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Bernoulli Numbers and a New Binomial Transform Identity

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Abstract

Let $(b_n)_{n\geq 0}$ be the binomial transform of $(a_n)_{n\geq 0}$. We show how a binomial transformation identity of Chen proves a symmetrical Bernoulli number identity attributed to Carlitz. We then modify Chen's identity to prove a new binomial transformation identity.

Carlitz [1] posed as a problem the remarkable symmetric Bernoulli number identity

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k}, \tag{1}$$

valid for arbitrary $m, n \geq 0$. The published solution by Shannon [2] used mathematical induction on m and n. The identity was rediscovered recently by Vassilev and Vassilev-Missana [10], but stated in the form

$$(-1)^m \sum_{k=0}^{m-1} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n-1} \binom{n}{k} B_{m+k}, \tag{2}$$

valid for arbitrary positive integers m and n. Identity (2) is equivalent to Identity (1) since $[(-1)^m - (-1)^n]\, B_{m+n} = 0$. Their proof used the symmetry of a function $f_k(x, y)$ involving Bernoulli numbers introduced in a separate paper [9]. They give no reference to Carlitz's or to Shannon's proof.

An alternative proof of Equation (1) is derived through an application of a binomial transformation identity discovered by Chen [3]. Let (a_n) be any sequence of numbers, and define the binomial transform of (a_n) to be the sequence (b_n), where $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. A corollary of [3, Thm. 2.1] is

$$\sum_{k=0}^{m} \binom{m}{k} a_{n+k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_{m+k}. \tag{3}$$

The Bernoulli numbers satisfy the recurrence $\sum_{k=0}^{n} \binom{n}{k} B_k = (-1)^n B_n$ for $n \geq 0$. Setting $a_k = B_k$, we then have $b_n = (-1)^n B_n$, so that Equation (3) becomes

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (-1)^{m+k} B_{m+k},$$

which is precisely Identity (1) of Carlitz.

Chen's proof of Equation (3) relies on certain properties of Seidel matrices. We present a direct proof which relies on the hypergeometric identity

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{x+k}{r} = (-1)^m \binom{x}{r-m}; \tag{4}$$

see [6, Identity 3.47, p. 27]. In Equation (4) we require that m and r be nonnegative integers and x be a complex number.

Since the binomial transform inverts to give $a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k$ we find that

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k} a_{n+k} &= \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{n+k} (-1)^{n+k-j} \binom{n+k}{j} b_j \\
&= \sum_{j=0}^{n+m} (-1)^{-j} b_j \sum_{k=j-n}^{m} (-1)^{n+k} \binom{m}{k} \binom{n+k}{j} \\
&= \sum_{j=0}^{n+m} (-1)^{-j} b_j \sum_{k=0}^{m} (-1)^{n+k} \binom{m}{k} \binom{n+k}{j} \\
&= \sum_{j=0}^{n+m} (-1)^{n+m-j} \binom{n}{j-m} b_j = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} b_{j+m}.
\end{aligned}$$

A careful analysis of this preceding proof yields a short proof of [3, Thm. 3.2], where Chen relies on lengthy induction arguments. We will instead use Equation (4).

Theorem 1. *[3, Thm. 3.2] Let b_n be the binomial transform of a_n. Then*

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{s} a_{n+k-s} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{m+k}{s} b_{m+k-s}, \tag{5}$$

for arbitrary nonnegative m, n, and s.

Proof. By definition $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. This implies that $a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k$. Hence

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{s} a_{n+k-s} &= \sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{s} \sum_{j=0}^{n+k-s} (-1)^{n+k-s-j} \binom{n+k-s}{j} b_j \\
&= \sum_{j=0}^{n+m-s} (-1)^{n-s-j} b_j \sum_{k=0}^{m} (-1)^k \binom{m}{k}\binom{n+k}{s}\binom{n+k-s}{j} \\
&= \sum_{j=0}^{n+m-s} (-1)^{n-s-j} \binom{s+j}{s} b_j \sum_{k=0}^{m} (-1)^k \binom{m}{k}\binom{n+k}{s+j} \\
&= \sum_{j=m-s}^{n+m-s} (-1)^{m+n-j-s} \binom{s+j}{s}\binom{n}{j+s-m} b_j \\
&= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j}\binom{m+j}{s} b_{m+j-s},
\end{aligned}$$

where the fourth equality follows by Equation (4). □

Equation (5) allows us to establish a generalization of the curious formula

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{n} x^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{n} (1+x)^k, \tag{6}$$

discovered by Simons [8]. A quick proof of this was given by Gould [7] using elementary properties of Legendre polynomials. Instead, choose $a_n = x^n$ for all $n \geq 0$. Then $b_n = (1+x)^n$ and Identity (5) tells us that

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{s} x^{n+k-s} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{m+k}{s} (1+x)^{m+k-s}.$$

Letting $m = s = n$ recovers Identity (6).

Through an induction argument Chen proves

Theorem 2. *[3, Thm. 3.1] Let b_n be the binomial transform of a_n. Then*

$$\begin{aligned}
\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n+k+s}{s}} a_{n+k+s} &= \sum_{k=0}^{n} (-1)^{n-k} \frac{\binom{n}{k}}{\binom{m+k+s}{s}} b_{m+k+s} \\
&+ \sum_{j=0}^{s-1} \sum_{i=0}^{s-1-j} \binom{s-1-j}{i}\binom{s-1}{j} \frac{(-1)^{n+1+i} s a_j}{(m+n+1+i)\binom{m+n+i}{n}},
\end{aligned} \tag{7}$$

where m, n, and s are nonnegative integers.

If we use Equation (4) and the following hypergeometric identity attributed to Frisch [4], [5, p. 337],

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{b+k}{c}}=\frac{c}{n+c}\frac{1}{\binom{n+b}{b-c}},\qquad b\geq c>0, \tag{8}$$

[6, Identity 4.2, p. 46], we are able to prove the following new binomial transformation identity.

Theorem 3. *Let b_n be the binomial transform of a_n. Let m, n, and s be nonnegative integers. Then*

$$\sum_{j=0}^{s}\frac{\binom{s}{j}a_j}{(m+n+s+1-j)\binom{m+n+s-j}{m}}=\sum_{j=0}^{s}\frac{(-1)^{s-j}\binom{s}{j}b_j}{(m+n+s+1-j)\binom{m+n+s-j}{n}}. \tag{9}$$

Proof. By definition $b_n=\sum_{k=0}^{n}\binom{n}{k}a_k$. Hence $a_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}b_k$ and

$$\begin{aligned}
\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+s}{s}}a_{n+k+s} &= \sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+s}{s}}\sum_{j=0}^{n+k+s}(-1)^{n+k+s-j}\binom{n+k+s}{j}b_j\\
&= \sum_{j=0}^{m+n+s}(-1)^{n+s-j}b_j\sum_{k=0}^{m}(-1)^k\frac{\binom{m}{k}}{\binom{n+k+s}{s}}\binom{n+k+s}{j}\\
&= \sum_{j=s}^{m+n+s}(-1)^{n+s-j}\frac{b_j}{\binom{j}{s}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{n+k}{j-s}+\sum_{j=0}^{s-1}(-1)^{n+s-j}b_j\sum_{k=0}^{m}(-1)^k\frac{\binom{m}{k}}{\binom{n+k+s}{s}}\binom{n+k+s}{j}\\
&= \sum_{j=s+m}^{m+n+s}(-1)^{m+n+s-j}\frac{\binom{n}{j-s-m}}{\binom{j}{s}}b_j+\sum_{j=0}^{s-1}(-1)^{n+s-j}b_j\sum_{k=0}^{m}(-1)^k\frac{\binom{m}{k}}{\binom{n+k+s}{s}}\binom{n+k+s}{j}\\
&= \sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}}{\binom{m+j+s}{s}}b_{m+j+s}+\sum_{j=0}^{s-1}(-1)^{n+s-j}b_j\sum_{k=0}^{m}(-1)^k\frac{\binom{m}{k}}{\binom{n+k+s}{s}}\binom{n+k+s}{j}\\
&= \sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}}{\binom{m+j+s}{s}}b_{m+j+s}+\sum_{j=0}^{s-1}(-1)^{n+s-j}\binom{s}{j}b_j\sum_{k=0}^{m}(-1)^k\frac{\binom{m}{k}}{\binom{n+k+s-j}{s-j}}\\
&= \sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}}{\binom{m+j+s}{s}}b_{m+j+s}+\sum_{j=0}^{s-1}(-1)^{n+s-j}\frac{(s-j)\binom{s}{j}}{(m+s-j)\binom{m+n+s-j}{n}}b_j\\
&= \sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n}{j}}{\binom{m+j+s}{s}}b_{m+j+s}+\sum_{j=0}^{s-1}(-1)^{n+s-j}\frac{s\binom{s-1}{j}}{(m+n+s-j)\binom{m+n+s-j-1}{n}}b_j.
\end{aligned}$$

The fourth line follows from Equation (4) while the seventh follows from Equation (8).

In summary, we have shown that

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+s}{s}}a_{n+k+s}=\sum_{j=0}^{n}\frac{(-1)^{n-j}\binom{n}{j}}{\binom{m+j+s}{s}}b_{m+j+s}+\sum_{j=0}^{s-1}\frac{(-1)^{n+s-j}s\binom{s-1}{j}}{(m+n+s-j)\binom{m+n+s-j-1}{n}}b_j, \tag{10}$$

If we compare Identity (7) to Identity (10), we conclude that

$$\sum_{j=0}^{s-1} \frac{(-1)^{n+s-j} s\binom{s-1}{j}}{(m+n+s-j)\binom{m+n+s-j-1}{n}} b_j$$
$$= \sum_{j=0}^{s-1}\sum_{i=0}^{s-1-j} \binom{s-1-j}{i}\binom{s-1}{j} \frac{(-1)^{n+1+i} s a_j}{(m+n+1+i)\binom{m+n+i}{n}}. \tag{11}$$

Equation (11) can be furthered simplified by applying Equation (8). In particular,

$$\sum_{j=0}^{s-1}\sum_{i=0}^{s-1-j} \binom{s-1-j}{i}\binom{s-1}{j} \frac{(-1)^{n+1+i} s a_j}{(m+n+1+i)\binom{m+n+i}{n}}$$
$$= \sum_{j=0}^{s-1} (-1)^{n+1} \frac{s}{n+1}\binom{s-1}{j} a_j \sum_{i=0}^{s-1-j} (-1)^i \frac{\binom{s-1-j}{i}}{\binom{m+n+1+i}{n+1}}$$
$$= \sum_{j=0}^{s-1} (-1)^{n+1} \frac{s}{n+1}\binom{s-1}{j} a_j \frac{n+1}{n+s-j} \frac{1}{\binom{m+n+s-j}{m}}$$
$$= (-1)^{n+1} \sum_{j=0}^{s-1} \frac{s\binom{s-1}{j}}{(n+s-j)\binom{m+n+s-j}{m}} a_j.$$

These calculations show that Equation (11) is equivalent to

$$-\sum_{j=0}^{s-1} \frac{\binom{s-1}{j}}{(n+s-j)\binom{m+n+s-j}{m}} a_j = \sum_{j=0}^{s-1} (-1)^{s-j} \frac{\binom{s-1}{j}}{(m+n+s-j)\binom{m+n+s-j-1}{n}} b_j. \tag{12}$$

Set $s \to s+1$ to obtain

$$\sum_{j=0}^{s} \frac{\binom{s}{j} a_j}{(n+s+1-j)\binom{m+n+s+1-j}{m}} = \sum_{j=0}^{s} \frac{(-1)^{s-j}\binom{s}{j} b_j}{(m+n+s+1-j)\binom{m+n+s-j}{n}}. \tag{13}$$

Since $(n+s+1-j)\binom{m+n+s+1-j}{m} = (m+n+s+1-j)\binom{m+n+s-j}{m}$, we see that Equation (13) is equivalent to Equation (9). □

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