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A Note on Catalan's Identity for the k-Fibonacci Quaternions

Emrah Polatlı and Seyhun Kesim
Department of Mathematics
Bülent Ecevit University
67100 Zonguldak
Turkey
emrahpolatli@gmail.com
seyhun.kesim@beun.edu.tr

Abstract

Ramírez recently conjectured a version of Catalan's identity for the k-Fibonacci quaternions. In this note we give a proof of (a suitably reformulated version of) this identity.

1 Introduction

For any positive real number k, define the k-Fibonacci and k-Lucas sequences, $(F_{k,n})_{n\in\mathbb{N}}$ and $(L_{k,n})_{n\in\mathbb{N}}$, as follows:

$$F_{k,0} = 0,\ F_{k,1} = 1,\ \text{and } F_{k,n} = kF_{k,n-1} + F_{k,n-2},\ n \geq 2,$$

and

$$L_{k,0} = 2,\ L_{k,1} = k,\ \text{and } L_{k,n} = kL_{k,n-1} + L_{k,n-2},\ n \geq 2,$$

respectively.

Let α and β be the roots of the characteristic equation $x^2 - kx - 1 = 0$. Then the Binet formulas for the k-Fibonacci and k-Lucas sequences are

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_{k,n} = \alpha^n + \beta^n,$$

where $\alpha = \left(k + \sqrt{k^2+4}\right)/2$ and $\beta = \left(k - \sqrt{k^2+4}\right)/2$.

A *quaternion* p, with real components a_0, a_1, a_2, a_3 and basis $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, is an element of the form

$$p = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = (a_0, a_1, a_2, a_3), \quad (a_0\mathbf{1} = a_0),$$

where

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,$$
$$\boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{ji},\ \boldsymbol{jk} = \boldsymbol{i} = -\boldsymbol{kj},\ \boldsymbol{ki} = \boldsymbol{j} = -\boldsymbol{ik}.$$

Ramírez [1] defined the nth k-Fibonacci quaternion, $D_{k,n}$, as follows:

$$D_{k,n} = F_{k,n} + F_{k,n+1}\boldsymbol{i} + F_{k,n+2}\boldsymbol{j} + F_{k,n+3}\boldsymbol{k},\ n \geq 0,$$

where $F_{k,n}$ is the nth k-Fibonacci number. Ramírez [1] also gave the Binet formula for the k-Fibonacci quaternion as follows:

$$D_{k,n} = \frac{\widehat{\alpha}\alpha^n - \widehat{\beta}\beta^n}{\alpha - \beta},$$

where $\widehat{\alpha} = 1 + \alpha\boldsymbol{i} + \alpha^2\boldsymbol{j} + \alpha^3\boldsymbol{k}$ and $\widehat{\beta} = 1 + \beta\boldsymbol{i} + \beta^2\boldsymbol{j} + \beta^3\boldsymbol{k}$.

Ramírez [1] conjectured that the Catalan identity for the k-Fibonacci quaternions is

$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = (-1)^{n-r}\left(2F_{k,r}D_{k,r} - G_{k,r}\boldsymbol{k}\right),$$

for $n \geq r \geq 1$, where $G_{k,r}$ is the sequence satisfying the following recurrence:

$$G_{k,0} = 0,\ G_{k,1} = k^2 + 2k,\ \text{and } G_{k,n} = \left(k^2+2\right)G_{k,n-1} - G_{k,n-2},\ n \geq 2.$$

However, in this short paper, we show that this conjecture is incorrect, by giving the correct Catalan identity and proving it.

2 Catalan identity for the k-Fibonacci quaternions

We need the following lemma.

Lemma 1. *For $r \geq 1$, we have*

$$\frac{\widehat{\alpha}\widehat{\beta}\beta^r - \widehat{\beta}\widehat{\alpha}\alpha^r}{\alpha - \beta} = -2D_{k,r} + L_{k,2}L_{k,r}\boldsymbol{k}.$$

Proof. Since

$$\widehat{\alpha}\widehat{\beta} = 2 + 2\beta\boldsymbol{i} + 2\beta^2\boldsymbol{j} + \left(\alpha^3 + \beta^3 + \alpha - \beta\right)\boldsymbol{k}$$

and

$$\widehat{\beta}\widehat{\alpha} = 2 + 2\alpha\boldsymbol{i} + 2\alpha^2\boldsymbol{j} + \left(\alpha^3 + \beta^3 + \beta - \alpha\right)\boldsymbol{k},$$

we get

$$\begin{aligned}
\frac{\widehat{\alpha}\widehat{\beta}\beta^r - \widehat{\beta}\widehat{\alpha}\alpha^r}{\alpha - \beta} &= -2F_{k,r} - 2F_{k,r+1}\boldsymbol{i} - 2F_{k,r+2}\boldsymbol{j} + \left(-2F_{k,r+3} + L_{k,2}L_{k,r}\right)\boldsymbol{k} \\
&= -2D_{k,r} + L_{k,2}L_{k,r}\boldsymbol{k}.
\end{aligned}$$

□

Theorem 2. *For $n \geq r \geq 1$, Catalan identity for the k-Fibonacci quaternions is*

$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = (-1)^{n-r+1}\left(2F_{k,r}D_{k,r} - L_{k,2}F_{k,2r}\boldsymbol{k}\right).$$

Proof. By considering the Binet formula for the k-Fibonacci quaternions, quaternion multiplication and Lemma 1, we obtain

$$\begin{aligned}
D_{k,n-r}D_{k,n+r} - D_{k,n}^2 &= \left(\frac{\widehat{\alpha}\alpha^{n-r} - \widehat{\beta}\beta^{n-r}}{\alpha - \beta}\right)\left(\frac{\widehat{\alpha}\alpha^{n+r} - \widehat{\beta}\beta^{n+r}}{\alpha - \beta}\right) - \left(\frac{\widehat{\alpha}\alpha^{n} - \widehat{\beta}\beta^{n}}{\alpha - \beta}\right)^2 \\
&= \frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left(\widehat{\alpha}\widehat{\beta}\left(1 - \frac{\beta^r}{\alpha^r}\right) + \widehat{\beta}\widehat{\alpha}\left(1 - \frac{\alpha^r}{\beta^r}\right)\right) \\
&= (\alpha\beta)^n\,\frac{\alpha^r - \beta^r}{(\alpha-\beta)^2}\left(\frac{\widehat{\alpha}\widehat{\beta}}{\alpha^r} - \frac{\widehat{\beta}\widehat{\alpha}}{\beta^r}\right) \\
&= (\alpha\beta)^{n-r}\,\frac{\alpha^r - \beta^r}{\alpha-\beta}\left(\frac{\widehat{\alpha}\widehat{\beta}\beta^r - \widehat{\beta}\widehat{\alpha}\alpha^r}{\alpha - \beta}\right) \\
&= (\alpha\beta)^{n-r}\,F_{k,r}\left(-2D_{k,r} + L_{k,2}L_{k,r}\boldsymbol{k}\right) \\
&= (-1)^{n-r+1}\left(2F_{k,r}D_{k,r} - L_{k,2}F_{k,2r}\boldsymbol{k}\right).
\end{aligned}$$

□

References

[1] J. L. Ramírez, Some combinatorial properties of the k -Fibonacci and the k-Lucas quaternions, *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.* **23** (2015), 201–212.

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