Journal of Integer Sequences, Vol. 22 (2019),
Article 19.3.6

The Riemann Zeta Function and the Fractional Part of Rational Powers

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Abstract

We use elementary methods to find an explicit formula for the second term of the continued fraction of the Riemann zeta function over positive integers, with only finitely many exceptions. As a corollary to this, we find connections between the Riemann zeta function and the fractional parts of rational powers.

1 Introduction

Given an irrational real number x we recall that the *simple continued fraction* of it is

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where all a_i are integers and $a_i \geq 1$ for $i \geq 1$. Thus, the first term $a_0 = \lfloor x \rfloor$ and the second term is $a_1 = \left\lfloor \frac{1}{x - a_0} \right\rfloor$.

The Riemann zeta function is defined for real $s > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For all positive integers $n \geq 2$ we have that $1 < \zeta(n) < 2$. Therefore, the first term of $\zeta(n)$ in its simple continued fraction is always 1.

Let $a(n)$ denote the second term in the continued fraction of $\zeta(n)$. According to Adams-Watters in the description of sequence A013697 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [3], "It appears that $a(n) = 2^n - \lfloor (\frac{4}{3})^n \rfloor - k$, where k is usually 2, but is sometimes 1. Up to $n = 1000$, the only values of n where $k = 1$ are 4, 5, 13, 14, and 17. That is,

$$\left\lfloor \frac{1}{\zeta(n) - 1} \right\rfloor = 2^n - \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor - k,$$

where $k = 1$ or $k = 2$". We first prove that this formula holds for all $n \geq 2$.

Theorem 1. *The equality*

$$\left\lfloor \frac{1}{\zeta(n) - 1} \right\rfloor = 2^n - \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor - 2,$$

holds for all natural numbers, $n \geq 2$, with finitely many exceptional cases. In the exceptional cases we instead have

$$\left\lfloor \frac{1}{\zeta(n) - 1} \right\rfloor = 2^n - \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor - 1.$$

We write $\{x\} = x - \lfloor x \rfloor$, for the fractional part of x. Given coprime integers $p > q > 1$, we consider the set $\left\{ \left\{ \left(\frac{p}{q}\right)^n \right\} \mid n \in \mathbb{N} \right\}$. Vijayaraghavan [4] proved that this set has infinitely many limit points, but otherwise not much is known about its distribution. For real s write $\varepsilon_x(s) = (2x)^s \left(\left(\frac{2}{3}\right)^s + \left(\frac{1}{2}\right)^s\right)^2$ and for simplicity we write $\varepsilon(s) = \varepsilon_1(s)$. As a corollary of Theorem 1 we obtain the following result.

Corollary 2. *For all natural numbers, $n \geq 2$, with finitely many exceptional cases, we have that*

$$1 < \left\{ \frac{1}{\zeta(n) - 1} \right\} + \left\{ \left(\frac{4}{3}\right)^n \right\} < 1 + \varepsilon(n).$$

In the exceptional cases we instead have

$$0 < \left\{ \frac{1}{\zeta(n) - 1} \right\} + \left\{ \left(\frac{4}{3}\right)^n \right\} < \varepsilon(n).$$

Corollary 2 implies the set $\left\{ \left\{ \frac{1}{\zeta(n)-1} \right\} \mid n \in \mathbb{N} \right\}$ has infinitely many limit points.

Corollary 3. *For a natural number, $n \geq 2$, $\zeta(n)$ can only be of the form $1 + \frac{1}{m}$, where m is a natural number, finitely many times.*

Conjecture 4. The finite exceptions of Theorem 1 only occur when $n = 4, 5, 13, 14, 17$.

Using Mathematica we checked when $\left\{ \left(\frac{4}{3}\right)^n \right\}$ is less than 10^{-9} for $n \leq 5,000,000$, and no examples were found. As $\varepsilon(n) < 10^{-9}$ for $n \geq 176$, Corollary 2 implies that the conjecture holds for $176 \leq n \leq 5,000,000$. Since Adams-Watters verified the conjecture for $n \leq 1000$, we conclude the conjecture holds for $n \leq 5,000,000$.

Using similar methods to Theorem 1 and Corollary 2, results can also be obtained for other functions of the form

$$\sum_{i=1}^{\infty} \frac{(-1)^{t_i}}{{u_i}^s},$$

where t_i and u_i are some sequence. The Riemann zeta function is such that $t_i = 0$ and $u_i = i$. For example for the Dirichlet beta function, $\beta(s)$, where $t_i = i - 1$ and $u_i = 2i - 1$, we can show

$$\left\{\left\{\frac{1}{1-\beta(n)}\right\} - \left\{\left(\frac{9}{5}\right)^n\right\} + \left\{\left(\frac{9}{7}\right)^n\right\} - \left\{\left(\frac{27}{25}\right)^n\right\}\right\} = 1 - \left(\frac{9}{11}\right)^n + \mathcal{O}\left(\left(\frac{27}{35}\right)^n\right).$$

For $P(s)$ where $t_i = 0$ and u_i is the i^{th} prime, we can show that

$$1 - \left(\frac{8}{9}\right)^s - 2\left(\frac{2}{3}\right)^s - \frac{1}{2^s} < \frac{1}{P(s)} - \frac{1}{\zeta(s)-1} < 1 + \left(\frac{8}{9}\right)^s + 2\left(\frac{8}{15}\right)^s + \left(\frac{8}{25}\right)^s.$$

The most interesting of these results using different functions is given in Theorem 5. This is using the function for when $t_i = 0$ and $u_i = \frac{2i}{3}$.

Theorem 5. *For n large enough we have*

$$1 - \left(\frac{8}{9}\right)^n - \left(\frac{2}{3}\right)^n < \left\{\frac{1}{\zeta(n)-1}\right\} + \left\{\frac{\left(\frac{2}{3}\right)^n}{\zeta(n)-1}\right\} <$$

$$1 - \left(\frac{8}{9}\right)^n - \left(\frac{2}{3}\right)^n + \varepsilon(n) + \varepsilon_{\frac{2}{3}}(n).$$

2 Analytic Results

Proposition 6. *For all $n \geq 2$ we have that*

$$1 < \frac{1}{\zeta(n)-1} - 2^n + \left(\frac{4}{3}\right)^n + 2.$$

Proof. For $n = 2$ and $n = 3$, it can be checked that the inequality is true. For $n \geq 4$ we prove the proposition by contradiction. Assume

$$1 \geq \frac{1}{\zeta(n)-1} - 2^n + \left(\frac{4}{3}\right)^n + 2.$$

We will first cancel out some terms from our inequality. Our assumption implies that

$$2^n - \left(\frac{4}{3}\right)^n - 1 \geq \frac{1}{\zeta(n)-1},$$

$$\frac{6^n-4^n-3^n}{3^n} \geq \frac{1}{\zeta(n)-1},$$

$$\frac{3^n}{6^n-4^n-3^n} \leq \zeta(n)-1,$$

$$3^n \leq (6^n-4^n-3^n)\sum_{i=2}^{\infty}\frac{1}{i^n},$$

$$3^n+\sum_{i=2}^{\infty}\frac{3^n}{i^n}+\sum_{i=2}^{\infty}\frac{4^n}{i^n} \leq \sum_{i=2}^{\infty}\frac{6^n}{i^n}. \tag{1}$$

On the other hand, the following three inequalities are obvious:

$$\sum_{i=2}^{\infty}\frac{3^n}{i^n} > \left(\frac{3}{2}\right)^n+1, \tag{2}$$

$$\sum_{i=2}^{\infty}\frac{4^n}{i^n} > 2^n+\left(\frac{4}{3}\right)^n+1, \tag{3}$$

$$\sum_{i=2}^{\infty}\frac{6^n}{i^n} < 3^n+2^n+\left(\frac{3}{2}\right)^n+\left(\frac{6}{5}\right)^n+1+\int_6^{\infty}\left(\frac{6}{t}\right)^n dt = 3^n+2^n+\left(\frac{3}{2}\right)^n+\left(\frac{6}{5}\right)^n+1+\frac{6}{n-1}. \tag{4}$$

Thus, substituting (2), (3), and (4) into (1), we have

$$3^n+\left(\frac{3}{2}\right)^n+1+2^n+\left(\frac{4}{3}\right)^n+1 < 3^n+2^n+\left(\frac{3}{2}\right)^n+\left(\frac{6}{5}\right)^n+1+\frac{6}{n-1},$$

$$\left(\frac{4}{3}\right)^n+1 < \left(\frac{6}{5}\right)^n+\frac{6}{n-1}.$$

It is clear that the above statement is false for $n \geq 7$. Specifically it can be checked that it is false for $n \geq 4$. This is a contradiction and thus,

$$1 < \frac{1}{\zeta(n)-1}-2^n+\left(\frac{4}{3}\right)^n+2,$$

for all $n \geq 2$. □

Proposition 7. *For $n \geq 2$ we have that*

$$\frac{1}{\zeta(n)-1}-2^n+\left(\frac{4}{3}\right)^n+2 < 1+\varepsilon(n).$$

Proof. Consider

$$\frac{1}{\zeta(n)-1} = \frac{1}{\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\cdots} < \frac{1}{\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}}.$$

Thus,

$$\frac{1}{\zeta(n)-1} < \frac{2^n}{1+\frac{2^n}{3^n}+\frac{1}{2^n}} =$$

$$2^n\left(1-\left(\frac{2^n}{3^n}+\frac{1}{2^n}\right)+\left(\frac{2^n}{3^n}+\frac{1}{2^n}\right)^2-\left(\frac{2^n}{3^n}+\frac{1}{2^n}\right)^3+\cdots\right) <$$

$$2^n\left(1-\left(\frac{2^n}{3^n}+\frac{1}{2^n}\right)+\left(\frac{2^n}{3^n}+\frac{1}{2^n}\right)^2\right) = 2^n-\left(\frac{4}{3}\right)^n-1+2^n\left(\frac{2^n}{3^n}+\frac{1}{2^n}\right)^2.$$

We conclude that

$$\frac{1}{\zeta(n)-1}-2^n+\left(\frac{4}{3}\right)^n+2 < 1+\varepsilon(n),$$

for $n \geq 2$. □

3 Proof of Results

The following lemma is obvious.

Lemma 8. *If*

$$A < x < B,$$

then

$$A-1 < \lfloor x \rfloor < B.$$

Proposition 9. *For all natural numbers $n \geq 2$, we have that*

$$\left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor = 2^n - \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor - k,$$

where $k=1$ or $k=2$.

Proof. It is easy to check the equality holds for $2 \leq n \leq 4$. We know from Proposition 6 and Proposition 7 that

$$1 < \frac{1}{\zeta(n)-1}-2^n+\left(\frac{4}{3}\right)^n+2 < 1+\varepsilon(n).$$

Applying Lemma 8 twice to this inequality we obtain that

$$-1 < \left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor - 2^n + \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor + 2 < 1+\varepsilon(n).$$

As $\lim_{n\to\infty} \varepsilon(n) = 0$, for $n \geq 5$ we have that

$$-1 < \left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor - 2^n + \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor + 2 < 2.$$

It is clear that $\left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor - 2^n + \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor + 2$ is an integer for $n \geq 2$. Thus,

$$\left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor - 2^n + \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor + 2 = 0 \text{ or } 1.$$

Hence,

$$\left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor = 2^n - \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor - k,$$

where $k = 1$ or $k = 2$ for all $n \geq 2$. □

Proposition 10. *For all natural numbers $n \geq 2$, where k is as in Proposition 9, we have that*

$$k - 1 < \left\{ \frac{1}{\zeta(n)-1} \right\} + \left\{ \left(\frac{4}{3}\right)^n \right\} < k - 1 + \varepsilon(n).$$

Proof. From Proposition 9 we have that

$$\left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor = 2^n - \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor - k,$$

and therefore,

$$\left\lfloor \frac{1}{\zeta(n)-1} \right\rfloor - 2^n + \left\lfloor \left(\frac{4}{3}\right)^n \right\rfloor + 2 = 2 - k.$$

Since the integral part plus the fractional part is the number itself, we have that

$$\frac{1}{\zeta(n)-1} - 2^n + \left(\frac{4}{3}\right)^n + 2 = \left\{ \frac{1}{\zeta(n)-1} \right\} + \left\{ \left(\frac{4}{3}\right)^n \right\} + 2 - k. \tag{5}$$

We know from Proposition 6 and Proposition 7 that

$$1 < \frac{1}{\zeta(n)-1} - 2^n + \left(\frac{4}{3}\right)^n + 2 < 1 + \varepsilon(n).$$

Substituting (5) into this inequality we obtain that

$$1 < \left\{ \frac{1}{\zeta(n)-1} \right\} + \left\{ \left(\frac{4}{3}\right)^n \right\} + 2 - k < 1 + \varepsilon(n).$$

We conclude that

$$k - 1 < \left\{ \frac{1}{\zeta(n)-1} \right\} + \left\{ \left(\frac{4}{3}\right)^n \right\} < k - 1 + \varepsilon(n).$$

□

Proposition 11. *Let k be as in Proposition 9. Then $k = 1$ occurs finitely many times.*

Proof. We know from Proposition 10 that

$$k - 1 < \left\{ \frac{1}{\zeta(n) - 1} \right\} + \left\{ \left(\frac{4}{3} \right)^n \right\} < k - 1 + \varepsilon(n).$$

It follows from Mahler's work [1] that if $p > q \geq 2$ are coprime integers and $\varepsilon > 0$, then

$$\left\{ \left(\frac{p}{q} \right)^n \right\} > e^{-\varepsilon n},$$

for all integers n, except for at most a finite number of exceptions.

As $\varepsilon(n) = \mathcal{O}\left(\left(\frac{8}{9} \right)^n \right)$, by taking $p = 4$, $q = 3$ and $\varepsilon = \log(\frac{10}{9})$ we obtain that only a finite number of n satisfy

$$0 < \left\{ \left(\frac{4}{3} \right)^n \right\} < \varepsilon(n).$$

Therefore, only a finite number of n satisfy

$$0 < \left\{ \frac{1}{\zeta(n) - 1} \right\} + \left\{ \left(\frac{4}{3} \right)^n \right\} < \varepsilon(n).$$

Thus, $k = 1$ occurs a finite number of times. $\square$

The best effective lower bound for the distance of $\left(\frac{4}{3}\right)^n$ to the nearest integer we are aware of is

$$\left\| \left(\frac{4}{3} \right)^n \right\| > 0.4910^n \quad \text{for} \quad n \geq 5\,868\,122\,745\,713\,241\,570,$$

given by Pupyrev [2], where $\|\cdot\|$ denotes the distance to the nearest integer. However this bound is not strong enough for our purposes. Therefore, Proposition 11 is not effective.

Proof of Theorem 1. Theorem 1 follows from combining both Proposition 9 and Proposition 11. $\square$

Proof of Corollary 2. Corollary 2 follows from combining both Proposition 10 and Proposition 11. $\square$

Proof of Corollary 3. We prove this by contradiction. Suppose $\zeta(n) = 1 + \frac{1}{m}$ for some integer m. Therefore, $\frac{1}{\zeta(n)-1} = m$ is an integer and thus, $\left\{ \frac{1}{\zeta(n)-1} \right\} = 0$. From Proposition 10 we know that

$$k - 1 < \left\{ \frac{1}{\zeta(n) - 1} \right\} + \left\{ \left(\frac{4}{3} \right)^n \right\} < k - 1 + \varepsilon(n).$$

Therefore,

$$k - 1 < \left\{ \left(\frac{4}{3} \right)^n \right\} < k - 1 + \varepsilon(n).$$

When $k = 2$ we have
$$1 < \left\{ \left(\frac{4}{3}\right)^n \right\} < 1 + \varepsilon(n).$$
However, this implies that a fractional part is greater than 1 which by definition is impossible. This is a contradiction. Therefore, if $\zeta(n) = 1 + \frac{1}{m}$ for some integer m, then $k = 1$. Thus, this happens finitely many times. $\square$

Proposition 12. *For all $\frac{1}{2} < x < \frac{3}{4}$ and n large enough,*
$$-\left(\frac{4x}{3}\right)^n - x^n - k < \left\{ \frac{x^n}{\zeta(n) - 1} \right\} - \{(2x)^n\} < \varepsilon_x(n) - \left(\frac{4x}{3}\right)^n - x^n - k,$$
where $k = 0$ or $k = -1$. When x is rational $k = 0$ except for a finite number of exceptions.

Proof. From Proposition 6 and Proposition 7 we know that
$$1 < \frac{1}{\zeta(n) - 1} - 2^n + \left(\frac{4}{3}\right)^n + 2 < 1 + \varepsilon(n).$$
So
$$0 < \frac{1}{\zeta(n) - 1} - 2^n + \left(\frac{4}{3}\right)^n + 1 < \varepsilon(n).$$
Recall that $\varepsilon_x(n) = (2x)^n \left(\left(\frac{2}{3}\right)^n + \left(\frac{1}{2}\right)^n\right)^2$. It is obvious that $x^n \varepsilon_y(n) = \varepsilon_{xy}(n)$. Therefore,
$$0 < \frac{x^n}{\zeta(n) - 1} - (2x)^n + \left(\frac{4x}{3}\right)^n + x^n < \varepsilon_x(n),$$
and thus,
$$-\left(\frac{4x}{3}\right)^n - x^n < \frac{x^n}{\zeta(n) - 1} - (2x)^n < \varepsilon_x(n) - \left(\frac{4x}{3}\right)^n - x^n. \tag{6}$$
Applying Lemma 8 twice to (6) and noticing that $\lfloor (2x)^n \rfloor$ appears with a negative sign we obtain that
$$-1 - \left(\frac{4x}{3}\right)^n - x^n < \left\lfloor \frac{x^n}{\zeta(n) - 1} \right\rfloor - \lfloor (2x)^n \rfloor < 1 + \varepsilon_x(n) - \left(\frac{4x}{3}\right)^n - x^n.$$
Notice that for $\frac{1}{2} < x < \frac{3}{4}$ both bounds of the above inequality tend to -1 and 1 respectively. Also notice that for n large enough we have that
$$\varepsilon_x(n) - \left(\frac{4x}{3}\right)^n - x^n = x^n \left(\varepsilon(n) - \left(\frac{4}{3}\right)^n - 1\right) < 0.$$
As $\left\lfloor \frac{x^n}{\zeta(n)-1} \right\rfloor - \lfloor (2x)^n \rfloor$ is an integer, for n large enough we have that
$$\left\lfloor \frac{x^n}{\zeta(n) - 1} \right\rfloor - \lfloor (2x)^n \rfloor = -1 \text{ or } 0.$$

Therefore,

$$\frac{x^n}{\zeta(n)-1} - (2x)^n = k + \left\{\frac{x^n}{\zeta(n)-1}\right\} - \{(2x)^n\},$$

where $k = -1$ or $k = 0$. Substituting this into (6) we get that

$$-\left(\frac{4x}{3}\right)^n - x^n < k + \left\{\frac{x^n}{\zeta(n)-1}\right\} - \{(2x)^n\} < \varepsilon_x(n) - \left(\frac{4x}{3}\right)^n - x^n.$$

Hence,

$$-\left(\frac{4x}{3}\right)^n - x^n - k < \left\{\frac{x^n}{\zeta(n)-1}\right\} - \{(2x)^n\} < \varepsilon_x(n) - \left(\frac{4x}{3}\right)^n - x^n - k.$$

One can use the same argument as in Proposition 11 to show that for a rational number $\frac{1}{2} < x < \frac{3}{4}$ we have that $k = -1$ occurs finitely many times. $\square$

One can apply the same methods for other values of x. For $0 < x \leq \frac{1}{2}$, both fractional parts converge to 0, so it is not interesting. For $x = \frac{3}{4}$, one can achieve the same result but with $k = -2$ or $k = -1$. For $\frac{3}{4} < x$, if we want the bounds to converge, there are two possibilities. The first is that there will be more than two fractional parts (apart for some exceptions like $x = 1$), and k will take more values, that is, the number of fractional parts. The second is that there will still be two fractional parts and k can only take two values, but one of the fractional parts will be of a finite sum of real numbers to natural powers rather than just one term. For example when $x = \frac{11}{10}$ we have that

$$-k < \left\{\frac{\left(\frac{11}{10}\right)^n}{\zeta(n)-1}\right\} - \left\{\left(\frac{11}{5}\right)^n\right\} + \left\{\left(\frac{22}{15}\right)^n\right\} + \left\{\left(\frac{11}{10}\right)^n\right\} < -k + \varepsilon_{\frac{11}{10}}(n),$$

where $k = -2$ or $k = -1$ or $k = 0$ or $k = 1$. Or

$$-k < \left\{\frac{\left(\frac{11}{10}\right)^n}{\zeta(n)-1}\right\} - \left\{\left(\frac{11}{5}\right)^n - \left(\frac{22}{15}\right)^n - \left(\frac{11}{10}\right)^n\right\} < -k + \varepsilon_{\frac{11}{10}}(n),$$

where $k = 0$ or $k = 1$.

Proof of Theorem 5. When $x = \frac{2}{3}$ in Proposition 12 we have

$$-\left(\frac{8}{9}\right)^n - \left(\frac{2}{3}\right)^n - k < \left\{\frac{\left(\frac{2}{3}\right)^n}{\zeta(n)-1}\right\} - \left\{\left(\frac{4}{3}\right)^n\right\} <$$

$$\varepsilon_{\frac{2}{3}}(n) - \left(\frac{8}{9}\right)^n - \left(\frac{2}{3}\right)^n - k,$$

where $k=-1$ or $k=0$. We know from Corollary 2 that

$$k'-1<\left\{\frac{1}{\zeta(n)-1}\right\}+\left\{\left(\frac{4}{3}\right)^n\right\}<k'-1+\varepsilon(n),$$

where $k'=1$ or $k'=2$. Adding the two inequalities we get

$$m-\left(\frac{8}{9}\right)^n-\left(\frac{2}{3}\right)^n<\left\{\frac{1}{\zeta(n)-1}\right\}+\left\{\frac{\left(\frac{2}{3}\right)^n}{\zeta(n)-1}\right\}<$$

$$m+\varepsilon(n)+\varepsilon_{\frac{2}{3}}(n)-\left(\frac{8}{9}\right)^n-\left(\frac{2}{3}\right)^n,$$

where $m=0$ or $m=1$ or $m=2$. We can see that $m=0$ or $m=2$ occur finitely many times. Thus, for n large enough we always have $m=1$. This proves the theorem. $\square$

4 Acknowledgments

I would like thank James McKee for his alternative proof for Proposition 7 using a geometric expansion. This improvement turned out to be crucial for this paper. I would also like to thank Jan-Christoph Schlage-Puchta for his helpful comments and discussions on previous drafts of this paper, as well as writing a Mathematica program. In addition, I thank Vsevolod Lev for a helpful suggestion on a previous draft and for Yann Bugeaud for pointing out Mahler's theorem, which made this paper much stronger. Finally I would like to thank my father, Yiftach Barnea, for helping me understand the previous literature, and with writing and organizing this paper.

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2010 *Mathematics Subject Classification*: Primary 11J71; Secondary 11A55, 11M99, 11Y65, 11K31.
Keywords: continued fraction, fractional part, zeta function, rational power.

(Concerned with sequence A013697.)

Received November 10 2018; revised version received March 9 2019. Published in *Journal of Integer Sequences*, May 20 2019.

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