Journal of Integer Sequences, Vol. 21 (2018),
Article 18.6.8

Factored Closed-form Expressions for the Sums of Cubes of Fibonacci and Lucas Numbers

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Abstract

We obtain factored closed-form expressions for the sums of cubes of Fibonacci and Lucas numbers.

1 Introduction

The Fibonacci numbers, F_n, and Lucas numbers, L_n, are defined, for $n \in \mathbb{Z}$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$, with $F_{-n} = (-1)^{n-1} F_n$ and $L_{-n} = (-1)^n L_n$.

Clary and Hemenway [2] derived the remarkable formulas

$$4\sum_{k=1}^{n} F_{2k}^3 = \begin{cases} F_n^2 L_{n+1}^2 F_{n-1} L_{n+2}, & \text{if } n \text{ is even;} \\ L_n^2 F_{n+1}^2 L_{n-1} F_{n+2}, & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

and

$$8\sum_{k=1}^{n} F_{4k}^3 = F_{2n}^2 F_{2n+2}^2 (L_{4n+2} + 6)\,. \tag{2}$$

In this present paper we will derive the following corresponding Lucas counterparts of (1) and (2):

$$4\sum_{k=1}^{n} L_{2k}^3 = \begin{cases} 5F_nF_{n+1}(L_nL_{n+1}L_{2n+1}+16), & \text{if } n \text{ is even;} \\ L_nL_{n+1}(5F_nF_{n+1}L_{2n+1}+16), & \text{if } n \text{ is odd,} \end{cases} \tag{3}$$

and

$$8\sum_{k=1}^{n} L_{4k}^3 = F_{2n}L_{2n+2}(5L_{2n}F_{2n+2}F_{4n+2}+32)\,. \tag{4}$$

In fact we will derive the following more general results:

- If r is odd, then

$$L_{3r}\sum_{k=1}^{n} F_{2rk}^3 = \begin{cases} F_{rn}^2L_{rn+r}^2(L_{rn}F_{rn+r}+F_r), & \text{if } n \text{ is even;} \\ L_{rn}^2F_{rn+r}^2(F_{rn}L_{rn+r}+F_r), & \text{if } n \text{ is odd,} \end{cases} \tag{5}$$

  and

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = \begin{cases} 5F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r}+4(L_{2r}+1)), & \text{if } n \text{ is even;} \\ L_{rn}L_{rn+r}(5F_{rn}F_{rn+r}L_{2rn+r}+4(L_{2r}+1)), & \text{if } n \text{ is odd.} \end{cases} \tag{6}$$

- If r is even, then

$$F_{3r}\sum_{k=1}^{n} F_{2rk}^3 = F_{rn}^2F_{rn+r}^2(L_{rn}L_{rn+r}+L_r) \tag{7}$$

  and

$$F_{3r}\sum_{k=1}^{n} L_{2rk}^3 = F_{rn}L_{rn+r}(5L_{rn}F_{rn+r}F_{2rn+r}+4(L_{2r}+1))\,. \tag{8}$$

As variations on identities (5) and (7) we will prove

- If r is odd, then

$$L_{3r}\sum_{k=1}^{n} F_{2rk}^3 = \begin{cases} F_{rn}L_{rn+r}(L_{rn}F_{rn+r}F_{2rn+r}-2F_r^2), & \text{if } n \text{ is even;} \\ L_{rn}F_{rn+r}(F_{rn}L_{rn+r}F_{2rn+r}-2F_r^2), & \text{if } n \text{ is odd.} \end{cases}$$

- If r is even, then

$$5F_{3r}\sum_{k=1}^{n} F_{2rk}^3 = F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r}-2L_r^2)\,.$$

2 Required identities and preliminary results

2.1 Telescoping summation identity

The following telescoping summation identity is a special case of more general identities proved by Adegoke [1].

Lemma 1. *If $f(k)$ is a real sequence and m, q and n are positive integers, then*

$$\sum_{k=1}^{n}\left[f(mk+mq)-f(mk)\right]=\sum_{k=1}^{q}f(mk+mn)-\sum_{k=1}^{q}f(mk)\,.$$

2.2 First-power Fibonacci summation identities

Lemma 2. *If r and n are integers, then*

(i) If r is even, then

$$F_r\sum_{k=1}^{n}F_{2rk}=F_{rn}F_{rn+r}\,.$$

(ii) If r is odd, then

$$L_r\sum_{k=1}^{n}F_{2rk}=\begin{cases}F_{rn}L_{rn+r}, & \text{if } n \text{ is even;}\\ L_{rn}F_{rn+r}, & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Setting $v=2r$ and $u=2rk$ in the identity

$$L_{u+v}-(-1)^vL_{u-v}=5F_uF_v \tag{9}$$

gives

$$L_{2rk+2r}-L_{2rk-2r}=5F_{2r}F_{2rk}\,. \tag{10}$$

Taking $f(k)=L_{k-2r}$, $q=2$ and $m=2r$ in Lemma 1 and employing identity (10) we have

$$\begin{aligned}5F_{2r}\sum_{k=1}^{n}F_{2rk}&=\sum_{k=1}^{2}L_{2rk+2rn-2r}-\sum_{k=1}^{2}L_{2rk-2r}\\&=L_{2rn+2r}+L_{2rn}-L_{2r}-2\,.\end{aligned} \tag{11}$$

If r is even, then on account of the identity

$$L_{u+v}+(-1)^vL_{u-v}=L_uL_v, \tag{12}$$

we have

$$L_{2rn+2r}+L_{2rn}=L_rL_{2rn+r},\quad L_{2r}+2=L_r^2\,,$$

and since

$$F_{2u} = F_u L_u\,, \tag{13}$$

identity (11) now becomes

$$\begin{aligned} 5F_r \sum_{k=1}^{n} F_{2rk} &= L_{2rn+r} - L_r \\ &= 5F_{rn}F_{rn+r}\,, \quad \text{by (9)}\,, \end{aligned} \tag{14}$$

that is,

$$F_r \sum_{k=1}^{n} F_{2rk} = F_{rn}F_{rn+r}, \quad r \text{ even}\,,$$

and the first part of Lemma 2 is proved.

If r is odd, then on account of the identities (9) and (12), we have

$$L_{2rn+2r} + L_{2rn} = 5F_rF_{2rn+r}, \quad L_{2r} + 2 = 5F_r^2\,,$$

and identity (11) reduces to

$$\begin{aligned} L_r \sum_{k=1}^{n} F_{2rk} &= F_{2rn+r} - F_r \\ &= \begin{cases} F_{rn}L_{rn+r}, & \text{if } n \text{ is even;} \\ L_{rn}F_{rn+r}, & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and the second part of Lemma 2 is proved. In the last stage of the above derivation we made use of the identities

$$F_{u+v} - (-1)^v F_{u-v} = F_v L_u \tag{15}$$

and

$$F_{u+v} + (-1)^v F_{u-v} = L_v F_u\,. \tag{16}$$

□

2.3 First-power Lucas summation identities

Lemma 3. *If r and n are integers, then*

(i) If r is even, then

$$F_r \sum_{k=1}^{n} L_{2rk} = F_{rn}L_{rn+r}\,.$$

(ii) If r is odd, then

$$L_r \sum_{k=1}^{n} L_{2rk} = \begin{cases} 5F_{rn}F_{rn+r}, & \textit{if } n \textit{ is even;} \\ L_{rn}L_{rn+r}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Setting $v = 2r$ and $u = 2rk$ in the identity (15) gives

$$F_{2rk+2r} - F_{2rk-2r} = F_{2r}L_{2rk}\,. \tag{17}$$

Taking $f(k) = F_{k-2r}$, $q = 2$ and $m = 2r$ in Lemma 1 and employing identity (17) we have

$$\begin{aligned} F_{2r} \sum_{k=1}^{n} L_{2rk} &= \sum_{k=1}^{2} F_{2rk+2rn-2r} - \sum_{k=1}^{2} F_{2rk-2r} \\ &= F_{2rn+2r} + F_{2rn} - F_{2r}\,. \end{aligned} \tag{18}$$

If r is even, then choosing $v = r$ and $u = 2rn + r$ in identity (16) gives

$$F_{2rn+2r} + F_{2rn} = L_r F_{2rn+r} \tag{19}$$

and, on account of identity (13), the identity (18) reduces to

$$\begin{aligned} F_r \sum_{k=1}^{n} L_{2rk} &= F_{2rn+r} - F_r \\ &= F_{rn+r+rn} - F_{rn+r-rn} \\ &= F_{rn}L_{rn+r}\,, \quad \text{by identity (15)}, \end{aligned}$$

and the first part of Lemma 3 is proved.

If r is odd, then choosing $v = r$ and $u = 2rn + r$ in identity (15) gives

$$F_{2rn+2r} + F_{2rn} = F_r L_{2rn+r} \tag{20}$$

and, again on account of identity (13), the identity (18) now reduces to

$$\begin{aligned} L_r \sum_{k=1}^{n} L_{2rk} &= L_{2rn+r} - L_r \\ &= L_{rn+r+rn} - L_{rn+r-rn} \\ &= \begin{cases} 5F_{rn}F_{rn+r}, & \text{if } n \text{ is even;} \\ L_{rn}L_{rn+r}, & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

where in the last step we used the identities (9) and (12).

□

2.4 Other identities

Lemma 4. *If r and n are integers, then*

$$\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} = L_{rn}L_{rn+r}L_{2rn+r} + L_{2r} + (-1)^{r-1}.$$

Proof. Using the identity Clary [2, Eq. (36)], or Dresel [3, Eq. (3.3)], namely,

$$F_{3u} = 5F_u^3 + 3(-1)^u F_u\,, \tag{21}$$

we have

$$\begin{aligned}\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} &= (5F_{rn}^2 + 3(-1)^{rn})(5F_{rn+r}^2 + 3(-1)^{rn+r})\\ &= (L_{rn}^2 - (-1)^{rn})(L_{rn+r}^2 - (-1)^{rn+r})\\ &= L_{rn}^2 L_{rn+r}^2 - (-1)^{rn+r}L_{rn}^2 - (-1)^{rn}L_{rn+r}^2 + (-1)^r\,,\end{aligned} \tag{22}$$

where we have also made use of the identity

$$5F_u^2 - L_u^2 = (-1)^{u-1}4\,. \tag{23}$$

Now,

$$\begin{aligned}L_{rn}^2 L_{rn+r}^2 &= L_{rn}L_{rn+r}(L_{rn}L_{rn+r})\\ &= L_{rn}L_{rn+r}(L_{2rn+r} + (-1)^{rn}L_r) \quad \text{by (12)}\\ &= L_{rn}L_{rn+r}L_{2rn+r} + (-1)^{rn}L_{rn}L_{rn+r}L_r\,.\end{aligned}$$

Therefore

$$\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} = L_{rn}L_{rn+r}L_{2rn+r} + (-1)^{rn}L_{rn+r}(L_{rn}L_r - L_{rn+r}) - (-1)^{rn+r}L_{rn}^2 + (-1)^r\,.$$

But

$$\begin{aligned}&(-1)^{rn}L_{rn+r}(L_{rn}L_r - L_{rn+r})\\ &= (-1)^{rn}L_{rn+r}(L_{rn+r} + (-1)^r L_{rn-r} - L_{rn+r}), \quad \text{by (12)}\\ &= (-1)^{rn+r}L_{rn+r}L_{rn-r}\\ &= (-1)^{rn+r}(L_{2rn} + (-1)^{rn-r}L_{2r}), \quad \text{again by (12)}\\ &= (-1)^{rn+r}L_{2rn} + L_{2r}\,.\end{aligned}$$

Thus

$$\begin{aligned}\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} &= L_{rn}L_{rn+r}L_{2rn+r} + (-1)^{rn+r}L_{2rn} + L_{2r} - (-1)^{rn+r}L_{rn}^2 + (-1)^r\\ &= L_{rn}L_{rn+r}L_{2rn+r} + (-1)^{rn+r}(L_{2rn} - L_{rn}^2) + L_{2r} + (-1)^r\,.\end{aligned}$$

Finally, using the identity

$$L_{2u} = L_u^2 + (-1)^{u-1}2\,, \tag{24}$$

obtained by setting $v = u$ in identity (12), we have the statement of the Lemma. □

Lemma 5. *If r and n are integers, then*

$$\frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} = 5F_{rn}F_{rn+r}L_{2rn+r} + L_{2r} + (-1)^{r-1}.$$

Proof. Using the following identity, of Dresel [3, Eq. (1.6)]

$$L_{3u} = L_u^3 - 3(-1)^u L_u\,, \tag{25}$$

we have

$$\begin{aligned}\frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} &= (L_{rn}^2 - 3(-1)^{rn})(L_{rn+r}^2 - 3(-1)^{rn+r})\\ &= (5F_{rn}^2 + (-1)^{rn})(5F_{rn+r}^2 + (-1)^{rn+r}), \quad \text{by (23)}\\ &= 25F_{rn}^2F_{rn+r}^2 + (-1)^{rn+r}5F_{rn}^2 + (-1)^{rn}5F_{rn+r}^2 + (-1)^r\,,\end{aligned}$$

and the rest of the calculation then proceeds as in the proof of Lemma 4, the basic required identities now being (9), (16) and the identity

$$L_{2u} = 5F_u^2 + (-1)^u 2, \tag{26}$$

obtained by setting $v = u$ in identity (9). □

Lemma 6. *If r and n are integers, then*

$$\frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} = 5F_{rn}L_{rn+r}F_{2rn+r} + L_{2r} + (-1)^r.$$

Lemma 7. *If r and n are integers, then*

$$\frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} = 5L_{rn}F_{rn+r}F_{2rn+r} + L_{2r} + (-1)^r.$$

Different but equivalent versions of Lemmas 4–7 are given below:

Lemma 8. *If r and n are integers, then*

$$\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} = L_{2rn+r}^2 + (-1)^{nr}L_{rn+r}^2 + (-1)^{(n-1)r}L_{rn}^2 + L_r^2 + (-1)^{r-1}7\,.$$

Proof. The proof is similar to that of Lemma 4, but here we use

$$\begin{aligned}L_{rn}^2L_{rn+r}^2 &= (L_{2rn+r} + (-1)^{rn}L_r)^2\\ &= L_{2rn+r}^2 + L_r^2 + 2(-1)^{rn}(L_rL_{2rn+r})\\ &= L_{2rn+r}^2 + L_r^2 + 2(-1)^{rn}(L_{2rn+2r} + (-1)^rL_{2rn})\\ &= L_{2rn+r}^2 + L_r^2 + 2(-1)^{rn}(L_{rn+r}^2 + (-1)^{rn+r-1}2 + (-1)^r(L_{rn}^2 + (-1)^{rn-1}2)),\end{aligned}$$

and substitute in (22). □

Lemma 9. *If r and n are integers, then*

$$\frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} = L_{2rn+r}^2 + (-1)^{nr-1}L_{rn+r}^2 - (-1)^{(n-1)r}L_{rn}^2 + L_r^2 + (-1)^r\,.$$

Lemma 10. *If r and n are integers, then*

$$\frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} = 5F_{2rn+r}^2 + (-1)^{nr-1}5F_{rn+r}^2 + (-1)^{(n-1)r}5F_{rn}^2 + 5F_r^2 + (-1)^r3\,.$$

Lemma 11. *If r and n are integers, then*

$$\frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} = 5F_{2rn+r}^2 + (-1)^{nr}5F_{rn+r}^2 - (-1)^{(n-1)r}5F_{rn}^2 + 5F_r^2 + (-1)^r3\,.$$

3 Main results

3.1 Sums of cubes of Fibonacci numbers

Theorem 12. *If r and n are integers such that r is odd, then*

$$L_{3r}\sum_{k=1}^{n}F_{2rk}^3 = \begin{cases} F_{rn}L_{rn+r}(L_{rn}F_{rn+r}F_{2rn+r} - 2F_r^2), & \text{if } n \text{ is even;}\\ L_{rn}F_{rn+r}(F_{rn}L_{rn+r}F_{2rn+r} - 2F_r^2), & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Setting $u = 2rk$ in identity (21) and summing, we have

$$5\sum_{k=1}^{n}F_{2rk}^3 = \sum_{k=1}^{n}F_{6rk} - 3\sum_{k=1}^{n}F_{2rk}\,,$$

so that,

$$\begin{aligned} 5L_{3r}\sum_{k=1}^{n}F_{2rk}^3 &= L_{3r}\sum_{k=1}^{n}F_{6rk} - 3\frac{L_{3r}}{L_r}L_r\sum_{k=1}^{n}F_{2rk}\\ &= L_{3r}\sum_{k=1}^{n}F_{6rk} - 3(L_r^2+3)L_r\sum_{k=1}^{n}F_{2rk}\,. \end{aligned} \tag{27}$$

- If n is even, then, by Lemma 2, identity (27) can be written as

  $$5L_{3r}\sum_{k=1}^{n}F_{2rk}^3 = F_{3rn}L_{3rn+3r} - 3(L_r^2+3)F_{rn}L_{rn+r}\,,$$

  so that

  $$\begin{aligned} \frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{F_{rn}L_{rn+r}} &= \frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} - 3(L_r^2+3)\\ &= 5L_{rn}F_{rn+r}F_{2rn+r} + L_{2r} - 1 - 3L_r^2 - 9, \quad \text{by Lemma 7}\\ &= 5L_{rn}F_{rn+r}F_{2rn+r} - 10F_r^2, \quad \text{by (23) and (24)}\,. \end{aligned}$$

- If n is odd, then, by Lemma 2, we have

$$5L_{3r}\sum_{k=1}^{n}F_{2rk}^3 = L_{3rn}F_{3rn+3r} - 3(L_r^2+3)L_{rn}F_{rn+r}\,,$$

so that

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{L_{rn}F_{rn+r}} &= \frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} - 3(L_r^2+3)\\ &= 5F_{rn}L_{rn+r}F_{2rn+r} + L_{2r} - 1 - 3L_r^2 - 9, \quad \text{by Lemma 7}\\ &= 5F_{rn}L_{rn+r}F_{2rn+r} - 10F_r^2, \quad \text{by (23) and (24)}\,.\end{aligned}$$

□

Theorem 13. *If r and n are integers such that r is even, then*

$$5F_{3r}\sum_{k=1}^{n}F_{2rk}^3 = F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r} - 2L_r^2)\,.$$

Proof.

$$\begin{aligned}5F_{3r}\sum_{k=1}^{n}F_{2rk}^3 &= F_{3r}\sum_{k=1}^{n}F_{6rk} - 3\frac{F_{3r}}{F_r}F_r\sum_{k=1}^{n}F_{2rk}\\ &= F_{3rn}F_{3rn+3r} - 3(5F_r^2+3)F_{rn}F_{rn+r},\\ &\text{by Lemma 2 and identity (21)}\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{5F_{3r}\sum_{k=1}^{n}F_{2rk}^3}{F_{rn}F_{rn+r}} &= \frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} - 3(5F_r^2+3)\\ &= L_{rn}L_{rn+r}L_{2rn+r} + L_{2r} - 1 - 15F_r^2 - 9\\ &\text{(by Lemma 4 and identity (21))}\,,\\ &= L_{rn}L_{rn+r}L_{2rn+r} - 2L_r^2, \quad \text{by (23), (24) and (26)}\,.\end{aligned}$$

□

Theorem 14. *If r and n are integers such that r is odd, then*

$$L_{3r}\sum_{k=1}^{n}F_{2rk}^3 = \begin{cases}F_{rn}^2L_{rn+r}^2(L_{rn}F_{rn+r}+F_r), & \text{if } n \text{ is even;}\\ L_{rn}^2F_{rn+r}^2(F_{rn}L_{rn+r}+F_r), & \text{if } n \text{ is odd.}\end{cases}$$

Proof. • If n is even, then from Lemma 2 and identity (27) we have

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{F_{rn}L_{rn+r}} &= \frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}} - 3(L_r^2+3)\\ &= 5F_{2rn+r}^2 + 5F_{rn+r}^2 + 5F_{rn}^2 + 5F_r^2 - 3 - 3L_r^2 - 9, \text{ by Lemma 11}\\ &= 5F_{2rn+r}^2 + 5F_{rn+r}^2 + 5F_{rn}^2 - 10F_r^2 \quad \text{by identity (23)}\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{F_{rn}L_{rn+r}} &= F_{2rn+r}^2 + F_{rn+r}^2 + F_{rn}^2 - 2F_r^2\\ &= (F_{2rn+r}^2 - F_r^2) + (F_{rn+r}^2 + F_{rn}^2) - F_r^2\,.\end{aligned}$$

Using the following identity, derived by Howard [4],

$$F_u^2 + (-1)^{u+v-1}F_v^2 = F_{u-v}F_{u+v}\,, \tag{28}$$

we have

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{F_{rn}L_{rn+r}} &= F_{2rn}F_{2rn+2r} + F_rF_{2rn+r} - F_r^2\\ &= F_{2rn}F_{2rn+2r} + F_r(F_{2rn+r} - F_r)\\ &= F_{2rn}F_{2rn+2r} + F_rF_{rn}L_{rn+r}, \text{ by identity (15)}\\ &= F_{rn}L_{rn+r}L_{rn}F_{rn+r} + F_rF_{rn}L_{rn+r}\\ &= F_{rn}L_{rn+r}(L_{rn}F_{rn+r} + F_r)\,.\end{aligned}$$

• If n is odd, then from Lemma 2 and identity (27) we have

$$\begin{aligned}\frac{5L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{L_{rn}F_{rn+r}} &= \frac{L_{3rn}F_{3rn+3r}}{L_{rn}F_{rn+r}} - 3(L_r^2+3)\\ &= 5F_{2rn+r}^2 + 5F_{rn+r}^2 + 5F_{rn}^2 + 5F_r^2 - 3 - 3L_r^2 - 9, \text{ by Lemma 10}\\ &= 5F_{2rn+r}^2 + 5F_{rn+r}^2 + 5F_{rn}^2 - 10F_r^2 \quad \text{by identity (23)}\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{L_{rn}F_{rn+r}} &= F_{2rn+r}^2 + F_{rn+r}^2 + F_{rn}^2 - 2F_r^2\\ &= (F_{2rn+r}^2 - F_r^2) + (F_{rn+r}^2 + F_{rn}^2) - F_r^2\,.\end{aligned}$$

Using identity (28), we have

$$\begin{aligned}\frac{L_{3r}\sum_{k=1}^{n}F_{2rk}^3}{L_{rn}F_{rn+r}} &= F_{2rn}F_{2rn+2r} + F_rF_{2rn+r} - F_r^2\\ &= F_{2rn}F_{2rn+2r} + F_r(F_{2rn+r} - F_r)\\ &= F_{2rn}F_{2rn+2r} + F_rL_{rn}F_{rn+r}, \text{ by identity (16)}\\ &= F_{rn}L_{rn+r}L_{rn}F_{rn+r} + F_rL_{rn}F_{rn+r}\\ &= L_{rn}F_{rn+r}(F_{rn}L_{rn+r} + F_r)\,.\end{aligned}$$

□

Theorem 15. *If r and n are integers such that r is even, then*

$$F_{3r}\sum_{k=1}^{n}F_{2rk}^{3}=F_{rn}^{2}F_{rn+r}^{2}(L_{rn}L_{rn+r}+L_{r})\,. \tag{29}$$

Proof.

$$\begin{aligned}5F_{3r}\sum_{k=1}^{n}F_{2rk}^{3}&=F_{3r}\sum_{k=1}^{n}F_{6rk}-3\frac{F_{3r}}{F_{r}}F_{r}\sum_{k=1}^{n}F_{2rk}\\&=F_{3rn}F_{3rn+3r}-3(5F_{r}^{2}+3)F_{rn}F_{rn+r}\,,\end{aligned}$$

so that

$$\begin{aligned}\frac{5F_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}F_{rn+r}}&=\frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}}-3(5F_{r}^{2}+3)\\&=L_{2rn+r}^{2}+L_{rn+r}^{2}+L_{rn}^{2}+L_{r}^{2}-7-15F_{r}^{2}-9,\quad\text{by Lemma 8}\\&=L_{2rn+r}^{2}+L_{rn+r}^{2}-2L_{r}^{2}+5F_{rn}^{2},\quad\text{by (23)}\\&=(L_{2rn+r}^{2}-L_{r}^{2})+(L_{rn+r}^{2}-L_{r}^{2})+5F_{rn}^{2}\,.\end{aligned}$$

Using the identity (derived by Howard [4])

$$L_{u}^{2}+(-1)^{u+v-1}L_{v}^{2}=5F_{u-v}F_{u+v}\,, \tag{30}$$

we see that

$$L_{2rn+r}^{2}-L_{r}^{2}=5F_{2rn}F_{2rn+2r}=5F_{rn}F_{rn+r}L_{rn}L_{rn+r} \tag{31}$$

and

$$L_{rn+r}^{2}-L_{r}^{2}=5F_{rn}F_{rn+2r}\,. \tag{32}$$

Thus,

$$\begin{aligned}\frac{F_{3r}\sum_{k=1}^{n}F_{2rk}^{3}}{F_{rn}F_{rn+r}}&=F_{rn}F_{rn+r}L_{rn}L_{rn+r}+F_{rn}F_{rn+2r}+F_{rn}^{2}\\&=F_{rn}F_{rn+r}L_{rn}L_{rn+r}+F_{rn}(F_{rn}+F_{rn+2r})\\&=F_{rn}F_{rn+r}L_{rn}L_{rn+r}+F_{rn}F_{rn+r}L_{r},\quad\text{by identity (16)}\\&=F_{rn}F_{rn+r}(L_{rn}L_{rn+r}+L_{r})\,.\end{aligned}$$

□

3.2 Sums of cubes of Lucas numbers

Theorem 16. *If r and n are integers such that r is odd, then*

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = \begin{cases} 5F_{rn}F_{rn+r}(L_{rn}L_{rn+r}L_{2rn+r} + 4(L_{2r}+1)), & \text{if } n \text{ is even;}\\ L_{rn}L_{rn+r}(5F_{rn}F_{rn+r}L_{2rn+r} + 4(L_{2r}+1)), & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Using identity (25) with $u = 2rk$, we have

$$\sum_{k=1}^{n} L_{2rk}^3 = \sum_{k=1}^{n} L_{6rk} + 3\sum_{k=1}^{n} L_{2rk}\,,$$

so that

$$\begin{aligned} L_{3r}\sum_{k=1}^{n} L_{2rk}^3 &= L_{3r}\sum_{k=1}^{n} L_{6rk} + 3\frac{L_{3r}}{L_r}L_r\sum_{k=1}^{n} L_{2rk}\\ &= L_{3r}\sum_{k=1}^{n} L_{6rk} + 3(L_r^2+3)L_r\sum_{k=1}^{n} L_{2rk}, \quad \text{by (25)}\,. \end{aligned}$$

- If n is even, then by Lemma 3 we have

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = 5F_{3rn}F_{3rn+3r} + 3(L_r^2+3)5F_{rn}F_{rn+r}\,, \tag{33}$$

  so that

$$\begin{aligned} \frac{L_{3r}\sum_{k=1}^{n} L_{2rk}^3}{5F_{rn}F_{rn+r}} &= \frac{F_{3rn}F_{3rn+3r}}{F_{rn}F_{rn+r}} + 3(L_r^2+3)\\ &= L_{rn}L_{rn+r}L_{2rn+r} + L_{2r} + 1 + 3L_r^2 + 9, \quad \text{by Lemma 4}\\ &= L_{rn}L_{rn+r}L_{2rn+r} + 4(L_{2r}+1), \quad \text{by (24)}\,. \end{aligned}$$

- If n is odd, then by Lemma 3 we have

$$L_{3r}\sum_{k=1}^{n} L_{2rk}^3 = L_{3rn}L_{3rn+3r} + 3(L_r^2+3)L_{rn}L_{rn+r}\,, \tag{34}$$

  so that

$$\begin{aligned} \frac{L_{3r}\sum_{k=1}^{n} L_{2rk}^3}{L_{rn}L_{rn+r}} &= \frac{L_{3rn}L_{3rn+3r}}{L_{rn}L_{rn+r}} + 3(L_r^2+3)\\ &= 5F_{rn}F_{rn+r}L_{2rn+r} + L_{2r} + 1 + 3L_r^2 + 9, \quad \text{by Lemma 5}\\ &= 5F_{rn}F_{rn+r}L_{2rn+r} + 4(L_{2r}+1), \quad \text{by (24)}\,. \end{aligned}$$

∎

Theorem 17. *If r and n are integers such that r is even, then*

$$F_{3r}\sum_{k=1}^{n}L_{2rk}^{3}=F_{rn}L_{rn+r}(5L_{rn}F_{rn+r}F_{2rn+r}+4(L_{2r}+1))\,.$$

Proof.

$$\begin{aligned}F_{3r}\sum_{k=1}^{n}L_{2rk}^{3}&=F_{3r}\sum_{k=1}^{n}L_{6rk}+3\frac{F_{3r}}{F_r}F_r\sum_{k=1}^{n}L_{2rk}\\&=F_{3r}\sum_{k=1}^{n}L_{6rk}+3(5F_r^2+3)F_r\sum_{k=1}^{n}L_{2rk},\quad\text{by identity (21)}\\&=F_{3rn}L_{3rn+3r}+3(5F_r^2+3)F_{rn}L_{rn+r},\quad\text{by Lemma 3}\,.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{F_{3r}\sum_{k=1}^{n}L_{2rk}^{3}}{F_{rn}L_{rn+r}}&=\frac{F_{3rn}L_{3rn+3r}}{F_{rn}L_{rn+r}}+3(5F_r^2+3)\\&=5L_{rn}F_{rn+r}F_{2rn+r}+L_{2r}+1+15F_r^2+9,\quad\text{by Lemma 7}\\&=5L_{rn}F_{rn+r}F_{2rn+r}+4(L_{2r}+1),\quad\text{by (24) and (26)}\,.\end{aligned}$$

∎

4 Acknowledgment

Thanks are due to the referee for a careful reading and for pointing out a couple of typographical errors.

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2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37.
Keywords: sum of cubes, Fibonacci number, Lucas number, telescoping summation.

(Concerned with sequences A000032 and A000045.)

Received February 11 2018; revised versions received March 8 2018; July 1 2018; July 3 2018.
Published in *Journal of Integer Sequences*, August 22 2018.

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