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Consecutive Integers Divisible by a Power of their Largest Prime Factor

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Abstract

We construct families of consecutive polynomials with integer coefficients that allow for the discovery of consecutive integers divisible by a power of their largest prime factor.

1 Introduction

Let $P(n)$ stand for the largest prime factor of an integer $n \geq 2$ and set $P(1) = 1$. Given an arbitrary positive integer ℓ and k distinct primes $p_0, p_1, \ldots, p_{k-1}$, the Chinese remainder theorem guarantees the existence of infinitely many integers n such that $p_i^\ell \mid n + i$ for $i = 0, 1, \ldots, k-1$. However, this theorem does not guarantee that such integers n will also have the property that $P(n+i) = p_i$ for $i = 0, 1, \ldots, k-1$, although such is the case in some particular instances. For example when $\ell = 2$, $k = 3$ and $n = 1\,294\,298$, we indeed have

$$\begin{aligned} 1\,294\,298 &= 2 \cdot 61 \cdot 103^2, \\ 1\,294\,299 &= 3^4 \cdot 19 \cdot 29^2, \\ 1\,294\,300 &= 2^2 \cdot 5^2 \cdot 7 \cdot 43^2. \end{aligned}$$

In fact, one can show that the above number n is the smallest positive integer with that property. This motivates the following definitions. Given fixed integers $k \geq 2$ and $\ell \geq 2$, set

$$\begin{aligned} E_{k,\ell} &:= \{n \in \mathbb{N} : P(n+i)^\ell \mid n+i \ \text{ for each } i = 0, 1, \ldots, k-1\} \\ E_{k,\ell}(x) &:= \#\{n \leq x : n \in E_{k,\ell}\}. \end{aligned}$$

Many elements of $E_{2,2}$, $E_{2,3}$, $E_{2,4}$, $E_{2,5}$ and $E_{3,2}$ are given in the 2009 book of the first author [2], whereas no elements of the sets $E_{3,3}$, $E_{2,6}$ and $E_{4,2}$ were known at that time. But, in 2014, Burcsi and Gévay (private communication), found the 77-digit number n_0 which satisfies

$$\begin{aligned} n_0 - 1 &= 2^7 \cdot 53 \cdot 4253 \cdot 27631 \cdot 27953 \cdot 1546327 \cdot 2535271 \\ &\quad \cdot 17603683 \cdot 1472289739 \cdot 16476952799^3, \\ n_0 &= 3^6 \cdot 19 \cdot 37 \cdot 787 \cdot 711163 \cdot 2181919 \cdot 137861107 \\ &\quad \cdot 318818473 \cdot 937617607 \cdot 7323090133^3, \\ n_0 + 1 &= 2 \cdot 12899 \cdot 133451 \cdot 421607 \cdot 2198029 \cdot 8046041 \\ &\quad \cdot 19854409 \cdot 555329197 \cdot 32953905599^3, \end{aligned}$$

thereby establishing that $n_0 - 1 \in E_{3,3}$. Perhaps, this number is the smallest element of $E_{3,3}$, but this has not been shown.

Even though no elements of $E_{k,\ell}$ for $k \geq 4$ and $\ell \geq 2$ are known, it seems reasonable to conjecture that, given any fixed integers $k \geq 2$ and $\ell \geq 2$, the corresponding set $E_{k,\ell}$ is infinite.

The fact that $\#E_{k,\ell} = \infty$ is certainly true in the particular case $k = \ell = 2$, as it is an immediate consequence of the fact that the Fermat-Pell equation $a^2 - 2b^2 = 1$ has infinitely many integer solutions (a, b), thereby also ensuring that $E_{2,2}(x) \gg \log x$. However, $E_{2,2}(x)$ can be proved to be much larger. Indeed, De Koninck, Doyon, and Luca [3] focused their attention on the size of $E_{2,2}(x)$ and proved that

$$x^{1/4}/\log x \ll E_{2,2}(x) \ll x \exp\{-c\sqrt{2 \log x \log\log x}\},$$

where $c = 25/24 \approx 1.042$. Note that de la Bretèche and Drappeau [5] have recently showed that one can choose $c = 4/\sqrt{5} \approx 1.789$, whereas, as we will see in Section 9, one can expect that the true order of $E_{2,2}(x)$ is $x \exp\{-(1 + o(1))2\sqrt{2 \log x \log\log x}\}$ as $x \to \infty$.

At this point, we introduce additional notation. Given k integers $\ell_0, \ell_1, \ldots, \ell_{k-1}$, each ≥ 2, consider the set

$$F(\ell_0, \ell_1, \ldots, \ell_{k-1}) := \{n \in \mathbb{N} : P(n+i)^{\ell_i} \mid n+i \text{ for } i = 0, 1, \ldots, k-1\},$$

so that in particular $E_{k,\ell} = F(\underbrace{\ell, \ldots, \ell}_{k})$. Also, for each integer $\ell \geq 2$, we set $G_\ell := \{n \in \mathbb{N} : P(n)^\ell \mid n\}$ and $G_\ell(x) := \#\{n \leq x : n \in G_\ell\}$.

Most likely, each set $F(\ell_0, \ell_1, \ldots, \ell_{k-1})$ is infinite, but besides the set $F(2,2)$, no such statement has been proved.

Here, we first show that if we assume that there exist infinitely many primes of the form $9k^2 + 6k + 2$ (respectively $4k^2 + 2k + 1$), then the set $F(3,2)$ (respectively $F(4,2)$) is infinite. We then explore some identities involving consecutive polynomials whose algebraic structure provides the potential for revealing infinitely many members of $E_{k,\ell}$ for any given pair of integers $k \geq 2$, $\ell \geq 2$ and of $F(\ell_0, \ell_1, \ldots, \ell_{k-1})$ for any given k-tuple of integers $\ell_0 \geq 2, \ell_1 \geq 2, \ldots, \ell_{k-1} \geq 2$.

2 Preliminary results and conjectures

2.1 Friable numbers and the Dickman function

For $2 \leq y \leq x$, the function $\Psi(x,y) := \#\{n \leq x : P(n) \leq y\}$, which counts the number of "y-friable" or "y-smooth" numbers not exceeding x, has been studied extensively. In particular, it is known (see for instance Hildebrand and Tenenbaum [7]), that, given $\varepsilon > 0$ and setting $u = \log x / \log y$,

$$\Psi(x,y) = x\rho(u)\left(1 + O_\varepsilon\left(\frac{\log(u+1)}{\log y}\right)\right)$$

uniformly for $x \geq 3$, $\exp\{(\log\log x)^{\frac{5}{3}+\varepsilon}\} \leq y \leq x$, where $\rho(u)$ stands for the Dickman function defined for $0 \leq u \leq 1$ by $\rho(u) = 1$ and for $u > 1$ by the differential equation $u\rho'(u) = -\rho(u-1)$. It can also be shown (see for instance Corollary 9.18 in the book of De Koninck and Luca [4]) that

$$\rho(u) = \exp\{-u(\log u + \log\log u - 1 + o(1))\} \qquad (u \to \infty),$$

indicating that $\rho(u)$ decreases very rapidly as $u \to \infty$. In fact the following table provides the approximate values of $\rho(u)$ for $u = 1, 2, \ldots, 7$.

u	1	2	3	4	5	6	7
$\rho(u)$	1.0	0.3068	0.0486	0.00491	0.000354	0.0000196	0.00000087

Table 1

Related to the above is the difficult problem of estimating the number of friable (or smooth) values of polynomials. To do so, given a polynomial $f \in \mathbb{Z}[x]$ with positive leading coefficient, we set

$$\Psi(f; x, y) := \#\{n \leq x : P(f(n)) \leq y\}.$$

Now, given k irreducible polynomials $f_1, \ldots, f_k \in \mathbb{Z}[x]$, consider the polyonimal $f(t) := f_1(t) \cdots f_k(t)$. Then, as stated by Martin [10], if we assume that the multiplicative properties of the various $f_i(n)$ are independent of one another, we are led to the probabilistic prediction that

$$\Psi(f; x, y) \sim x \prod_{i=1}^{k} \rho\left(\frac{\log f_i(x)}{\log y}\right) \qquad (x \to \infty). \tag{1}$$

As we will later see in Section 9, this probabilistic relation will prove useful in estimating the expected size of the smallest elements of the various sets $E_{k,\ell}$.

2.2 Estimates for the size of $G_\ell(x)$

It was established by Ivić and Pomerance [9] that

$$G_2(x) = x \exp\{-(1 + o(1))\sqrt{2 \log x \log \log x}\} \qquad (x \to \infty). \tag{2}$$

Observe that a more explicit expression for the right hand side of (2) was later obtained by Ivić [8]. Now, the technique used in [9] can be used to establish a more general result, namely that, for any fixed integer $\ell \geq 2$,

$$G_\ell(x) = x \exp\{-(1 + o(1))\sqrt{2(\ell - 1) \log x \log \log x}\} \qquad (x \to \infty). \tag{3}$$

2.3 The Bunyakovsky conjecture

As we will see in the next section, there is an unexpected connection between the size of the sets $F(3, 2)$ and $F(4, 2)$ and a particular case of an old conjecture of Bunyakovsky [1], which essentially says that any irreducible polynomial with no fixed prime divisor contains infinitely many prime values.

Conjecture A (Bunyakovsky). *Let $f \in \mathbb{Z}[x]$ be an irreducible polynomial of positive degree and with positive leading coefficient such that the greatest common divisor of $f(1), f(2), f(3), \ldots$ is 1. Then there exist infinitely many values of n for which $f(n)$ is prime.*

In trying to prove that $F(4, 2)$ is infinite and as will be seen in Section 3, it would be helpful if we could say that there are infinitely many primes p such that $P(p^2 + 1) < p$. However, no such claim has been proved, so far. Interestingly, although one can easily show that the sequence $(n^2 + 1)_{n \geq 1}$ is such that $P(n^2 + 1) < n$ for infinitely many integers n (simply consider the subsequence $n = 2m^2$, $m = 1, 2, \ldots$, for which $n^2 + 1 = 4m^4 + 1 = (2m^2 + 2m + 1)(2m^2 - 2m + 1)$, and observe that $5 \mid 2m^2 + 2m + 1$ provided $m \equiv 1, 3 \pmod 5$, in which case $P(2m^2+2m+1)<n$, whereas $P(2m^2-2m+1)<n$ for all m), it seems to be much more difficult to prove that $P(p^2+1)<p$ for infinitely many primes p. However, if Conjecture A is true, this will indeed be the case.

3 On the infinitude of the sets $F(3,2)$ and $F(4,2)$

Using a computer one can check that the smallest four elements of $F(3,2)$ are 8, 6 859, 12 167 and 101 250, whereas the smallest four elements of $F(4,2)$ are 101 250, 11 859 210, 23 049 600 and 32 580 250. Further calculations seem to indicate that the two sets $F(3,2)$ and $F(4,2)$ are indeed infinite. However, no such claim has yet been proved. Nevertheless, we can prove the following.

Theorem 1. *Assuming that Conjecture A is true, then each of the sets $F(3,2)$ and $F(4,2)$ is infinite.*

Proof. First consider the identity

$$(2m^3+1)^2-1=4m^3(m+1)(m^2-m+1) \qquad (m=1,2,\ldots). \tag{4}$$

It is clear that if $m=p$, a prime, and if the largest prime factor of p^2-p+1 is less than p, then it follows from (4) that $n=4p^3(p^3+1)\in F(3,2)$. Now, assuming Conjecture A, there exist infinitely many positive integers k such that $9k^2+6k+2$ is prime. For each such k, write $p=9k^2+6k+2$, in which case $p-1=(3k+1)^2$. Since $p\equiv 2 \pmod 3$, we have that $p+(3k+1)\equiv 0 \pmod 3$. We may thus write

$$p^2-p+1=(p-\sqrt{p-1})(p+\sqrt{p-1})=3(p-(3k+1))\frac{p+(3k+1)}{3}.$$

Clearly $p-(3k+1)<p$. On the other hand,

$$\frac{p+(3k+1)}{3}=\frac{1}{3}(p+\sqrt{p-1})<p,$$

thereby implying that $P(p^2-p+1)<p$, thus completing the proof of the first assertion of the theorem.

Using a similar argument, one can show that $F(4,2)$ is infinite. Indeed, first observe that for every prime p,

$$(2p^4-1)^2-1=4p^4(p+1)(p-1)(p^2+1). \tag{5}$$

Assuming Conjecture A, there exist infinitely many integers k such that $4k^2+2k+1$ is prime. For each such k, write

$$p=4k^2+2k+1, \tag{6}$$

in which case we have $p^2+1=2(4k^2+1)(2k^2+2k+1)$. Now clearly, $4k^2+1<p$ and $2k^2+2k+1<p$, implying that $P(p^2+1)<p$ for each prime $p\geq 3$, thereby implying that

if we set $n = (2p^4 - 1)^2 - 1$ as p runs through the primes of the form (6), it follows that $P(n)^4 \mid n$ and $P(n+1)^2 \mid n$, and therefore that $n \in F(4,2)$ for infinitely integers n, thereby establishing the second assertion of the theorem.

□

4 Searching for elements of $E_{3,2}$

In order to generate elements of $E_{3,2}$, one can use a computer to find all the members of that set say with fifteen digits or less, thus obtaining the 60 numbers

1 294 298, 9 841 094, 158 385 500, 1 947 793 550, 5 833 093 013, 11 587 121 710, 20 944 167 840, 22 979 821 310, 24 604 784 814, 267 631 935 500, 290 672 026 412, 956 544 588 350, 987 988 937 343, 2 399 283 556 900, 2 816 075 601 855, 4 174 608 151 758, 4 322 550 249 043, 6 789 218 799 999, 10 617 595 679 778, 16 036 630 184 409, 22 869 997 335 620, 23 153 476 981 634, 23 480 833 955 320, 23 614 828 289 298, 24 126 198 551 098, 24 694 738 692 960, 31 456 704 045 166, 51 739 297 269 174, 52 898 121 606 525, 58 983 108 265 025, 71 709 481 909 254, 85 685 045 024 449, 113 707 706 201 375, 121 263 390 681 828, 122 169 948 877 430, 131 369 477 978 033, 133 959 037 005 774, 173 673 369 470 573, 176 664 623 046 273, 182 814 446 304 023, 209 744 971 905 458, 233 128 603 089 248, 237 464 160 321 408, 255 379 708 116 026, 280 778 107 745 620, 295 087 727 328 448, 313 232 585 684 886, 329 032 104 424 099, 360 853 931 895 982, 366 044 187 876 124, 467 683 999 401 022, 472 490 089 634 815, 480 138 936 005 168, 508 162 109 136 976, 593 047 972 159 008, 628 665 479 832 194, 638 506 456 514 625, 660 115 890 581 849, 906 165 826 118 135, 931 393 753 411 195.

The above list can be generated in a few hours using a powerful computer. However, in order to generate say thousands of members of $E_{3,2}$, one needs another approach. One efficient way is to identify polynomials $f(x)$, $g(x)$ and $h(x)$ with squared factors and such that $g(x) = f(x) + 1$ and $h(x) = f(x) + 2$, with the hope that adequate choices of x will reveal members of $E_{3,2}$. Our first choice for such consecutive polynomials is given by

$$\begin{aligned} f(x) &= x^2(2x^3 + 5x^2 - 5), \\ g(x) = f(x) + 1 &= (x^2 + x - 1)^2(2x + 1), \\ h(x) = f(x) + 2 &= (x + 1)^2(2x^3 + x^2 - 4x + 2). \end{aligned} \tag{7}$$

Setting $x = 3802$ reveals the three consecutive 19-digit integers

$$\begin{aligned} 1\,589\,922\,788\,612\,140\,124 &= 2^2 \cdot 59 \cdot 61 \cdot 71 \cdot 239 \cdot 1801 \cdot 1901^2, \\ 1\,589\,922\,788\,612\,140\,125 &= 3^2 \cdot 5^3 \cdot 11^2 \cdot 13^2 \cdot 151^2 \cdot 1741^2, \\ 1\,589\,922\,788\,612\,140\,126 &= 2 \cdot 103 \cdot 701 \cdot 809 \cdot 941 \cdot 3803^2. \end{aligned}$$

As x runs through positive integers up to 2 000 000, we thus find a total of 33 members of $E_{3,2}$; much work for little outcome. Other approaches can be more fruitful. Indeed, first consider the system

$$\begin{aligned} f(x) &= (2x^2+1)^2(x-1)(x+1), \\ g(x)=f(x)+1 &= x^2(4x^4-3), \\ h(x)=f(x)+2 &= (2x^2-1)^2(x^2+1). \end{aligned} \tag{8}$$

With $x = 5087$, we find the somewhat larger 23-digit numbers

$$\begin{aligned} 69\,315\,509\,064\,481\,032\,011\,329 &= 2^6 \cdot 3^7 \cdot 53 \cdot 53 \cdot 2543 \cdot 1916857^2, \\ 69\,315\,509\,064\,481\,032\,011\,330 &= 11 \cdot 71 \cdot 769 \cdot 1163 \cdot 1321 \cdot 2903 \cdot 5087^2, \\ 69\,315\,509\,064\,481\,032\,011\,331 &= 2 \cdot 5 \cdot 7^2 \cdot 17 \cdot 29^2 \cdot 167^2 \cdot 181 \cdot 44273^2. \end{aligned}$$

However, letting x run up to $2\,000\,000$ reveals 252 elements of $E_{3,2}$, a score better than the one obtained through system (7). Many more elements of $E_{3,2}$ can be obtained with a small modification to system (8). Indeed, in (8), replacing x^2 by x, we find the new system

$$\begin{aligned} g(x)-1 &= (2x+1)^2(x-1), \\ g(x) &= x(4x^2-3), \\ g(x)+1 &= (2x-1)^2(x+1). \end{aligned} \tag{9}$$

Although at first sight, system (9) appears to be useless in the quest to find elements of $E_{3,2}$, observe that since $mP(m) \in G_2$ for each integer $m \geq 2$, if we substitute $x = mP(m)$ in system (9) and let m run up to $2\,000\,000$, we find 261 elements of $E_{3,2}$ and running m up to $30\,000\,000$, we find $4\,473$ elements of $E_{3,2}$. One reason for the numerous elements of $E_{3,2}$ found using system (9) is that the polynomials involved are of degree 3 while those in systems (7) and (8) are of degree 5 and 6 respectively.

Remark 2. It has recently come to the attention of the authors that in 1986 Hildebrand [6] also came up with system (9) but for a different purpose and in a different context.

5 Searching for elements of $F(2, \ell, 2)$

Consider the consecutive polynomials

$$\begin{aligned} g(x)-1 &= 2(x+1)^2(24x^3+12x^2+2x-1), \\ g(x) &= (2x+1)^3(6x^2+6x-1), \\ g(x)+1 &= 2x^2(24x^3+60x^2+50x+15). \end{aligned} \tag{10}$$

This set up is of particular interest because it allows for a rapid search for elements in $F(2,3,2)$. For instance, using a computer and letting x run through the primes p, we find that the prime $p = 6\,158\,923$ yields the 36-digit number n_1 which is such that

$$\begin{aligned} n_1 - 1 &= 2^5 \cdot 139 \cdot 331 \cdot 1627 \cdot 3457 \cdot 73019 \cdot 296729 \cdot 1539731^2, \\ n_1 &= 3^3 \cdot 11 \cdot 137 \cdot 40433 \cdot 3735181 \cdot 4105949^3, \\ n_1 + 1 &= 2 \cdot 17 \cdot 23 \cdot 463 \cdot 107251 \cdot 433259 \cdot 666529 \cdot 6158923^2. \end{aligned}$$

Many other elements of $F(2,3,2)$ can also be obtained from the polynomials in (10).

A smaller element of $F(2,3,2)$ can be obtained by considering another configuration of the same kind but with greater symmetry, namely

$$\begin{aligned} g(x)-1 &= (2x+1)^2(12x^3-12x^2+4x-1), \\ g(x) &= 4x^3(12x^2-5), \\ g(x)+1 &= (2x-1)^2(12x^3+12x^2+4x+1). \end{aligned} \tag{11}$$

This scenario allows us to find the 34-digit number $n_2 \in F(2,3,2)$ which is such that

$$\begin{aligned} n_2-1 &= 3^3 \cdot 17 \cdot 113 \cdot 239 \cdot 12829 \cdot 13679 \cdot 321109 \cdot 1242121^2, \\ n_2 &= 2^2 \cdot 24121 \cdot 32203 \cdot 53629 \cdot 1863181^3, \\ n_2+1 &= 7211 \cdot 17977 \cdot 689237 \cdot 868691 \cdot 3726361^2. \end{aligned}$$

Remark 3. The fact that the numbers n_1 and n_2 are large is in part due to the fact that they each are solutions of a system of polynomials with a very specific algebraic structure. Therefore, one will not be surprised to learn that smaller elements of $F(2,3,2)$ do exist. For instance, a judicious computer search reveals that the much smaller 16 and 20 digit numbers 3 858 290 162 662 516 and 67 500 618 671 796 179 920 are both elements of $F(2,3,2)$.

The three polynomials given in (11) can also be put to good use to find an element of $F(2,6,2)$. Indeed by considering positive integers r of the form $r = mP(m)$, that is members of G_2, we automatically get that $r^3 \in G_6$, implying that if $2r-1 =: p$ and $(2r+1)/3 =: q$ are both primes and if the three conditions $P(12r^3-12r^2+4r-1) < q$, $P(12r^3+12r^2+4r+1) < p$ and $P(12r^2-5) < P(m)$ are simultaneously satisfied, then we are guaranteed that the number $4r^3(12r^2-5)-1$ belongs to $F(2,6,2)$. In fact by choosing $m = 79\,311\,205$, that is $r = 5 \cdot 17 \cdot 933\,073^2$ and thus substituting the value $x = 74\,003\,143\,982\,965$ in (11), we find the 72-digit integer n which satisfies

$$\begin{aligned} n &= 3^4 \cdot 3171439637 \cdot 5982121049 \cdot 91023709109 \cdot 312912616603 \cdot 49335429321977^2, \\ n+1 &= 2^2 \cdot 5^4 \cdot 7^2 \cdot 17^3 \cdot 43 \cdot 223 \cdot 599 \cdot 3607 \cdot 66977 \cdot 227089 \cdot 851231 \cdot 933073^6, \\ n+2 &= 121327 \cdot 13747920817 \cdot 49470595193983 \cdot 58937287195613 \cdot 148006287965929^2. \end{aligned}$$

Of course, this also reveals a number in $F(2,5,2)$ and in $F(2,4,2)$. However, for the same reason as the one mentioned in Remark 3, smaller elements of $F(2,5,2)$ and $F(2,4,2)$ most certainly exist.

6 Elements of $E_{3,3}$

In the hope of finding elements of $E_{3,3}$ using consecutive polynomials, one might search for three consecutive polynomials $f(x)-1, f(x), f(x)+1$ which are respectively divisible by, say, $(2x-1)^3, x^3, (2x+1)^3$. If this is possible, then, since the original polynomials only differ by a constant, they must share the same derivative $f'(x)$. But then, clearly, $(2x-1)^2, x^2, (2x+1)^2$ are three factors of $f'(x)$. This means that assuming that $f(x)$ is of degree 7, then we must have that for some constant $a \in \mathbb{N}$,

$$f(x) = \int a \cdot (2x-1)^2 \cdot x^2 \cdot (2x+1)^2 \, dx = \int ax^2(4x^2-1)^2 \, dx.$$

It turns out that, by choosing $a = 105$, we find the three consecutive polynomials

$$\begin{aligned} f(x) - 1 &= (2x-1)^3(30x^4 + 45x^3 + 24x^2 + 6x + 1), \\ f(x) &= x^3(240x^4 - 168x^2 + 35), \\ f(x) + 1 &= (2x+1)^3(30x^4 - 45x^3 + 24x^2 - 6x + 1), \end{aligned} \tag{12}$$

which certainly have the potential of revealing several elements of $E_{3,3}$. In fact, setting $x = 39\,682\,272\,446$ in (12), we find the 77-digit number n_1 which is such that

$$\begin{aligned} n_1 - 1 &= 3^3 \cdot 137 \cdot 251 \cdot 49253 \cdot 6892241 \cdot 1400173417 \cdot 1749071927 \cdot 2602138829 \cdot 26454848297^3, \\ n_1 &= 2^3 \cdot 1162 \cdot 31 \cdot 3301 \cdot 92639 \cdot 376627 \cdot 474994139 \cdot 573384841 \cdot 5057839271 \cdot 19841136223^3, \\ n_1 + 1 &= 13 \cdot 23 \cdot 41 \cdot 61^3 \cdot 233 \cdot 3767 \cdot 9551 \cdot 977719 \cdot 5076637 \cdot 367839041 \cdot 396464197 \cdot 1301058113^3, \end{aligned}$$

so that $n_1 - 1 \in E_{3,3}$. Observe that although n_1 is larger than the number n_0 appearing in Section 1, it has the same number of digits. Similarly, setting $x = 39\,874\,762\,919$ in (12), we find another 77-digit number belonging to $E_{3,3}$. Many more can be obtained with larger values of x.

7 The particular case $E_{2,\ell}$

Given an integer $\ell \geq 2$, are there two consecutive polynomials that can help one find positive integers n such that $P(n)^\ell \mid n$ and $P(n+1)^\ell \mid n+1$? The answer is "yes" and an explicit answer is given by the following result.

Theorem 4. *Let $\ell \geq 2$ be a fixed integer. Then, there exist $g_1(x), g_2(x) \in \mathbb{Z}[x]$ each of degree $\ell - 1$ such that*

$$x^\ell \cdot g_1(x) + (-1)^\ell = (x-1)^\ell \cdot g_2(x). \tag{13}$$

Proof. First observe that, using repeated integration by parts one easily obtains that

$$\int_0^1 t^r (t-1)^r \, dt = (-1)^r \frac{r! \cdot r!}{(2r+1)!} = (-1)^r C \qquad (r = 1, 2, \ldots) \tag{14}$$

where $C = \dfrac{r! \cdot r!}{(2r+1)!}$.

In order to prove (13), we first consider the case ℓ even and set $r = \ell - 1$. Computing the primitive of the function $x^r(x-1)^r$, that is

$$g(x) := \int_0^x \frac{1}{C}\, t^r (t-1)^r \, dt, \tag{15}$$

we obtain, since r is odd,

$$\begin{aligned}
g(x) &= \int_0^x \frac{1}{C}\, t^r \left(t^r - \binom{r}{1} t^{r-1} + \binom{r}{2} t^{r-2} - \cdots + \binom{r}{r-1} t - 1 \right) dt \\
&= \int_0^x \frac{1}{C} \left(t^{2r} - \binom{r}{1} t^{2r-1} + \binom{r}{2} t^{2r-2} - \cdots + \binom{r}{r-1} t^{r+1} - t^r \right) dt \\
&= \frac{1}{C} \left(\frac{x^{2r+1}}{2r+1} - \binom{r}{1} \frac{x^{2r}}{2r} + \binom{r}{2} \frac{x^{2r-1}}{2r-1} - \cdots + \binom{r}{r-1} \frac{x^{r+2}}{r+2} - \frac{x^{r+1}}{r+1} \right) \\
&= x^{r+1} \left\{ \frac{1}{C} \left(\frac{x^r}{2r+1} - \binom{r}{1} \frac{x^{r-1}}{2r} + \binom{r}{2} \frac{x^{r-2}}{2r-1} - \cdots + \binom{r}{r-1} \frac{x}{r+2} - \frac{1}{r+1} \right) \right\} \\
&= x^{r+1} g_1(x),
\end{aligned} \tag{16}$$

which corresponds to the first term on the left hand side of (13). Setting

$$h(x) := g(x) + 1, \tag{17}$$

it follows from (16) and (14) that

$$h(1) = g(1) + 1 = \int_0^1 \frac{1}{C}\, t^r (t-1)^r \, dt + 1 = \frac{1}{C}\, (-1)^r\, C + 1 = -1 + 1 = 0,$$

thereby implying that $x - 1$ is a factor of $h(x)$. In order to complete the proof of (13), it is sufficient to prove that $(x-1)^{\ell-1} = (x-1)^r$ is a factor of the derivative of $h(x)$. But recalling the definitions of g and h given by (15) and (17), it follows that

$$h'(x) = g'(x) = x^r (x-1)^r,$$

thereby implying that $(x-1)^r$ is indeed a factor of $h'(x)$ as requested.

The case ℓ odd can be treated in a similar manner.

□

Since, as mentioned in Section 1, no elements of $E_{2,6}$ have been previously discovered, let us illustrate the above method in the case $\ell = 6$. Applying Theorem 4, we find the two consecutive polynomials

$$\begin{aligned}
g(x) &= x^6(252x^5 - 1386x^4 + 3080x^3 - 3465x^2 + 1980x - 462), \\
g(x) + 1 &= (x-1)^6(252x^5 + 126x^4 + 56x^3 + 21x^2 + 6x + 1).
\end{aligned}$$

Choosing $x = 20\,905\,825\,364$, we obtain the 116-digit integer n which satisfies

$$\begin{aligned} n &= 2^{13} \cdot 5 \cdot 83 \cdot 22157 \cdot 127139 \cdot 12177577 \cdot 17565259 \cdot 372289003 \\ &\qquad 1659308773 \cdot 3257215037 \cdot 5226456341^6, \\ n+1 &= 569 \cdot 32939 \cdot 122489 \cdot 146359 \cdot 50300881 \cdot 919974911 \\ &\qquad 4166729363 \cdot 15532846993 \cdot 20905825363^6. \end{aligned}$$

Choosing $x = 86\,459\,129\,774$, we find another element of $E_{2,6}$, this time with 123 digits.

8 Consecutive polynomials generating elements of the sets $E_{k,\ell}$

In the previous sections, we were successful in generating two or three consecutive integer coefficient polynomials each divisible by a power of some linear polynomial. We will now show that this can be done for an arbitrarily long sequence of consecutive polynomials each divisible by the ℓ-th power of some linear polynomial.

Theorem 5. *Given fixed integers $k \geq 2$ and $\ell \geq 2$, there exist k consecutive polynomials $L_i(x) \in \mathbb{Z}[x]$, $i = 0, 1, \ldots, k-1$, that is $L_{i+1}(x) - L_i(x) = 1$ for $i = 0, 1, \ldots, k-2$, such that each $L_i(x)$ is divisible by the ℓ-th power of some linear polynomial.*

Proof. We only give the proof in the case $\ell = 2$ and arbitrary $k \geq 2$, the general case being similar. Our proof is constructive. First, choose $k-1$ linear polynomials $c_i x + d_i$, where $c_i, d_i \in \mathbb{Q}$ and $c_i > 0$ for $i = 1, \ldots, k-1$, in such a way that the k linear polynomials $x, c_1 x + d_1, \ldots, c_{k-1}x + d_{k-1}$ are pairwise linearly independent over $\mathbb{Q}$. We shall first construct k consecutive polynomials $Q_i(x)$ with rational coefficients, namely

$$Q_0(x) = x^2 \sum_{r=0}^{2k-1} a_r x^r, \qquad Q_i(x) = (c_i x + d_i)^2 \sum_{r=0}^{2k-1} b_{i,r} x^r \qquad (i = 1, \ldots, k-1), \tag{18}$$

where each $Q_i(x) \in \mathbb{Q}[x]$ and $Q_i(x) = Q_0(x) + i$ for $i = 1, \ldots, k-1$.

In order for the equation $Q_1(x) = Q_0(x) + 1$ to hold for all x, that is for the equation

$$(c_1 x + d_1)^2 (b_{1,0} + b_{1,1} x + \cdots + b_{1,2k-1} x^{2k-1}) = 1 + a_0 x^2 + a_1 x^3 + \cdots + a_{2k-1} x^{2k+1} \tag{19}$$

to hold, we need to equate the respective coefficients of x^r, $r = 0, 1, \ldots, 2k+1$, on both sides of the above identity. Equating the coefficients of x^0 and of x^1, we find

$$d_1^2 b_{1,0} = 1 \qquad \text{and} \qquad d_1^2 b_{1,1} + 2c_1 d_1 b_{1,0} = 0,$$

which allows us to express the values of $b_{1,0}$ and $b_{1,1}$ in terms of c_1 and d_1. Equating the coefficients of $x^2, x^3, \ldots, x^{2k+1}$ on both sides of (19), we find that

$$\begin{aligned}
a_0 &= d_1^2 b_{1,2} + 2c_1 d_1 b_{1,1} + c_1^2 b_{1,0}, \\
a_1 &= d_1^2 b_{1,3} + 2c_1 d_1 b_{1,2} + c_1^2 b_{1,1}, \\
&\vdots \\
a_{2k-3} &= d_1^2 b_{1,2k-1} + 2c_1 d_1 b_{1,2k-2} + c_1^2 b_{1,2k-3}, \\
a_{2k-2} &= 2c_1 d_1 b_{1,2k-1} + c_1^2 b_{1,2k-2}, \\
a_{2k-1} &= c_1^2 b_{1,2k-1}.
\end{aligned}$$

We then move to equation $Q_2(x) = Q_0(x) + 2$ and again equate coefficients. Equating the coefficients of x^0 and of x^1, we find

$$d_2^2 b_{2,0} = 2 \qquad \text{and} \qquad d_2^2 b_{2,1} + 2c_2 d_2 b_{2,0} = 0,$$

which allows us to express $b_{2,0}$ and $b_{2,1}$ in terms of c_2 and d_2. Equating the coefficients of $x^2, x^3, \ldots, x^{2k+1}$, we find that

$$\begin{aligned}
a_0 &= d_2^2 b_{2,2} + 2c_2 d_2 b_{2,1} + c_2^2 b_{2,0}, \\
a_1 &= d_2^2 b_{2,3} + 2c_2 d_2 b_{2,2} + c_2^2 b_{2,1}, \\
&\vdots \\
a_{2k-3} &= d_2^2 b_{1,2k-1} + 2c_2 d_2 b_{2,2k-2} + c_2^2 b_{2,2k-3}, \\
a_{2k-2} &= 2c_2 d_2 b_{2,2k-1} + c_2^2 b_{2,2k-2}, \\
a_{2k-1} &= c_2^2 b_{2,2k-1}.
\end{aligned}$$

Similarly, we construct $k-3$ other systems each with $2k$ equations. Then, from these $k-1$ systems, we see that

$$a_0 = d_1^2 b_{1,2} + 2c_1 d_1 b_{1,1} + c_1^2 b_{1,0} = d_2^2 b_{2,2} + 2c_2 d_2 b_{2,1} + c_2^2 b_{2,0} = \cdots = d_2^2 b_{k-1,2} + 2c_2 d_2 b_{k-1,1} + c_2^2 b_{k-1,0}$$

and obtain analogous identities for $a_1, a_2, \ldots, a_{2k-1}$. Hence, recalling that $c_1, \ldots, c_{k-1}$ and $d_1, \ldots, d_{k-1}$ are given, in all we obtain $2k(k-2)$ equations involving a total of $2(k-1)^2$ unknowns, namely $b_{i,j}$ where $1 \le i \le k-1$ and $2 \le j \le 2k-1$. To summarize, we have constructed a system of $2k^2 - 4k$ linear equations involving $2k^2 - 4k + 2$ unknowns. This means that if we fix any two of these unknowns, we will obtain a unique solution for the set of $b_{i,j}$'s.

Finally, having obtained the values of $b_{i,j}$ for $1 \le i \le k-1$ and $2 \le j \le 2k-1$, we can use any of the $k-1$ systems of equations to determine the unique values of $a_0, a_1, \ldots, a_{2k-1}$.

Moreover, since all of the above equations are linear and involve rational coefficients, the coefficients a_r are also rational. Let us then write each a_r as

$$a_r = \frac{p_r}{q_r} \text{ where } p_r, q_r \in \mathbb{Z},\ (p_r, q_r) = 1,\ q_r > 0 \quad \text{ and set } D := \operatorname{lcm}[q_1, \ldots, q_r]$$

and consider the polynomials

$$L_i(x) := Q_i(D\,x) \qquad (i = 0, \dots, k-1).$$

We then have

$$L_0(x) = Q_0(Dx) = D^2\,x^2 \sum_{r=0}^{2k-1} a_r D^r \cdot x^r = D \cdot x^2 \sum_{r=0}^{2k-1} D \cdot a_r \cdot (Dx)^r.$$

By the nature of D, the coefficients of $L_0(x)/x^2$ are therefore all integers. This implies that each of the polynomials $L_0(x), \dots, L_{k-1}(x)$ has integer coefficients. Moreover, these polynomials are clearly consecutive. We have thus created an infinite family of consecutive polynomials each with a squared factor, as required. $\square$

Example. In the case $(k, \ell) = (4, 2)$, the above construction yields the four consecutive polynomials

$$\begin{aligned} L_0(x) \;&=\; x^2\Big(17220 + a - 5b + (815440 + 52a - 248b)x + (14339520 + 1108a - 4916b)x^2 \\ &\quad +(120865536 + 12384a - 48624b)x^3 + (494346240 + 76608a - 234432b)x^4 \\ &\quad +(789626880 + 248832a - 324864b)x^5 + (331776a + 1327104b)x^6 + 3981312bx^7\Big) \end{aligned}$$

and $L_1(x)$, $L_2(x)$ and $L_3(x)$ whose squared factors are $(6x+1)^2$, $(8x+1)^2$ and $(12x+1)^2$, respectively. We have thus constructed infinitely many such quadruples with parameters a and b. Choosing $a = b = 0$ allows for the more simple quadruple

$$\begin{aligned} L_0(x) \;&=\; x^2\Big(17220 + 815440x + 14339520x^2 + 120865536x^3 + 494346240x^4 + 789626880x^5\Big) \\ L_1(x) \;&=\; L_0(x) + 1 = (1+6x)^2\Big(1 - 12x + 17328x^2 + 607936x^3 + 6420480x^4 + 21934080x^5\Big) \\ L_2(x) \;&=\; L_0(x) + 2 = 2(1+8x)^2\Big(1 - 16x + 8802x^2 + 267912x^3 + 2319840x^4 + 6168960x^5\Big) \\ L_3(x) \;&=\; L_0(x) + 3 = (1+12x)^2\Big(3 - 72x + 18516x^2 + 381424x^3 + 2519040x^4 + 5483520x^5\Big) \end{aligned}$$

Using the method of proof of Theorem 5, the following result also holds.

Theorem 6. *Given arbitrary integers $\ell_i \geq 2$, $i = 0, 1, \dots, k-1$, there exist k consecutive polynomials $L_0(x), L_1(x), \dots, L_{k-1}(x) \in \mathbb{Z}[x]$ such that each $L_i(x)$, $i = 0, 1, \dots, k-1$, is divisible by the ℓ_i-th power of some linear polynomial.*

Proof. We only provide a sketch of the proof. The idea is to let $s := \ell_0 + \cdots + \ell_{k-1}$, to set

$$Q_0(x) := x^{\ell_0} \sum_{r=0}^{s-\ell_0-1} a_r x^r \quad \text{and} \quad Q_i(x) := (c_i x + d_i)^{\ell_i} \sum_{r=0}^{s-\ell_i-1} b_{i,r} x^r \text{ for } i = 1, \dots, k-1$$

and then to search for the coefficients $a_r \in \mathbb{Q}$ and $b_{i,r} \in \mathbb{Q}$ in the same manner as we did in the proof of Theorem 5. This allows us to obtain k consecutive polynomials $Q_0(x)$, $Q_1(x)$, $\ldots$, $Q_{k-1}(x) \in \mathbb{Q}[x]$ with the property that $Q_0(x)$ is divisible by x^{ℓ_0} whereas each $Q_i(x)$, for $i = 1, \ldots, k-1$, is divisible by the ℓ_i-th power of a linear polynomial $c_i x + d_i$ with $c_i, d_i \in \mathbb{Q}$, $c_i > 0$. Using these k polynomials with rational coefficients, we proceed as in the proof of Theorem 5 and obtain k consecutive polynomials $L_0(x), L_1(x), \ldots, L_{k-1}(x) \in \mathbb{Z}[x]$ with the same properties. $\square$

Remark 7. It follows from Theorems 5 and 6 that if Martin's probabilistic prediction (1) is true, then each one of the sets $E_{k,\ell}$ and $F(\ell_0, \ell_1, \ldots, \ell_{k-1})$ is infinite.

9 Final remarks and heuristics

Let us now examine the expected size of the smallest elements of $E_{k,\ell}$.

A consequence of estimate (3) is that the probability that a given large integer n is such that $P(n)^\ell$ divides n is approximately $1/e^{\sqrt{2(\ell-1)\log n \log\log n}}$.

On the other hand, given an arbitrary integer $k \geq 2$, it is reasonable to assume that $P(n), P(n+1), \ldots, P(n+k-1)$ are independent events and therefore to conclude that the probability that $P(n+i)^\ell \mid n+i$ for $i = 0, 1, \ldots, k-1$ is around $1/e^{k\sqrt{2(\ell-1)\log n \log\log n}}$.

Using this approach, one can expect the smallest element of $E_{3,3}$ to have around 82 digits (that is, roughly the size of the numbers n_0 and n_1 obtained in Sections 1 and 6, respectively) and that the smallest element of $E_{4,2}$ to have around 71 digits.

On the other hand, in line with our algebraic approach, the following system (similar to the one displayed in Section 8, but with smaller coefficients for each of the four degree 5 polynomials) clearly has the potential of generating infinitely many elements of $E_{4,2}$:

$$\begin{aligned}
f(x) &= 4x^2(184896x^5 + 292320x^4 + 172500x^3 + 46500x^2 + 5501x + 195), \\
f(x)+1 &= (2x+1)^2(184896x^5 + 107424x^4 + 18852x^3 + 792x^2 - 4x + 1), \\
f(x)+2 &= 2(4x+1)^2(23112x^5 + 24984x^4 + 7626x^3 + 438x^2 - 8x + 1), \\
f(x)+3 &= (6x+1)^2(20544x^5 + 25632x^4 + 10052x^3 + 1104x^2 - 36x + 3).
\end{aligned}$$

Unfortunately, in order to find a number $n_1 \in E_{4,2}$ using the above four polynomials, one would need much computer time since, in light of the conjectured estimate (1) and of Table 1, one can expect, as x runs through the positive integers, that the probability that each of the above degree 5 co-factors has its largest prime factor smaller than the largest prime factor of their respective squared factors is smaller than $\rho(5) \approx 0.000354$, implying that the smallest integer x meeting these four requirements would be larger than $1/\rho(5)^4 > 10^{14}$ and therefore that

$$n_1 = f(x) > 4 \cdot x^2 \cdot 184896 \cdot x^5 > 739584 \cdot (10^{14})^7 > 10^{103}.$$

Of course, one could perhaps come up with a smaller element of $E_{4,2}$ using a totally new approach.

Finally, if Martin's probabilistic estimate (1) could be proved, not only would each set $E_{k,\ell}$ be infinite (as already mentioned in Remark 7), but one could hope to find the approximate size of $E_{k,\ell}(x)$.

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