$A(W(x)) = B(x)/V(x), \tag{9}$$

then for any non-negative integer n, there holds

$$\det(M(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1},$$

where $M(n)$ is the $(n+1) \times (n+1)$ upper left corner matrix of $M(\infty)$.

We are now in a position to prove the following theorem which is the second result of this paper.

Theorem 3. *Let $A(n)$ be defined as in Theorem 1 and let c be a constant. In the case when $u = v = 1$ and $\lambda_i = (2^i - 1)c + 1$, we have the following statements:*

(a) *if* $\mu_i = \left(2^i + \frac{(i-2)(i+1)}{2}\right)c - \frac{i(i-3)}{2}$, *then* $\det(A(n)) = F_{n+1}$.

(b) *if* $\mu_i = \left(\frac{5\cdot 3^i}{4} - 2^i - \frac{2i+1}{4}\right)c + \frac{5(3^i-1)}{4} + \frac{i}{2}$, *then* $\det(A(n)) = L_{n+1}$.

(c) *if* $\mu_i = i^2c - i^2 + 2i$, *then* $\det(A(n)) = J_{n+1}$.

(d) *if* $\mu_i = \left(2^{i+1} + \frac{(i+1)(i-4)}{2}\right)c + \frac{(5-i)i}{2}$, *then* $\det(A(n)) = P_{n+1}$.

Proof. Let $\mu = (\mu_i)_{i\geqslant 0}$ be a sequence with $\mu_0 = 0$ and let c be a constant. Let $\lambda = (\lambda_i)_{i\geqslant 0}$ be a sequence with $\lambda_i = (2^i-1)c+1$. We consider the infinite matrices $A(\infty) = [A_{i,j}]_{i,j\geqslant 0}$ whose entries satisfy

$$A_{i,j} = A_{i-1,j} + A_{i,j-1} - (2^i-1)c - 1 + \mu_i - \mu_{i-1} \quad \text{for} \quad i,j \geqslant 1, \tag{10}$$

with the initial conditions $A_{i,0} = (2^i-1)c+1$ and $A_{0,i} = 1+i$, $i \geqslant 0$. By Theorem 2, we observe that

$$A(n) = L(n)\cdot H(n)\cdot U(n),$$

where

$$H(n) = \left(\begin{array}{c|ccc} 1 & 1 \quad 0 & \cdots & 0 \\ \hline c & & & \\ c & & & \\ \vdots & & T_{(\hat{\mu}_1,\hat{\mu}_2,\hat{\mu}_3,\ldots),(\hat{\mu}_1,v,0,0,\ldots)}(n-1) & \\ c & & & \end{array}\right).$$

Evidently $\det(A(n)) = \det(H(n))$, so it suffices to find $\det(H(n))$. From the structure of matrix $H(\infty)$, we have

$$\mathrm{C}_0(H(\infty)) = (b_i)_{i\geqslant 0} = (1,c,c,\ldots)^t,$$

whose generating function is

$$B(x) = \frac{1+(c-1)x}{1-x}.$$

(a) Let $\mu_i = (2^i + \frac{(i-2)(i+1)}{2})c - \frac{i(i-3)}{2}$. In this case, we have the following infinite dimensional matrices:

$$A(\infty) = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ c+1 & 2c+3 & 3c+6 & 4c+10 & \cdots \\ 3c+1 & 7c+3 & 12c+8 & 18c+17 & \cdots \\ 7c+1 & 17c+2 & 32c+8 & 53c+23 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$H(\infty) = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & & \cdots & 0 \\ \hline c & & & & & \\ c & & & T_{(c+1,2c-1,c,c,\ldots),(c+1,1,0,0,\ldots)}(\infty) & & \\ \vdots & & & & & \end{array}\right).$$

Note that the submatrix $H(\infty)^{[1]}$ is the convolution of sequences

$$(v_i)_{i\geqslant 0} = (1, c+1, 2c-1, c, c, \ldots), \quad \text{and} \quad (w_i)_{i\geqslant 0} = (0, 1, 0, 0, 0, \ldots),$$

whose generating functions are

$$V(x) = \frac{1 + cx + (c-2)x^2 - (c-1)x^3}{1-x} \quad \text{and} \quad W(x) = x,$$

respectively. Plugging these generating functions into (9) yields

$$A(W(x)) = A(x) = \frac{\frac{1+(c-1)x}{1-x}}{\frac{1+cx+(c-2)x^2-(c-1)x^3}{1-x}} = 1 - x + 2x^2 - 3x^3 + \cdots + (-1)^n F_{n+1} x^n + \cdots,$$

and it follows by Proposition 2 that

$$\det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = F_{n+1},$$

as required.

(b) Let $\mu_i = \left(\frac{5\cdot 3^i}{4} - 2^i - \frac{2i+1}{4}\right) c + \frac{5(3^i-1)}{4} + \frac{i}{2}$. The infinite dimensional matrices created in this case are as follows:

$$A(\infty) = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ c+1 & 2c+5 & 3c+10 & 4c+16 & \cdots \\ 3c+1 & 9c+13 & 16c+30 & 24c+53 & \cdots \\ 7c+1 & 31c+36 & 62c+88 & 101c+163 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$H(\infty) = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & & \cdots & 0 \\ \hline c & & & & & \\ c & & & T_{(c+3,4c+5,9c+10,19c+20,\ldots),(c+3,1,0,0,\ldots)}(\infty) & & \\ \vdots & & & & & \end{array}\right).$$

Again, one can easily see that the submatrix $H(\infty)^{[1]}$ is the convolution of sequences

$$(v_i)_{i\geqslant 0} = (1, c+3, 4c+5, 9c+10, 19c+20, \ \ldots),$$

(with general form $v_0 = 1$, $v_1 = c+3$ and $v_i = (5\cdot 2^{i-2})(c+1) - c$ for $i \geqslant 2$), and

$$(w_i)_{i\geqslant 0} = (0, 1, 0, 0, 0, \ldots).$$

The generating functions for these sequences are

$$V(x) = \frac{(1+(c-1)x)(-x^2+x+1)}{(1-x)(1-2x)}, \quad \text{and} \quad W(x) = x,$$

respectively. If $B(x)$, $V(x)$ and $W(x)$ are substituted in (9), then we obtain

$$A(W(x)) = A(x) = \frac{\frac{1+(c-1)x}{1-x}}{\frac{(1+(c-1)x)(-x^2+x+1)}{(1-x)(1-2x)}} = 1-3x+4x^2-7x^3+11x^4+\cdots+(-1)^n L_{n+1}x^n+\cdots,$$

and by Proposition 2, it follows that

$$\det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = L_{n+1},$$

as required.

(c) Let $\mu_i = i^2c - i^2 + 2i$. In this case, we have the following infinite dimensional matrices:

$$A(\infty) = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ c+1 & 2c+3 & 3c+6 & 4c+10 & \cdots \\ 3c+1 & 7c+2 & 12c+6 & 18c+14 & \cdots \\ 7c+1 & 16c-1 & 30c+1 & 50c+11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$H(\infty) = \left(\begin{array}{c|c} 1 & 1 \quad 0 \quad \cdots \quad 0 \\ \hline c & \\ c & T_{(c+1,2c-2,0,0,\ldots),(c+1,1,0,0,\ldots)}(\infty) \\ \vdots & \end{array}\right).$$

Moreover, from the structure of $H(\infty)$, we see that the submatrix $H(\infty)^{[1]}$ is the convolution of sequences

$$(v_i)_{i\geqslant 0} = (1, c+1, 2c-2, 0, 0, \ \ldots), \quad \text{and} \quad (w_i)_{i\geqslant 0} = (0, 1, 0, 0, \ldots),$$

with generating functions $V(x) = 1 + (c+1)x + (2c-2)x^2$ and $W(x) = x$, respectively. Substituting the obtained generating functions in (9), we obtain

$$A(W(x)) = A(x) = \frac{\frac{1+(c-1)x}{1-x}}{1+(c+1)x+(2c-2)x^2} = 1-x+3x^2-5x^3+\cdots+(-1)^n J_{n+1}x^n+\cdots.$$

Therefore, it follows from Proposition 2 that

$$\det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = J_{n+1},$$

as required.

(d) Let $\mu_i = \left(2^{i+1} + \frac{(i+1)(i-4)}{2}\right) c + \frac{(5-i)i}{2}$. This time, we will deal with the following matrices:

$$A(\infty) = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ c+1 & 2c+4 & 3c+8 & 4c+13 & \cdots \\ 3c+1 & 8c+5 & 14c+13 & 21c+26 & \cdots \\ 7c+1 & 21c+5 & 41c+17 & 68c+42 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$H(\infty) = \left(\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ \hline c & & & & \\ c & & T_{(c+2,3c-1,2c,2c,\ldots),(c+2,1,0,0,\ldots)}(\infty) & & \\ \vdots & & & & \end{array}\right).$$

In addition, the submatrix $H(\infty)^{[1]}$ of $H(\infty)$ is the convolution of sequences:

$$(v_i)_{i\geqslant 0} = (1, c+2, 3c-1, 2c, 2c, \ \ldots) \quad \text{and} \quad (w_i)_{i\geqslant 0} = (0, 1, 0, 0, \ldots).$$

Note that the generating functions of these sequences are

$$V(x) = \frac{1 + (1+c)x + (2c-3)x^2 - (c-1)x^3}{1-x} \quad \text{and} \quad W(x) = x.,$$

respectively. After having substituted these generating functions in (9), we obtain

$$A(W(x)) = A(x) = \frac{\frac{1+(c-1)x}{1-x}}{\frac{1+(1+c)x+(2c-3)x^2-(c-1)x^3}{1-x}} = 1-2x+5x^2-12x^3+\cdots+(-1)^n P_{n+1}x^n+\cdots$$

Now, by Proposition 2, we deduce that

$$\det(H(n)) = (-1)^n v_0^{n+1} w_1^{n(n+1)/2} a_{n+1} = (-1)^n a_{n+1} = P_{n+1},$$

as required.

This completes the proof. $\square$

3 Some remarks

In this section, we will explain how the sequences $(\lambda_i)_{i\geqslant 0}$ and $(\mu_i)_{i\geqslant 0}$ in Theorem 3, are determined. Consider the following lower Hessenberg matrix

$$H(\infty)=[H_{i,j}]_{i,j\geqslant 0}=\begin{pmatrix} h_{0,0} & h_{0,1} & 0 & 0 & 0 & \cdots \\ h_{1,0} & h_{1,1} & h_{1,2} & 0 & 0 & \cdots \\ h_{2,0} & h_{2,1} & h_{1,1} & h_{1,2} & 0 & \cdots \\ h_{3,0} & h_{3,1} & h_{2,1} & h_{1,1} & h_{1,2} & \cdots \\ h_{4,0} & h_{4,1} & h_{3,1} & h_{2,1} & h_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $H(n)=[H_{i,j}]_{0\leqslant i,j\leqslant n}$, and let d_n be the nth determinant of $H(n)$. In what follows, we show that the sequence of principal minors of $H(\infty)$, i.e., $D(H(\infty))=(d_n)_{n\geqslant 0}$, satisfies a recurrence relation.

Proposition 4. *With the above notation, we have*

$$d_n=\begin{cases} h_{0,0}, & \text{if } n=0, \\ (-1)^n h_{0,1}(h_{1,2})^{n-1}h_{n,0}+\sum\limits_{k=0}^{n-1} h_{n-k,1}(-h_{1,2})^{n-k-1}d_k, & \text{if } n\geqslant 1. \end{cases}$$

Proof. Obviously, $d_0=h_{0,0}$. Hence, from now on we assume $n>1$. First, we apply the following row operations:

$$\begin{aligned} H_1(n) &= \Big(\prod_{i=1}^{n} O_{i,0}(\tfrac{-h_{i,1}}{h_{0,1}})\Big)H(n), \\ H_2(n) &= \Big(\prod_{i=1}^{n-1} O_{i+1,1}(\tfrac{-h_{i,1}}{h_{1,2}})\Big)H_1(n), \\ H_3(n) &= \Big(\prod_{i=1}^{n-2} O_{i+2,2}(\tfrac{-h_{i,1}}{h_{1,2}})\Big)H_2(n), \\ &\vdots \\ H_n(n) &= \Big(\prod_{i=1}^{1} O_{i+(n-1),n-1}(\tfrac{-h_{i,1}}{h_{1,2}})\Big)H_{n-1}(n). \end{aligned}$$

It is obvious that, step by step, the columns are "emptied" until finally the following matrix

$$H_n(n)=\begin{pmatrix} \tilde{h}_{0,0} & h_{0,1} & 0 & 0 & 0 & \cdots & 0 \\ \tilde{h}_{1,0} & 0 & h_{1,2} & 0 & 0 & \cdots & 0 \\ \tilde{h}_{2,0} & 0 & 0 & h_{1,2} & 0 & \cdots & 0 \\ \tilde{h}_{3,0} & 0 & 0 & 0 & h_{1,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{n-1,0} & 0 & 0 & 0 & 0 & \cdots & h_{1,2} \\ \tilde{h}_{n,0} & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n+1)\times(n+1)},$$

is obtained, where

$$\tilde{h}_{i,0} = \begin{cases} h_{0,0}, & \text{if} \quad i = 0; \\ h_{1,0} - \frac{h_{1,1}}{h_{0,1}} h_{0,0}, & \text{if} \quad i = 1; \\ h_{i,0} - \frac{h_{i,1}}{h_{0,1}} h_{0,0} - \frac{1}{h_{1,2}} \sum_{k=1}^{i-1} h_{i-k,1} \tilde{h}_{k,0}, & \text{if} \quad i \geqslant 2. \end{cases} \tag{11}$$

Evidently, $d_n = \det(H_n(n))$. Expanding the determinant along the last row of $\det(H_n(n))$, we obtain

$$d_n = (-1)^n \tilde{h}_{n,0} h_{0,1} (h_{1,2})^{n-1}, \quad (n \geqslant 1). \tag{12}$$

Finally, after some simplification, it follows that

$$\begin{aligned} d_n &= (-1)^n \tilde{h}_{n,0} h_{0,1} (h_{1,2})^{n-1} \\ &= (-1)^n h_{0,1} (h_{1,2})^{n-1} \Big[h_{n,0} - \frac{h_{n,1}}{h_{0,1}} h_{0,0} - \frac{1}{h_{1,2}} \sum_{k=1}^{n-1} h_{n-k,1} \tilde{h}_{k,0} \Big] \quad \text{(by (11))} \\ &= (-1)^n h_{0,1} (h_{1,2})^{n-1} h_{n,0} + (-1)^{n+1} (h_{1,2})^{n-1} h_{n,1} h_{0,0} + (-1)^{n+1} h_{0,1} (h_{1,2})^{n-2} \sum_{k=1}^{n-1} h_{n-k,1} \tilde{h}_{k,0} \\ &= (-1)^n h_{0,1} (h_{1,2})^{n-1} h_{n,0} + (-1)^{n+1} (h_{1,2})^{n-1} h_{n,1} h_{0,0} + \sum_{k=1}^{n-1} h_{n-k,1} (-h_{1,2})^{n-k-1} d_k \quad \text{(by (12))} \\ &= (-1)^n h_{0,1} (h_{1,2})^{n-1} h_{n,0} + \sum_{k=0}^{n-1} h_{n-k,1} (-h_{1,2})^{n-k-1} d_k. \end{aligned}$$

and the result follows. □

In Proposition 4, if we take $h_{0,0} = h_{0,1} = 1$, $h_{1,2} = 1$, $h_{i,0} = c$ and $h_{i,1} = \hat{\mu}_i$ for $i \geqslant 1$, then we obtain

$$d_n = \begin{cases} 1, & \text{if} \quad n = 0; \\ (-1)^n c + \sum_{k=0}^{n-1} \hat{\mu}_{n-k} (-1)^{n-k-1} d_k, & \text{if} \quad n \geqslant 1. \end{cases}$$

Now, if $(d_n)_{n \geqslant 0} \in \{\mathcal{F}, \mathcal{L}, \mathcal{J}, \mathcal{P}\}$, then

$$\hat{\mu}_n = c + (-1)^{n-1} d_n + \sum_{k=1}^{n-1} (-1)^{k+1} \hat{\mu}_{n-k} d_k,$$

from which we determine the sequence $(\hat{\mu}_i)_{i \geqslant 1}$. Now, we form

$$H(\infty) = \left(\begin{array}{c|c} 1 & 1 \quad 0 \quad \cdots \\ \hline c & \\ c & \\ \vdots & T_{(\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \ldots), (\hat{\mu}_1, 1, 0, 0, \ldots)}(\infty) \\ c & \end{array} \right).$$

Finally, the sequences $(\lambda_i)_{i\geqslant 0}$ and $(\mu_i)_{i\geqslant 0}$ are determined by the equation $A(n) = L(n) \cdot H(n) \cdot U(n)$.

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