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Seeds for Generalized Taxicab Numbers

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Abstract

The generalized taxicab number $T(n, m, t)$ is equal to the smallest number that is the sum of n positive mth powers in t ways. This definition is inspired by Ramanujan's observation that $1729 = 1^3+12^3 = 9^3+10^3$ is the smallest number that is the sum of two cubes in two ways and thus $1729 = T(2, 3, 2)$. In this paper we prove that for any given positive integers m and t, there exists a number s such that $T(s+k, m, t) = T(s, m, t)+k$ for every $k \geq 0$. The smallest such s is termed the seed for the generalized taxicab number. Furthermore, we find explicit expressions for this seed number when the number of ways t is 2 or 3 and present a conjecture for $t \geq 4$ ways.

1 Introduction

Hardy relays the following story about visiting Ramanujan during his illness (see [4, p. xxxv]):

> I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

Indeed, $1729 = 1^3 + 12^3$ and $1729 = 9^3 + 10^3$ and it is the smallest such number that is the sum of two cubes in two different ways. In honor of the Ramanujan–Hardy conversation, the smallest number expressible as the sum of two cubes in t different ways is known as the t^{th} taxicab number and is denoted $\text{Taxicab}(t)$. Therefore, with this notation, $\text{Taxicab}(2) = 1729$.

There has been considerable effort expended in finding these taxicab numbers. The interested reader is referred to [1] for information about these numbers. That paper contains interesting information about the history of the problem as well as a discussion about the techniques used to find certain values of $\text{Taxicab}(t)$. Further information about taxicab numbers and their variants can also be found in [2, 3, 5, 6, 7]. Basically, $\text{Taxicab}(t)$ is known for $2 \le t \le 6$ and upper bounds for $\text{Taxicab}(t)$ have been given for $7 \le t \le 22$.

In this paper we generalize the definition of taxicab numbers, as there is really nothing special about using exactly two cubes (except for historical reasons). We will be concerned with finding the smallest number that is the sum of n positive m^{th} powers in at least t ways.

Let $T(n, m, t)$ denote the least number that is the sum of n positive m^{th} powers in at least t ways provided such a number exists.[1] The reason the definition above says that $T(n, m, t)$ is the least number that is the sum of n positive m^{th} powers in *at least* t ways is motivated by the following equation

$$28 = 3^3 + 3^2 + 3^2 + 1^2 = 4^2 + 2^2 + 2^2 + 2^2 = 5^2 + 1^2 + 1^2 + 1^2.$$

Indeed, 28 is the smallest number that is the sum of 4 squares in 2 ways as well as being the smallest number that is the sum of 4 squares in 3 ways. Hence we have that $T(4, 2, 2) = 28$ and also $T(4, 2, 3) = 28$. This is not the only example, in Section 4 we show $T(8, 2, 4) = T(8, 2, 3) = 32$.

So, as noted above, $T(2, 3, 2) = 1729$ and in general $T(2, 3, t)$ is the taxicab number

[1] It should be noted that in Wikipedia, the generalized taxicab number $\text{Taxicab}(k, j, n)$ is the smallest number which can be expressed as the sum of j k^{th} positive powers in n different ways. However, since there have been no published papers with this notation, we use the notation in the definition given above.

Taxicab(t). It is also easy to verify that

$$\begin{aligned}
T(2,2,2) &= 50 = 5^2+5^2 = 7^2+1 \\
T(3,2,2) &= 27 = 5^2+1+1 = 3^2+3^2+3^2 \\
T(4,2,2) &= 28 = 5^2+1+1+1 = 3^2+3^2+3^2+1 \\
T(5,2,2) &= 20 = 4^2+1+1+1+1 = 2^2+2^2+2^2+2^2+2^2.
\end{aligned}$$

Note that by adding 1 to the two sums in the last example we obtain

$$T(6,2,2) \le 21 = 4^2+1+1+1+1+1 = 2^2+2^2+2^2+2^2+2^2+1.$$

Doing this again we obtain

$$T(7,2,2) \le 22 = 4^2+1+1+1+1+1+1 = 2^2+2^2+2^2+2^2+2^2+1+1.$$

In fact it is indeed true that $T(6,2,2)=21$ and $T(7,2,2)=22$.

Considering cubes now, it is straightforward to verify that

$$\begin{aligned}
T(2,3,2) &= 1729 = 12^3+1 = 10^3+9^3 \\
T(3,3,2) &= 251 = 6^3+3^3+2^3 = 5^3+5^3+1 \\
T(4,3,2) &= 219 = 6^3+1+1+1 = 4^3+4^3+4^3+3^3 \\
T(5,3,2) &= 157 = 5^3+2^3+2^3+2^3+2^3 = 4^3+4^3+3^3+1+1 \\
T(6,3,2) &= 158 = 5^3+2^3+2^3+2^3+2^3+1 = 4^3+4^3+3^3+1+1+1 \\
T(7,3,2) &= 131 = 5^3+1+1+1+1+1+1 = 4^3+3^3+2^3+2^3+2^3+2^3+2^3 \\
T(8,3,2) &= 132 = 5^3+1+1+1+1+1+1+1 = 4^3+3^3+2^3+2^3+2^3+2^3+2^3+1 \\
T(9,3,2) &= 72 = 4^3+1+1+1+1+1+1+1+1 = 2^3+2^3+2^3+2^3+2^3+2^3+2^3+2^3+2^3 \\
T(10,3,2) &= 73 = 4^3+1+1+1+1+1+1+1+1+1 = \\
&\quad 2^3+2^3+2^3+2^3+2^3+2^3+2^3+2^3+2^3+1.
\end{aligned}$$

One also can check that $T(11,3,2)=74$ and that this solution results from adding 1 to both of the (equal) sums in the case of $T(10,3,2)$.

From the examples above it seems plausible that there exists a number, say s_0, such that $T(n+1,m,t) = T(n,m,t)+1$ for all $n \ge s_0$ or equivalently, that $T(s_0+k,m,t) = T(s_0,m,t)+k$ for all $k \ge 0$. This motivates the following definition.

Definition 1. If s_0 is the smallest positive integer such that $T(n+1,m,t) = T(n,m,t)+1$ for all $n \ge s_0$, then we call s_0 the *seed number* for m^{th} powers in t ways and denote this number by $S(m,t) = s_0$. We also call $T(S(m,t),m,t)$ the *seed value* of m^{th} powers in t ways and denote it by $V(m,t)$.

This paper proceeds as follows. In Section 2 we show that for every m and t there exists a seed number. In Section 3 we give an explicit value for the seed of the sum of m^{th} powers in 2 ways. We prove there that the seed number for squares in 2 ways is indeed 5 and the seed value is 20. In our notation, this says $S(2,2) = 5$ and $V(2,2) = 20$ and hence $T(5+k, 2, t) = 20 + k$ for all $k \geq 0$. In Section 4 we give an explicit value of $V(m,3)$. Finally, in Section 5, we give a general theorem and a conjecture about the seed for m^{th} powers in t ways for all $t \geq 2$.

2 Seeds exist

In this section we prove that for every m and t there exists a seed number. We first show that for every m and t there exist some positive integer n and some value v such that v is the sum of n m^{th} powers in t ways. From that we then show that there is a least such n and hence there exists a seed number (and a seed value).

Lemma 2. *For all positive integers $m, t \geq 1$, there exist positive integers n and v such that v is the sum of n m^{th} powers in t ways.*

Proof. If $m = 1$ or $t = 1$, the result is obvious, so assume that $m, t > 1$.

We give a direct construction of t different sums of m^{th} powers, each sum having the same number of terms. The first sum is $t^m + t^m + \cdots + t^m$ for a suitable number of terms. For each $1 \leq i \leq t-1$ we construct the sum S_i from the terms $(t-i)^m$ and $(t+i)^m$. So each $S_i = \underbrace{(t-i)^m + \cdots + (t-i)^m}_{x_i} + \underbrace{(t+i)^m + \cdots + (t+i)^m}_{y_i}$ for suitable x_i and y_i.

In order to define the sums, we first need to define some values. For each $1 \leq i \leq t-1$ define

$$\begin{array}{lll} a_i = t^m - (t-i)^m & b_i = (t+i)^m - t^m & l_i = \operatorname{lcm}(a_i, b_i); \\ \alpha_i = l_i / a_i & \beta_i = l_i / b_i & \gamma_i = \alpha_i + \beta_i \\ n = \operatorname{lcm}(\gamma_1, \gamma_2, \ldots, \gamma_{t-1}) & \delta_i = n / \gamma_i. & \end{array}$$

Using these values, define

$$S_0 = \underbrace{t^m + \cdots + t^m}_{n}$$

and for each $1 \leq i \leq t-1$ define

$$S_i = \underbrace{(t-i)^m + \cdots + (t-i)^m}_{\alpha_i \delta_i} + \underbrace{(t+i)^m + \cdots + (t+i)^m}_{\beta_i \delta_i}.$$

Obviously all of these sums are different (in fact no two sums contain any terms in common). We must prove two things: first, that each of the sums $S_0, S_1, \ldots, S_{t-1}$ contain the same number of terms and second, that $S_0 = S_1 = \cdots = S_{t-1}$.

To show the first, we note that S_0 contains n terms and for each $1 \le i \le t-1$, S_i contains $\alpha_i\delta_i + \beta_i\delta_i$ terms. But now

$$\alpha_i\delta_i + \beta_i\delta_i = (\alpha_i + \beta_i)\delta_i = \gamma_i\delta_i = n$$

as desired.

Next we compute the sums. Clearly $S_0 = nt^m$. For each $1 \le i \le t-1$,

$$\begin{aligned} S_i &= \alpha_i\delta_i(t-i)^m + \beta_i\delta_i(t+i)^m \\ &= (\alpha_i(t^m - a_i) + \beta_i(t^m + b_i))\delta_i \\ &= ((\alpha_i + \beta_i)t^m + (\beta_i b_i - \alpha_i a_i))\delta_i \\ &= ((\alpha_i + \beta_i)t^m + (l_i - l_i))\delta_i \\ &= \gamma_i t^m \ \frac{n}{\gamma_i} \\ &= nt^m. \end{aligned}$$

Thus we conclude that $S_0 = S_1 = \cdots = S_{t-1} = nt^m$. This completes the proof. $\square$

By adding a 1 to each multiset of terms in the above sums, we see that $T(n+1,m,t) \le T(n,m,t)+1$.

As an immediate consequence of this observation and Lemma 2 we can conclude that $T(n,m,t)$ exists for every $m,t \ge 1$ when n is large enough. In general, we do not know whether $T(n,m,t)$ exists for all n less than the value of $n(= \operatorname{lcm}(\gamma_1,\gamma_2,\ldots,\gamma_{t-1}))$ given above.

In the next theorem we prove that there is a seed number for every m and t.

Theorem 3. *For every $m,t \ge 1$, there is a smallest number s_0 such that $T(s_0+k,m,t) = T(s_0,m,t)+k$ for every $k \ge 0$. Hence for every $m,t \ge 1$ the seed number $s_0 = S(m,t)$ exists.*

Proof. Fix m and t. Note that for some n large enough, $T(n,m,t)$ exists by Lemma 2. Now let n_0 be the smallest integer for which $T(n_0,m,t)$ exists.

To shorten notation let $T'(n) = T(n,m,t)$. Notice that for any n, a (very naive) lower bound for $T'(n)$ is n since

$$T'(n) \ge \underbrace{1^m + 1^m + \cdots + 1^m}_{n} = n.$$

The *gap* between the value of $T'(n_0)$ and the naive lower bound on $T'(n_0)$ is $g = T'(n_0) - n_0 \ge 0$. We also have that $T'(n_0+k) \le T'(n_0)+k$ for every $k \ge 0$. Thus $n_0 + k \le T'(n_0+k) \le T'(n_0)+k$.

We say that the function T' *drops* at n if $T'(n+1) < T'(n)+1$ (note that n is the *location* of the "drop" and not the amount T' drops). Let $D = \{n \ge n_0 \mid T'(n+1) < T'(n)+1\}$ be the set of all drops of T'. We claim that D is a finite set and in fact we show that $|D| \le g$.

Let $D = \{n_1, n_2, \ldots\}$. (We should note that possibly $D = \emptyset$, in which case $S(m,t)$ is just n_0.) If D is nonempty, say that $n_k = n_0 + x_k$, then we see that since T' is dropped by at least one at each n_i we have

$$T'(n_k + 1) = T'(n_0 + x_k + 1) \leq T'(n_0) + x_k + 1 - k.$$

So since $n_0 + x_k + 1 \leq T'(n_0 + x_k + 1)$, we have $n_0 \leq T'(n_0) - k$ and hence $k \leq T'(n_0) - n_0 = g$. This implies that $|D| \leq g$ and hence there are at most g drops in the function T'. So if $D = \{n_1, n_2, \ldots n_i\}$ is the set of all drops, then since n_i is the last drop in the function T', we have $T'(n_i + 1 + k) = T'(n_i + 1) + k$ for all $k \geq 0$ and hence $n_i + 1$ is the seed number $S(m,t)$. □

As an example of the above theorem we consider the values of $T(2,3,2), T(3,3,2), \ldots, T(10,3,2)$ given in Section 1. Notice that $n_0 = 2$ since $T(2,3,2) = 1729$ (and clearly T(1,3,2) does not exist). We also see that $n_1 = 2, n_2 = 3, n_3 = 4, n_4 = 6$, and $n_5 = 8$. We prove in Section 4 that indeed $D = \{2,3,4,6,8\}$. Thus we conclude that $S(3,2) = 9$ and hence $T(9+k,3,2) = T(9,3,2) + k = 72 + k$ for all $k \geq 0$.

3 Seeds for two ways

In this section we give the explicit value for the seed numbers for two ways. We assume that all variables are positive integers except where noted. We begin with three easy lemmas that hold for any number of sums. The first lemma says that if all the sums share a common term, then that term must be equal to 1.

Lemma 4. *If $x = T(n,m,t)$ and $x = \sum_{i=1}^{n} a_i^m = \sum_{i=1}^{n} b_i^m = \cdots = \sum_{i=1}^{n} t_i^m$, and if $a_{i_1} = b_{i_2} = \cdots = t_{i_t}$ for some choice of i's, then $a_{i_1} = b_{i_2} = \cdots = t_{i_t} = 1$.*

Proof. If not, then replace each of $a_{i_1}, b_{i_2}, \ldots, t_{i_t}$ with a 1 and get a contradiction to $x = T(n,m,t)$ since the new sums of the m^{th} powers are still the same and less than before, a contradiction. □

The next lemma says that the t sums adding to the seed value can't all have a 1 as a term.

Lemma 5. *If $x = V(m,t)$ and $x = \sum_{i=1}^{n} a_i^m = \sum_{i=1}^{n} b_i^m = \cdots = \sum_{i=1}^{n} t_i^m$, and if $a_{i_1} = b_{i_2} = \cdots = t_{i_t}$ for some choice of i's, then $a_{i_1} = b_{i_2} = \cdots = t_{i_t} \neq 1$.*

Proof. If each sum has a 1 as a term, then by simply deleting the 1 in each of these sums we would obtain a smaller seed value, a contradiction. □

In the next lemma we show that any seed value for m^{th} powers in t ways must always be greater or equal to $n2^m$ where n is the seed number. This essentially says that we can always assume that one of the sums is

$$\underbrace{2^m + 2^m + \cdots + 2^m}_{n}.$$

This fact is of fundamental importance in finding seeds for 2 and 3 ways.

Lemma 6. *If $V(m,t) = \sum_{i=1}^{s_0} a_i^m = \sum_{i=1}^{s_0} b_i^m = \cdots = \sum_{i=1}^{s_0} t_i^m$ is the seed value for m^{th} powers in t ways, then $V(m,t) \geq s_0 2^m$. Further, if n is any number which provides a solution to $\sum_{i=1}^{n} a_i^m = \sum_{i=1}^{n} b_i^m = \cdots = \sum_{i=1}^{n} t_i^m = n2^m$, then $V(m,t) = s_0 2^m$, where s_0 is the smallest such n (and hence $s_0 = S(m,t)$).*

Proof. Let $V(m,t) = \sum_{i=1}^{s_0} a_i^m = \sum_{i=1}^{s_0} b_i^m = \cdots = \sum_{i=1}^{s_0} t_i^m$ be the seed value for m^{th} powers in t ways and assume that each sum is written in nonincreasing order.

Now if $a_{s_0} = 1$, then from Lemma 5 we have without loss of generality that $b_{s_0} \neq 1$. Thus for all $1 \leq i \leq s_0$ it must be that $b_i \geq 2$. Hence in this case we have $V(m,t) \geq s_0 2^m$. If $a_{s_0} \geq 2$, then since $a_i \geq a_{i+1}$ for all $1 \leq i \leq s_0 - 1$, then clearly $V(m,t) \geq s_0 2^m$.

The second part of this lemma now follows immediately. □

Lemma 6 implies that the seed value is the sum of s_0 2^m's (where s_0 are equal to the seed number $S(m,t)$). So we are interested in this value for the sum. In the next two lemmas we consider two different sums that are equal to the sum of 2^m's. The verification of the first is a straightforward calculation.

Lemma 7. *If $\alpha 4^m + (n-\alpha) = n2^m$, then $n = (2^m+1)\alpha$ and hence $n \geq 2^m + 1$.*

A comment is in order concerning Lemma 7. This lemma deals with the case when two sums are equal and one of the sums is all 2^m's while the other is 4^m's and 1^m's. Specifically, it says that if

$$\underbrace{4^m + 4^m + \cdots + 4^m}_{\alpha} + \underbrace{1 + 1 + \cdots + 1}_{n-\alpha} = \underbrace{2^m + 2^m + \cdots + 2^m}_{n},$$

then $n \geq 2^m + 1$. The next lemma deals with the case when two sums are equal and one of the sums is all 2^m's and the other is 3^m's and 1^m's.

Lemma 8. *If $\alpha 3^m + (n-\alpha) = n2^m$ and if $d = \gcd(3^m - 2^m, 2^m - 1)$, then $\frac{3^m-1}{d} \mid n$ and hence $n \geq \frac{3^m-1}{d}$.*

Proof. Assume that $\alpha 3^m + (n-\alpha) = n2^m$ and that $d = \gcd(3^m - 2^m, 2^m - 1)$. Then

$$\alpha\left(\frac{3^m-1}{d}\right) = n\left(\frac{2^m-1}{d}\right).$$

Now since $d = \gcd(3^m - 2^m, 2^m - 1) = \gcd(3^m - 1, 2^m - 1)$, then $1 = \gcd(\frac{3^m-1}{d}, \frac{2^m-1}{d})$ and thus we have $\frac{2^m-1}{d} \mid \alpha$. So $\alpha(\frac{d}{2^m-1})$ is an integer. Now since $\alpha(\frac{d}{2^m-1})\,(\frac{3^m-1}{d}) = n$ it follows that $(\frac{3^m-1}{d}) \mid n$ and hence $n \geq \frac{3^m-1}{d}$. □

The application of this lemma is similar to that of Lemma 7. In this case we have the situation where

$$\underbrace{3^m + 3^m + \cdots + 3^m}_{\alpha} + \underbrace{1 + 1 + \cdots + 1}_{n-\alpha} = \underbrace{2^m + 2^m + \cdots + 2^m}_{n}.$$

So here we have that $n = \alpha(\frac{d}{2^m-1})\ (\frac{3^m-1}{d})$ and our main application is that in this case $n \geq \frac{3^m-1}{d}$.

We now obtain our characterization of the seed number and the seed value for sums in two ways.

Theorem 9. *Let $d = \gcd(3^m - 2^m, 2^m - 1)$. The seed number $S(m, 2) = \min(\frac{3^m-1}{d}, 2^m + 1) = s_0$ and the seed value $V(m, 2) = s_0 2^m$. Hence $T(s_0 + j, m, 2) = s_0 2^m + j$ for every $j \geq 0$.*

Proof. Consider the two equations

$$4^m + \underbrace{1 + \cdots + 1}_{2^m} = \underbrace{2^m + \cdots + 2^m}_{2^m+1} = (2^m + 1)2^m \tag{1}$$

and

$$\underbrace{3^m + \cdots + 3^m}_{\frac{2^m-1}{d}} + \underbrace{1 + \cdots + 1}_{\frac{3^m-2^m}{d}} = \underbrace{2^m + \cdots + 2^m}_{\frac{3^m-1}{d}} = (\frac{3^m - 1}{d})2^m. \tag{2}$$

In view of Equations (1) and (2) and Lemma 6, if $V(m, t) = a_1^m + a_2^m + \cdots + a_n^m = b_1^m + b_2^m + \cdots + b_n^m$ is the seed value for m^{th} powers in 2 ways, then we can assume that $b_i = 2$ for all i. We next show that $a_1 \leq 4$.

Assume that $V(m, t) = a_1^m + a_2^m + \cdots + a_n^m = 2^m + 2^m + \cdots + 2^m$ is the seed value for m^{th} powers in 2 ways with $a_i \geq a_{i+1}$ for all $1 \leq i \leq n - 1$.

Assume that $a_1 \geq 5$. Clearly, if $n \geq 2^m + 1$, then in view of Equation (1) above this is a contradiction (since $a_1^m + a_2^m + \cdots + a_n^m > 4^m + 1 + \cdots + 1$). Assume $n < 2^m + 1$, then

$$a_1^m + a_2^m + \cdots + a_n^m + \underbrace{1 + 1 + \cdots + 1}_{2^m+1-n} > 4^m + \underbrace{1 + \cdots + 1}_{2^m}$$

which is again a contradiction to the assumption that $V(m, t) = a_1^m + a_2^m + \cdots + a_n^m$. Assuming that $a_1 = 4$ and $a_2 > 1$ yields a similar contradiction.

So either $\{a_1, a_2, \ldots a_n\} = \{1, 4\}$ or $\{a_1, a_2, \ldots a_n\} = \{1, 3\}$ since by Lemma 4 no a_i can be equal to any $b_i = 2$. In the first case we obtain Equation (1) since no smaller sum can have only 4^m's and 1's as its terms. In the second case we can assume that $s3^m + (n - s) = n2^m$ for some s.

From Lemma 8, the minimum value of n is $\frac{3^m-1}{d}$, which leads to Equation (2). The seed number is therefore be the minimum length of the sums in either Equation (1) or Equation (2). Thus the minimum of $2^m + 1$ and $\frac{3^m-1}{d}$ is the seed number $S(m, 2)$. □

In Table 3 we compute seeds for m^{th} powers in 2 ways for $m \leq 20$.

The interested reader may note that $S(m,2) = 2^m + 1$ in every case above except when $m = 1, 4, 12$. This says that $2^m + 1 \leq (3^m - 1)/d$ for every $m \leq 20$ with $m \neq 1, 4, 12$. We computed values of $2^m + 1$ and $(3^m - 1)/d$ for all $m \leq 200,000$ and found that $2^m + 1 \leq (3^m - 1)/d$ for all m in that range except for $m = 1, 4, 12$ and 36. We do not conjecture that this is true for all $m > 36$; however, it certainly appears to be true.

In Table 3 we give explicit values from Theorem 9. Remember that $S(m,2)$ is the number of terms in the seed, while $V(m,2)$ is the exact value of the seed.

m	d	$S(m,2)$	$V(m,2)$
1	1	2	4
2	1	5	20
3	1	9	72
4	5	16	256
5	1	33	1056
6	7	65	4160
7	1	129	16512
8	5	257	65792
9	1	513	262656
10	11	1025	1049600
11	23	2049	4196352
12	455	1168	4784128
13	1	8193	67117056
14	1	16385	268451840
15	1	32769	1073774592
16	85	65537	4295032832
17	1	131073	17180000256
18	133	262145	68719738880
19	1	524289	274878431232
20	275	1048577	1099512676352

Table 1: Seed values and size

4 Seeds for three ways

In this section we give an explicit value for $V(m,3)$, the seed value for the smallest number that can be written as the sum of m^{th} powers in 3 ways.

We first need a preliminary lemma which says that no term in a sum that is a seed value (for powers $m \geq 4$) can exceed the number 4.

Lemma 10. *The seed value* $V(m,3)=\sum_{i=1}^{s}a_i^m=\sum_{i=1}^{s}b_i^m=s2^m$ *where* $s=S(m,3)$ *is the seed number. If* $m\geq 4$*, then* $a_i,b_i\leq 4$ *for all* $1\leq i\leq s$*.*

Proof. We first note the following equalities:

$$\begin{aligned}\underbrace{4^m+1+\cdots+1}_{2^m+1}+\underbrace{4^m+1+\cdots+1}_{2^m+1}&=\underbrace{4^m+1+\cdots+1}_{2^m+1}+\underbrace{2^m+\cdots+2^m}_{2^m+1}\\&=\underbrace{2^m+\cdots+2^m}_{2^m+1}+\underbrace{2^m+\cdots+2^m}_{2^m+1}.\end{aligned}\tag{3}$$

From this equation and Lemma 6 we have $V(m,3)=s2^m$ for $s=S(m,3)$. We also see from this equation

that the taxicab number $T(2(2^m+1)+j,m,3)\leq 2(2^m+1)2^m+j$ for all $j\geq 0$ and that the seed number $S(m,3)\leq 2(2^m+1)$. So in particular, when $j=0$ we have $T(2(2^m+1),m,3)\leq 2(2^m+1)2^m$.

Now, assume $V(m,3)=\sum_{i=1}^{s}a_i^m=\sum_{i=1}^{s}b_i^m=s2^m$ where $s=S(m,3)$ is the seed number. Then $s\leq 2(2^m+1)$. Assume that $a_1\geq 5$. Since $a_i\geq 1$ for all $i>1$, when extending the sums to have $2(2^m+2)$ terms by adding sufficiently many 1's, we get

$$5^m+2(2^m+1)-1\leq\sum_{i=1}^{s}a_i^m+(2(2^m+1)-s)\leq T(2(2^m+1),m,3)$$

and hence

$$5^m+2(2^m+1)-1\leq T(2(2^m+1),m,3)\leq 2(2^m+1)2^m.$$

Thus

$$\begin{aligned}5^m+2(2^m+1)-1&\leq 2(2^m+1)2^m\\&\leq(2^m+1)2^{m+1}\end{aligned}$$

and hence

$$\begin{aligned}5^m&\leq 2^m+1)(2^{m+1}-2)+1\\&\leq 2^{2m+1}-1\\&\leq 2\times 4^m-1.\end{aligned}$$

This last inequality implies $m=1,2,$ or 3, but by hypothesis $m\geq 4$, so we obtain a contradiction. Hence $a_i\leq 4$ (similarly $b_i\leq 4$) for all $1\leq i\leq s$. □

We are now in position to obtain our characterization of the seed number and the seed value for sums in three ways. We begin with the small values of m.

Theorem 11. (a) $S(1,3)=3$ *and the seed value* $V(1,3)=3\times 2^1=6$*,*

(b) $S(2,3)=8$ *and the seed value* $V(2,3)=8\times 2^2=32$*,* (c) $S(3,3)=18$ *and the seed value* $V(3,3)=18\times 2^3=144$*.*

Proof. The sums are given below. It is straightforward to check that they are minimal.

(a) $6 = 4 + 1 + 1 = 3 + 2 + 1 = 2 + 2 + 2$.

(b) $32 = 4^2 + 2^2 + 2^2 + 2^2 + 1 + 1 + 1 + 1 = 3^2 + 3^2 + 3^2 + 1 + 1 + 1 + 1 + 1 = \underbrace{2^2 + \cdots + 2^2}_{8}$.

(c) $144 = 4^3 + 4^3 + \underbrace{1 + \cdots + 1}_{16} = 4^3 + \underbrace{1 + \cdots + 1}_{8} + \underbrace{2^3 + \cdots + 2^3}_{9} = \underbrace{2^3 + \cdots + 2^3}_{18}$.

□

Theorem 12. *Assume that $m \geq 4$ and let $d = \gcd(3^m - 2^m, 2^m - 1)$. Also let $l_3 = \frac{3^m-1}{d}$ and $l_4 = 2^m + 1$.*

Given the four values $l_3, l_4, 2l_3, 2l_4$, the second smallest of these values is the seed number $S(m,3)$ and the seed value $V(m,3) = S(m,3) \times 2^m$.

Remark 13. Then

1. if $2l_4 < l_3$, then the seed number $S(m,3) = 2l_4$ and the seed value $V(m,3) = 2l_4 2^m$.
2. if $l_3 \leq l_4 \leq 2l_3$, then the seed number $S(m,3) = l_4$ and the seed value $V(m,3) = l_4 2^m$,
3. if $l_4 < l_3 \leq 2l_4$, then the seed number $S(m,3) = l_3$ and the seed value $V(m,3) = l_3 2^m$,
4. if $2l_3 < l_4$, then the seed number $S(m,3) = 2l_3$ and the seed value $V(m,3) = 2l_3 2^m$,

Proof. Considering Equation (3) in the proof of Lemma 10, in all cases the seed number $S(m,3) \leq 2l_4$. Also, we can assume that the seed value $V(m,3) = \sum_{i=1}^{s} a_i^m = \sum_{i=1}^{s} b_i^m = s2^m$ where $s = S(m,3)$ is the seed number and (from Lemma 10) $a_i, b_i \leq 4$ for all i. Let $A = \{a_1, a_2, \ldots, a_s\} = \{1^{\alpha_1}, 2^{\alpha_2}, 3^{\alpha_3}, 4^{\alpha_4}\}$ be the multiset containing all the terms in the sum $\sum_{i=1}^{s} a_i^m$ (so A contains the term i^m exactly α_i times for $1 \leq i \leq 4$), and let $B = \{b_1, b_2, \ldots, b_s\} = \{1^{\beta_1}, 2^{\beta_2}, 3^{\beta_3}, 4^{\beta_4}\}$ be the multiset containing the terms in the sum $\sum_{i=1}^{s} b_i^m$. From Lemma 4 we can assume without loss of generality that $\beta_2 = 0$, since it can not be the case that both an $a_i = 2$ and a $b_j = 2$, so we can assume that $\beta_2 = 0$.

There are four cases to consider, depending on which of the four values in the statement of the theorem is the second smallest value.

Remark 14.

Case 1: $2l_4$ is the second smallest value;

Case 2: l_4 is the second smallest value;

Case 3: l_3 is the second smallest value;

Case 4: $2l_3$ is the second smallest value.

Case 1. $2l_4$ is the second smallest value. This assumption implies $2l_4 \leq l_3$. Assume $s = S(m, 3) < 2l_4$. We see first that $0 \leq \alpha_4, \beta_4 \leq 1$, since if (say) $\alpha_1 \geq 2$, then $\sum_{i=1}^{s} a_i^m \geq 4^m + 4^m + (s-2)1^m$ and so $\sum_{i=1}^{s} a_i^m + (2l_4 - s) \geq 4^m + 4^m + (2l_4 - 2)$ which (because of Equation (3)) says $\sum_{i=1}^{s} a_i^m$ can not be a seed value unless $\alpha_3 = \alpha_2 = 0$ in which case we are led to one of the sums in Equation (3). However since we assumed $s < 2l_4$, this is a contradiction.

Now, if $\beta_4 = 0$ we obtain the equation $\beta_3 3^m + \beta_1 1^m = s2^m$. By Lemma 8 we thus have $s \geq l_3$. So $s \geq l_3 > 2l_4 > s$ a contradiction. If $\alpha_4 = 0$, then by subtracting α_2 2^m's from each side of the equation $\sum_{i=1}^{s} a_i^m = s2^m$ we obtain a similar contradiction. Hence we can assume that $\alpha_4 = \beta_4 = 1$.

So we have

$$4^m + \alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = 4^m + \beta_3 3^m + \beta_1 1^m$$

subtracting 4^m from both sides yields

$$\alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = \beta_3 3^m + \beta_1 1^m.$$

This implies that

$$\alpha_2 2^m = (\beta_3 - \alpha_3)3^m + (\beta_1 - \alpha_1).$$

Now since $(\beta_3 - \alpha_3) + (\beta_1 - \alpha_1) = \alpha_2$ and since $\alpha_2 > 0$ (else the equation is degenerate), then by Lemma 8 we have $\alpha_2 \geq l_3$, a clear contradiction.

So in this case we have $s \geq 2l_4$. Equation (3) then proves that indeed in this case $s = S(m, 3) = 2l_4$ and hence the seed value $V(m, 3) = 2l_4 2^m$.

Case 2. l_4 is the second smallest value. This assumption implies $l_3 \leq l_4 \leq 2l_3$.

First consider the following equation:

$$\underbrace{4^m + 1 + \cdots + 1}_{l_4} = \underbrace{3^m + \cdots + 3^m}_{(2^m-1)/d} + \underbrace{1^m + \cdots + 1^m}_{(3^m-2^m)/d)} + \underbrace{2^m + \cdots + 2^m}_{l_4-l_3} = \underbrace{2^m + \cdots + 2^m}_{l_4}. \tag{4}$$

From this equation we see that in this case $s = S(m, 3) \leq l_4$. Assume $s = S(m, 3) < l_4$.

If $\alpha_4 \geq 1$ (or $\beta_4 \geq 1$), then $\sum_{i=1}^{s} a_i^m \geq 4^m + (s-1)1^m$ and so $\sum_{i=1}^{s} a_i^m + (l_4 - s) \geq 4^m + (l_4 - 1)$ which (because of Equation (4) says $\sum_{i=1}^{s} a_i^m$ can not be a seed value, unless $\alpha_3 = \alpha_2 = 0$ in which case we are led to the first sum in Equation (4). However, since we assumed that $s < l_4$ we see that this is a contradiction. So $\alpha_4 = \beta_4 = 0$. Hence

$$\alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = \beta_3 3^m + \beta_1 1^m = s2^m \tag{5}$$

and subtracting $\alpha_3 3^m$ and $\alpha_1 1^m$ from the first two sums yields

$$\alpha_2 2^m = (\beta_3 - \alpha_3)3^m + (\beta_1 - \alpha_1)1^m$$

and so by Lemma 8 we have $\alpha_2 \geq l_3$. Also, subtracting $\alpha_2 2^m$ from the first and third sums in Equation (5) we have

$$\alpha_3 3^m + \alpha_1 1^m = (s - \alpha_2)2^m.$$

So, again by Lemma 8 we have $s - \alpha_2 = \alpha_1 + \alpha_3 \geq l_3$. Thus $s = \alpha_1 + \alpha_2 + \alpha_3 \geq 2l_3$, a clear contradiction to our assumption that $s < l_4 \leq 2l_3$.

So in this case we have $s \geq l_4$. Equation (4) then proves that in this case $s = S(m,3) = l_4$ and hence the seed value $V(m,3) = l_4 2^m$.

Case 3. l_3 **is the second smallest value.** So in this case $l_4 \leq l_3 \leq 2l_4$. First note the following equation:

$$\underbrace{4^m + 1 + \cdots + 1}_{l_4} + \underbrace{2^m + \cdots + 2^m}_{l_3 - l_4} = \underbrace{3^m + \cdots + 3^m}_{(2^m-1)/d)} + \underbrace{1^m + \cdots + 1^m}_{(3^m-2^m)/d)} = \underbrace{2^m + \cdots + 2^m}_{l_3}. \quad (6)$$

From this we see that $s = S(m,3) \leq l_3$. Assume $s = S(m,3) < l_3$ and consider $T(2l_4, m, 3)$. We see that $T(2l_4, m, 3) = \sum_{i=1}^{s} a_i^m + (2l_4 - s) < 2l_4 2^m$ since $\sum_{i=1}^{s} a_i^m$ is the seed value and $l_3 \leq 2l_4$. However, if $\alpha_4 \geq 2$ (or $\beta_4 \geq 2$), then $T(2l_4, m, 3) \geq 4^m + 4^m + (l_4 - 2)1^m = 2l_4 2^m$, a contradiction.

So we can assume $0 \leq \alpha_4, \beta_4 \leq 1$. We have

$$\alpha_4 4^m + \alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = \beta_4 4^m + \beta_3 3^m + \beta_1 1^m = s2^m.$$

If $\beta_4 = 0$, then we have $\beta_3 3^m + \beta_1 1^m = s2^m$, but from Lemma 8 this implies $s \geq l_3$, a contradiction our assumption that $s < l_3$. So now we have

$$\alpha_4 4^m + \alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = 4^m + \beta_3 3^m + \beta_1 1^m = s2^m.$$

If $\alpha_4 = 1$, then by subtracting 4^m from the first two sums in the equation above, we obtain $\alpha_2 2^m = (\beta_3 - \alpha_3)3^m + (\beta_1 - \alpha_1)1^m$. But from Lemma 8 we have $\alpha_2 \geq l_3$, a contradiction. So $\alpha_4 = 0$. So now we have

$$\alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = 4^m + \beta_3 3^m + \beta_1 1^m = s2^m.$$

Finally, from $\alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = s2^m$ subtract $\alpha_2 2^m$ from each side to obtain $\alpha_3 3^m + \alpha_1 1^m = (s - \alpha_2)2^m$. From Lemma 8 we obtain $s - \alpha_2 \geq l_3$ a clear contradiction to our assumption that $s < l_3$.

So we have shown that $s \geq l_3$. Equation (6) then proves that in this case $s = S(m,3) = l_3$ and hence the seed value $V(m,3) = l_3 2^m$.

Case 4. $2l_3$ **is the second smallest value**. This assumption implies $2l_3 \leq l_4$. Consider the following equation:

$$\begin{aligned} &\underbrace{3^m + \cdots + 3^m}_{(2^m-1)/d)} + \underbrace{1^m + \cdots + 1^m}_{(3^m-2^m)/d)} + \underbrace{3^m + \cdots + 3^m}_{(2^m-1)/d)} + \underbrace{1^m + \cdots + 1^m}_{(3^m-2^m)/d)} \\ &= \underbrace{3^m + \cdots + 3^m}_{(2^m-1)/d)} + \underbrace{1^m + \cdots + 1^m}_{(3^m-2^m)/d)} + \underbrace{2^m + \cdots + 2^m}_{l_3} = \underbrace{2^m + \cdots + 2^m}_{2l_3}. \end{aligned} \quad (7)$$

From this we see $s = S(m,3) \leq 2l_3$. Assume that $s = S(m,3) < 2l_3$. Since $s < l_4$ (as in the proof of Case 2) we have $\alpha_4 = \beta_4 = 0$. So we have

$$\alpha_3 3^m + \alpha_2 2^m + \alpha_1 1^m = \beta_3 3^m + \beta_1 1^m = s2^m. \quad (8)$$

Subtracting $\alpha_3 3^m$ and $\alpha_1 1^m$ from the first two sums yields

$$\alpha_2 2^m = (\beta_3 - \alpha_3)3^m + (\beta_1 - \alpha_1)1^m$$

and so by Lemma 8 we have $\alpha_2 \geq l_3$. Also, subtracting $\alpha_2 2^m$ from the first and third sums in Equation (8) we have

$$\alpha_3 3^m + \alpha_1 1^m = (s - \alpha_2)2^m.$$

So, again by Lemma 8 we get $s - \alpha_2 = \alpha_1 + \alpha_3 \geq l_3$. Thus $s = \alpha_1 + \alpha_2 + \alpha_3 \geq 2l_3$ a contradiction to our assumption that $s \leq 2l_3$.

So in this case $s \geq 2l_3$. Equation (7) then proves that $s = S(m, 3) = 2l_3$ and hence the seed value $V(m, 3) = l_4 2^m$. □

As was done after the proof of Theorem 9 we wish to compute the exact value of $S(m, 3)$ using the results of Theorem 12.

We found that for every $1 \leq m \leq 200,000$ with $m \neq 1, 2, 4, 6, 12, 36$, that $S(m, 3) = 2l_4$ and hence $V(m, 3) = 2l_4 2^m$.

This is Case 1 above. This calculation shows $2l_4 \leq l_3$ for all powers $36 < m \leq 200,000$. Again we do not conjecture that $2l_4 \leq l_3$ for all $m > 36$, but certainly the evidence is very strong.

When $m = 4$ we are in Case 2, so $l_3 \leq l_4 \leq 2l_3$ and hence $S(4, 3) = l_4 = 2^4 + 1$. We should note that $m = 1$ also has the property $l_3 \leq l_4 \leq 2l_3$, and although it doesn't follow from the general proof, it is indeed true that $S(1, 3) = l_4 = 3$ and so $V(1, 3) = 3 \times 2^1 = 6$, since $6 = 1 + 2 + 3 = 4 + 1 + 1 = 2 + 2 + 2$.

When $m = 6$ we are in Case 3 so $l_4 \leq l_3 \leq 2l_4$, and hence $S(6, 3) = l_3 = 104$.

Also note $m = 2$ also has the property that $l_4 \leq l_3 \leq 2l_4$, and indeed $S(2, 3) = l_3 = 8$ and so $V(1, 3) = 8 \times 2^2 = 32$, since $32 = 4^2 + 2^2 + 2^2 + 2^2 + 1 + 1 + 1 + 1 = 3^2 + 3^2 + 3^2 + 1 + 1 + 1 + 1 + 1 = 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2$. As a bonus here we see that $32 = 5^2 + 7 \times 1^2$ and hence four different sums of 8 squares are equal to 32. So we get $S(2, 4) = 8$ and hence $T(8, 2, 4) = T(8, 2, 3) = 32$.

Finally when $m = 12$ or $m = 36$, we have $2l_3 < l_4$ and thus both these values fall into Case 4.

5 More than 3 ways

In this section we present a general theorem and a conjecture about seeds. Again let $d = \gcd(3^m - 2^m, 2^m - 1)$, $l_3 = \frac{3^m-1}{d}$ and $l_4 = 2^m + 1$. Also let the sums

$$S_3 = \underbrace{3^m + \cdots + 3^m}_{\frac{2^m-1}{d}} + \underbrace{1 + \cdots + 1}_{\frac{3^m-2^m}{d}} \text{ and } S_4 = 4^m + \underbrace{1 + \cdots + 1}_{2^m}.$$

Theorem 15. *Given t, there exists a number m_0, such that if $m \geq m_0$, then the seed number $S(m, t)$ is bounded above by the $t - 1$st smallest of the values $al_3 + bl_4$ over all $a, b \geq 0$. The seed value $V(m, 3) = S(m, 3) \times 2^m$.*

Proof. Let $n = \min(l_3, l_4)$ and let $m_0 = \max\{m \mid 5^m < (t-1)n2^m\}$. Let n_0 be the $t-1$st smallest of the values $al_3 + bl_4$ over all $a, b \geq 0$. Finally define the sum $aS_3 + bS_4 + \overline{2^m}$ to be a copies of S_3 added to b copies of S_4 added to $n_0 - (al_3 + bl_4)$ copies of 2^m. Now it is clear that for all $al_3 + bl_4 \leq n_0$, the sum $aS_3 + bS_4 + \overline{2^m} = n_0 2^m$. But also, if $m \geq m_0$, then $k^m \geq (t-1)n2^m \geq n_0 2^m$ for all $k \geq 5$ and hence no sum with n_0 terms and equal to $n_0 2^m$ can contain any k^m for $k \geq 5$. Thus the $t-1$ sums $aS_3 + bS_4 + \overline{2^m}$ with the sum of m_0 2^m's are all equal, proving our upper bound. □

Note that in the proof of the previous theorem in order to obtain the upper bound it was not necessary to prove that no sum could contain a k^m for any $k \geq 5$. We included that fact in order to add credence to our conjecture below.

Indeed, we believe that the number presented in Theorem 15 is the actual seed number. We state this in the following conjecture. One can see that both Theorem 9 and Theorem 12 follow from this conjecture.

Conjecture 16. Given t, there exists a number m_0, such that if $m \geq m_0$, then the $t-1$st smallest of the values $al_3 + bl_4$ over all $a, b \geq 0$ is the seed number $S(m, t)$ and the seed value $V(m, 3) = S(m, 3) \times 2^m$.

6 Conclusion

The generalized taxicab number $T(n, m, t)$ is equal to the smallest number that is the sum of n m^{th} powers in t ways. This definition is inspired by Ramanujan's observation that $1729 = 1^3 + 12^3 = 9^3 + 10^3$ is the smallest number that is the sum of two cubes in two ways and thus $1729 = T(2, 3, 2)$. In this paper we first proved that for any given positive integers m and t, there exist a seed for the generalized taxicab number, i.e., there exists a number $s = S(m, t)$ such $T(s + k, m, t) = T(s, m, t) + k$ for every $k \geq 0$. We then found explicit expressions for this seed number when the number of ways t is 2 or 3. We ended with a general theorem and conjecture about the seed number $S(m, t)$ for all t.

Addendum: Research for this paper was mostly undertaken while the authors were together in their first year of graduate school in mathematics at The Ohio State University in 1974. This paper should have appeared shortly after that, but at least it is finally finished now. (It only took another 45 years).

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