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A Bijective Answer to a Question of Simion

Richard Ehrenborg
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
USA
richard.ehrenborg@uky.edu

Gábor Hetyei
Department of Mathematics and Statistics
UNC-Charlotte
Charlotte, NC 28223-0001
USA
ghetyei@uncc.edu

Margaret Readdy
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
USA
margaret.readdy@uky.edu

Abstract

We present a bijection between balanced Delannoy paths of length $2n$ and the faces of the n-dimensional Simion type B associahedron. This polytope is also known as the Bott-Taubes polytope and the cyclohedron. This bijection takes a path with k up steps (and k down steps) to a $(k-1)$-dimensional face of the Simion type B associahedron. We give two presentations of this bijection, one recursive and one non-recursive.

1 Introduction

Simion constructed a type B associahedron Q_n^B whose facets correspond to centrally symmetric triangulations of a regular $(2n+2)$-gon; see [10]. The vertices correspond to B-diagonals, that is, pairs of diagonals that are centrally symmetric. This polytope is also known as the Bott-Taubes polytope [4] and the cyclohedron [12]. We follow Simion's convention and view this polytope Q_n^B as a simplicial polytope.

Simion observed that the number of $(k-1)$-dimensional faces of the type B associahedron is given by

$$f_{k-1}(Q_n^B) = \binom{n}{k} \cdot \binom{n+k}{k}. \tag{1}$$

Simion asked if there is a combinatorial proof of this identity. A combinatorial interpretation for the right-hand side of equation (1) is the number of lattices paths between $(0,0)$ and $(2n,0)$ taking k up steps $(1,1)$, k down steps $(1,-1)$, and $n-k$ horizontal steps $(2,0)$. Such paths are known as *balanced Delannoy paths*. We provide a bijection between the faces of the type B associahedron and balanced Delannoy paths.

Our approach is based upon the paper [7], where the Simion type B associahedron Q_n^B is shown to be a pulling triangulation of the Legendre polytope. This polytope, also known as the full type A root polytope, is the convex hull of the set $\{e_i - e_j : 1 \le i,j \le n+1, i \neq j\}$. Hence it is natural to encode the vertices of the polytope Q_n^B with the pairs (i,j) where $i,j \in \{1,2,\ldots,n+1\}$ and $i \neq j$. We view these pairs as directed edges on the vertex set $\{1,2,\ldots,n+1\}$. We call these directed edges *arrows* to be consistent with [7] and not to overuse the term edge. Thus a face of the type B associahedron Q_n^B corresponds to a digraph on the vertex set $\{1,2,\ldots,n+1\}$.

In the paper [7] a condition is given on pairs of arrows when their associated B-diagonals cross. Reformulating this condition yields a description of those digraphs that correspond to faces of Q_n^B; see Proposition 2. We call these digraphs *facial*.

After preliminaries, in Section 3 we give a bijection between paths that never go below the x-axis, known as Schröder paths, and facial digraphs with the extra condition that all the arrows are directed backwards. Next, we extend this bijection to all digraphs and facial digraphs. This presentation of the bijection is recursive. In Section 4 we reformulate this presentation to obtain a non-recursive description of the bijection. Lastly, we end with concluding remarks.

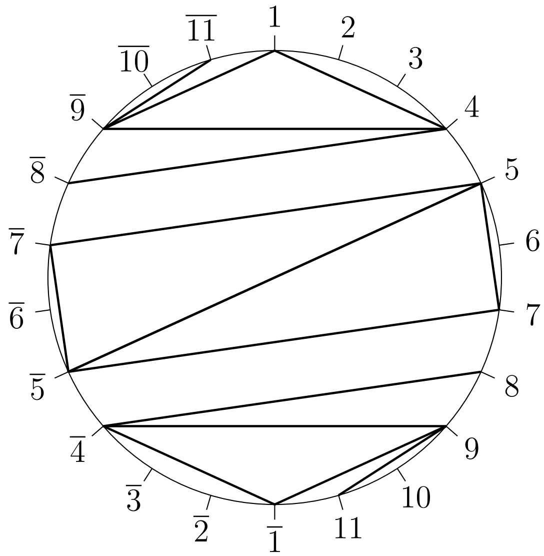

Figure 1: A set of B-diagonals corresponding to a 7-dimensional face of Q_{10}^B.

2 Preliminaries

2.1 The Simion type B associahedron

Following Simion [10], for a centrally symmetric convex $(2n+2)$-gon label its vertices in the clockwise order with $1, 2, \ldots, n, n+1, \overline{1}, \overline{2}, \ldots, \overline{n}, \overline{n+1}$. Let Γ_n^B denote the simplicial complex whose $n \cdot (n+1)$ vertices arise from the B-diagonals of the $(2n+2)$-gon, that is, (i) diagonals joining antipodal pairs of points, and (ii) antipodal pairs of noncrossing diagonals. In the first case, the diagonals are pairs of the form $\{i, \overline{i}\}$ satisfying $1 \leq i \leq n+1$. In the second case, the B-diagonals are either of the form $\{\{i, j\}, \{\overline{i}, \overline{j}\}\}$ with $1 \leq i < i+1 < j \leq n+1$ or of the form $\{\{i, \overline{j}\}, \{\overline{i}, j\}\}$ with $1 \leq j < i \leq n+1$. Throughout the paper we identify $\overline{i}$ with $i+n+1$. The simplicial complex Γ_n^B is then the family of sets of pairwise noncrossing B-diagonals.

Simion [10, Theorem 1] showed that the simplicial complex Γ_n^B is the boundary complex of an n-dimensional convex polytope Q_n^B, now known as the *Simion type B associahedron*. See Proposition 1 and Corollary 1 in [10] for Simion's computation of the f- and h-vectors of this polytope.

We define an *arrow* to be pair (i, j) where $1 \leq i, j \leq n+1$ and $i \neq j$. We call i the *tail* of the arrow (i, j) and j the *head* of the arrow. An arrow (i, j) is a *forward arrow* if $i < j$ and a *backward arrow* if $i > j$. Two undirected arcs $\{a, b\}$ and $\{c, d\}$ *cross* if $a < c < b < d$ or $c < a < d < b$ where we assume $a < b$ and $c < d$. Furthermore, the arc $\{a, b\}$ *nests* the arc $\{c, d\}$ if $a < c < d < b$ where we again assume $a < b$ and $c < d$. These two concepts naturally lift to arrows.

Define a bijection between B-diagonals and arrows as follows.

Definition 1. For a B-diagonal $\{\{u, v\}, \{u+n+1, v+n+1\}\}$ where $1 \leq u \leq n+1$, $u < v \leq u+n+1$ and the addition $v+n+1$ is modulo $2n+2$, let the associated arrow be (w, u) where $w \equiv v-1 \bmod n+1$ and $1 \leq w \leq n+1$. The case of two antipodal points is covered when $v = u+n+1$. The resulting arrow is $(u-1, u)$ when $2 \leq u \leq n+1$ and the arrow is $(n+1, 1)$ when $u = 1$.

From the proof of Proposition 5.5 in [7], we have the following result.

Proposition 2. *A non-empty digraph on the node set $\{1, 2, \ldots, n+1\}$ represents a proper face of the Simion type B associahedron Q_n^B exactly when the following conditions are satisfied:*
(1) There are no crossings between arrows.

(2) Forward arrows nest.

(3) A backward arrow cannot nest a forward arrow.

(4) No head of an arrow is the tail of another arrow.

We call such a set of arrows a *facial* digraph. An example of a facial digraph is shown in Figure 2.

Let P be a polytope with a vertex v. We can *pull* the vertex v, that is, we consider the convex hull of the polytope P and a new point w, where $\overrightarrow{vw}$ is a small vector in general position pointing outwards from the polytope. Given a polytope P and an ordering of its vertices $v_1, v_2, \ldots, v_m$, we can pull all the vertices in this given order. Namely, pick a real number $0 < \epsilon < 1$. When pulling the ith vertex, pull it a distance of ϵ^i. The resulting polytope Q is simplicial, that is, the boundary of Q is a triangulation of the boundary of the polytope P. Furthermore, the combinatorics of Q only depends on the original polytope P and the pulling order of the vertices. The idea of pulling triangulation is due to Hudson [9, Lemma 1.4]. See Athanasiadis [2, End of Section 2], Hetyei [8, Section 2.3] and Stanley [11, Lemma 1.1], for modern treatments, as well as the book by de Loera, Rambau, and Santos [6] for other types of triangulations.

We identify the arrow (i, j) with the point $e_j - e_i$ in $(n+1)$-dimensional Euclidean space. The convex hull of these $n \cdot (n+1)$ points forms a convex polytope known as the *Legendre polytope* [8]. This polytope is also known as the full root polytope of type A; see [1, 5]. Hetyei proved that any pulling triangulation of the Legendre polytope yields a simplicial polytope whose f-vector is given by (1); see [8]. In the paper [7] the authors show that there is a pulling triangulation of Legendre polytope such that the resulting polytope is the Simion type B associahedron. Proposition 2 arises from understanding this pulling triangulation.

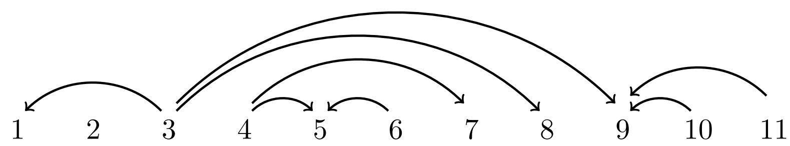

Figure 2: The facial digraph corresponding to the set of B-diagonals in Figure 1.

3 A bijection between the faces of Γ_n^B and Delannoy paths

In this section we establish a bijection between the faces of Γ_n^B and balanced Delannoy paths of length $2n$ in such a way that $(k-1)$-dimensional faces correspond to Delannoy paths with k up steps, thus answering Simion's question. The study of Delannoy paths has a long history. See [3] for a comprehensive survey.

Definition 3. A *balanced Delannoy path* of length $2n$ is a lattice path starting at $(0,0)$, ending at $(2n,0)$ and using only steps of the following three types: up steps $(1,1)$, down steps $(1,-1)$ and horizontal steps $(2,0)$. A *Schröder path* is a balanced Delannoy path that never goes below the horizontal axis.

More generally, a *Delannoy path* is a path taking the steps $(1,1)$, $(1,-1)$ and $(2,0)$ with no condition where the path ends. In this paper we only work with balanced Delannoy paths.

Let U, D and H denote the up, down and horizontal steps, respectively. Let the *length* of the letters U and D each to be 1, whereas the length of the letter H be 2. Note that the length of a word is the sum of the lengths of its letters. For example, the word $DUUHH$ has length 7. A Delannoy path is uniquely encoded by a word in these three letters. We call the associated word a *Delannoy word*. A balanced Delannoy path of length $2n$ corresponds to a word in which the number k of occurrences of U is the same as the number of occurrences of D, and the number of occurrences of H is $n-k$. We call such a word *balanced*. Hence the number of Delannoy paths of length $2n$ with k up steps is given by the multinomial coefficient $\binom{n+k}{k,k,n-k} = \binom{n+k}{k}\binom{n}{k}$, which is the same as the number of $(k-1)$-dimensional faces in the Simion type B associahedron Q_n^B; see equation (1). A *Schröder word* is a balanced Delannoy word such that the associated path never goes below the horizontal axis. Let $\alpha \cdot \beta$ denote the concatenation of the two words α and β.

Let $\mathbb{P}$ be the set of positive integers ordered by the natural order. We will consider facial digraphs A on a finite subset V of the positive integers. Recall that the definition of a facial digraph is the conditions appearing in Proposition 2. We view a digraph A as a pair consisting of a node set $V = V(A)$ and an arrow set $E = E(A)$. For a facial digraph A on a non-empty finite set of nodes V and $W \subseteq V$ let A_W denote the induced subgraph on W.

Let A be a facial digraph of backward arrows on a non-empty finite set of nodes $V \subseteq \mathbb{P}$

and let v be the minimal element of V. We define the *Schröder word* $\mathrm{SP}(A)$ of the facial digraph A by the following recursive definition.

(i) If the set V only consists of one node, that is, $V = \{v\}$, let $\mathrm{SP}(A)$ be the empty word ϵ.

(ii) If the minimal node v is an isolated node of A, let $\mathrm{SP}(A)$ be the concatenation $H \cdot \mathrm{SP}(A_{V-\{v\}})$.

(iii) Lastly, when the node v is not isolated then since v is the least element, there is necessarily a backward arrow (x, v) going into v. Let w be the smallest node such that (w, v) is an arrow of A, that is, $w = \min(\{x \in V \,:\, (x, v) \in A\})$. Define $\mathrm{SP}(A)$ by

$$\mathrm{SP}(A) = U \cdot \mathrm{SP}(A_{V\cap(v,w]}) \cdot D \cdot \mathrm{SP}(A_{V-(v,w]}).$$

Lemma 4. *Let A be a facial digraph on a set of nodes of size $n + 1$ consisting of exactly k backward arrows. The Schröder word $\mathrm{SP}(A)$ then has length $2n$ and has exactly k copies of the letters U and k copies of the letters D.*

Proof. Both statements follow by induction on n. The first statement follows from the fact that the two sets of nodes $V \cap (v, w]$ and $V - (v, w]$ are complements of each other. The second statement follows by observing that the noncrossing property ensures that each arrow in A is either the arrow (w, v), an arrow in $A_{V\cap(v,w]}$, or an arrow in $A_{V-(v,w]}$. □

Proposition 5. *The map SP is a bijection between the set of all facial digraphs consisting only of backward arrows on the set of $n + 1$ nodes and all Schröder words of length $2n$.*

Proof. Given a Schröder word α of length $2n$ and a node set $V = \{v_1 < v_2 < \cdots < v_{n+1}\}$, the inverse map is computed recursively as follows. If α is the empty word ϵ then the inverse image is the isolated node v_1. If the word α begins with H, that is, $\alpha = H \cdot \beta$ where β has length $2n - 2$, compute the inverse image of β on the nodes $\{v_2, v_3, \ldots, v_{n+1}\}$ and add the node v_1 as an isolated node. Otherwise factor α uniquely as $U \cdot \beta \cdot D \cdot \gamma$, where β and γ are Schröder words of lengths $2p$ and $2n - 2p - 2$, respectively. Compute the inverse image of β on the nodes $\{v_2, v_3, \ldots, v_{p+2}\}$. Similarly, compute the inverse image of γ on the nodes $\{v_1, v_{p+3}, v_{p+4}, \ldots, v_{n+1}\}$. Take the union of these two digraphs and add the backward arrow (v_{p+2}, v_1). □

We now extend the bijection SP to all facial digraphs. In order to do so we introduce a *twisting operation* tw on each digraph A that has a forward arrow from the least element $v \in V$ to the largest element $w \in V$. The *twisted digraph* $\mathrm{tw}(A)$ is a digraph on the node set $V - \{v\}$ with arrow set

$$E(\mathrm{tw}(A)) = E(A_{V-\{v\}}) \,\cup\, \{(w, z) \,:\, (v, z) \in A, z \neq w\}.$$

In other words, we remove the least node v and replace each forward arrow (v, z) starting at v with a backward arrow (w, z). Note that there is no backward arrow (z, v) in A as v is the tail of the forward arrow (v, w).

Lemma 6. *Let V be a node set whose smallest (respectively, largest) node is v (respectively, w). The twisting map is a bijection from the set of facial digraphs on the set V containing the forward arrow (v, w) to the set of facial digraphs on the set $V - \{v\}$.*

Proof. We claim that the twisted digraph $\mathrm{tw}(A)$ is a facial digraph. Note that the restriction $A_{V-\{v\}}$ is a facial digraph. If there are no forward arrows in A of the form (v, z) where $z < w$, the equality $\mathrm{tw}(A) = A_{V-\{v\}}$ holds and the claim is true. Hence assume that there are forward arrows of the form (v, z). We need to show that the digraph remains facial after adding the backward arrow (w, z). We verify conditions (1) through (4) of Proposition 2 in order. If the arrow (w, z) crosses an arrow (x, y) then the arrow (v, z) already crossed this arrow in A, verifying condition (1). Since we did not introduce any new forward arrows, condition (2) holds vacuously. Assume that the backward arrow (w, z) nests a forward arrow (x, y). Then the forward arrows (v, z) and (x, y) did not nest in A, verifying (3). Condition (4) holds directly, proving the claim.

The twisting map is one-to-one. The inverse has the node set $V(\mathrm{tw}^{-1}(B)) = V(B) \cup \{v\}$ and the arrow set is given by

$$E(\mathrm{tw}^{-1}(B)) = \{(v, w)\} \;\cup\; E(B_{V\cap(v,w)}) \;\cup\; \{(v, z) \;:\; (w, z) \in B\},$$

where $V \cap (v, w)$ denotes the intersection of the set V with the open interval (v, w). Note that in case the node w was a 'tail node', the inverse map switches it back to being a 'head node'. □

We now extend the bijection SP to a map DP which applies to all facial digraphs. Let A be facial digraph on the node set V. Furthermore, let v and w be the minimal, respectively, the maximal node of the node set V, that is, $v = \min(V)$ and $w = \max(V)$.

(i) If the facial digraph A has no forward arrows, let $\mathrm{DP}(A) = \mathrm{SP}(A)$.

(ii) If the facial digraph A has a forward arrow, let (x, y) be the forward arrow which nests the other forward arrows. Since $v \leq x < y \leq w$, observe that the digraphs $A_{V\cap[v,x]}$ and $A_{V\cap[y,w]}$ have no forward arrows. Define $\mathrm{DP}(A)$ to be the concatenation

$$\mathrm{DP}(A) = \mathrm{SP}(A_{V\cap[v,x]}) \cdot D \cdot \mathrm{DP}(\mathrm{tw}(A_{V\cap[x,y]})) \cdot U \cdot \mathrm{SP}(A_{V\cap[y,w]}).$$

As an extension of Lemma 4 we have the following lemma.

Lemma 7. *Let A be a facial digraph on a set V of $n+1$ nodes and assume that A consists of k arrows. Then the balanced Delannoy word $\mathrm{DP}(A)$ has length $2n$ and has exactly k copies of the letters U and D, respectively.*

Proof. We only have to prove the lemma for the second case defining DP. Assume that $V \cap [v, x]$ and $V \cap [y, w]$ have cardinalities a and b, respectively. Then the middle part $V \cap [x, y]$ has size $n - a - b + 3$. Thus $\mathrm{tw}(A_{V\cap[x,y]})$ has $n - a - b + 2$ nodes. Hence the total

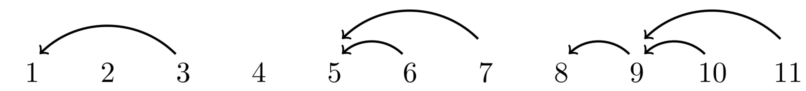

Figure 3: The modified set of arrows obtained from the set in Figure 2.

length is $2\cdot(a-1)+2\cdot(n-a-b+1)+2\cdot(b-1)+2=2n$. For the second case of the lemma, assume that the restricted digraphs $A_{V\cap[v,x]}$ and $A_{V\cap[y,w]}$ have c and d arrows, respectively. Since there is no arrow nesting the forward arrow (x,y), the middle digraph $A_{V\cap[x,y]}$ has $k-c-d$ arrows. Thus the twisted digraph has one less arrow, namely $k-c-d-1$. Hence the total number of U letters is $c+(k-c-d-1)+d+1=k$, proving the second claim. $\square$

We now extend Proposition 5 to all facial digraphs.

Theorem 8. *The map* DP *is a bijection between the set of all facial digraphs on a set of* $n+1$ *nodes and the set of all balanced Delannoy words of length* $2n$.

Proof. The inverse of the map DP is computed as follows. Let α be a balanced Delannoy word of length $2n$ and V be a node set $\{v_1 < v_2 < \cdots < v_{n+1}\}$. If the Delannoy word α is a Schröder word, apply the inverse map of Proposition 5. Otherwise, we can factor the Delannoy word α uniquely as $\beta\cdot D\cdot\gamma\cdot U\cdot\delta$, where γ is a balanced Delannoy word, and β and δ are Schröder words. Note that the factor $\beta\cdot D$ is uniquely determined by the fact it is the shortest initial segment in which the number of D letters exceeds the number of U letters, in other words, the encoded path ends with the first down step going below the horizontal axis. By a symmetric argument, $U\cdot\delta$ is the shortest final segment with one more U than the number of D's. Assume that β, γ and δ have lengths $2p$, $2q$ and $2n-2p-2q-2$, respectively. Apply the inverse map from the proof of Proposition 5 to the words β and δ to obtain digraphs on $\{v_1, v_2, \ldots, v_{p-1}\}$, respectively $\{v_{p+q+2}, v_{p+q+3}, \ldots, v_{n+1}\}$.

By recursion, apply the inverse of DP to γ to obtain a facial digraph on the node set $\{v_{p+2}, v_{p+3}, \ldots, v_{p+q+2}\}$. Then apply the inverse of the twisting map to the digraph thus obtained. Finally, take the union of these three digraphs. $\square$

4 A non-recursive description of the bijection

We now give non-recursive description of the bijection DP. First we encode a facial digraph with a multiset of indexed letters.

Definition 9. Given a facial digraph A on the node set V, define the *associated multiset* $M(A)$ with elements from the set of indexed letters $\{D_x, U_x, H_x : x \in \mathbb{P}\}$ by the following three steps:

1. For each $x < \max(V)$ which is a maximum element in a weakly connected component we add the letter H_x to the multiset $M(A)$.

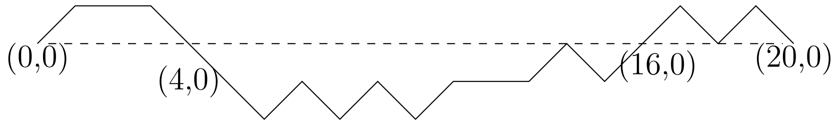

Figure 4: The lattice path associated to the digraph in Figure 2.

2. For each tail x of a forward arrow, consider the set $\mathrm{Head}(x) = \{y > x : (x, y) \in A\}$, that is, the set of heads of forward arrows with tail x. Add a copy of the letter D_x to $M(A)$, also add a copy of the letter U_w for $w = \max(\mathrm{Head}(x))$. Remove the forward arrow (x, w). For each remaining $y \in \mathrm{Head}(x)$ that is less than w, replace the forward arrow (x, y) with the backward arrow (w, y). The resulting set of arrows has only backward arrows.

3. For each head y of a backward arrow, consider the set $\mathrm{Tail}(y) = \{x > y : (x, y) \in A\}$, that is, the set of tails of backward arrows with head y. Add a copy of U_y to $M(A)$. Also add a copy of D_x for $x \in \mathrm{Tail}(y)$ and add a copy of U_x for all but the maximum element of $\mathrm{Tail}(y)$.

Example 10. For the facial set of arrows A shown in Figure 2, in the first step we add the letters H_2 and H_7 to $M(A)$. In the second step we add the letters D_3, U_9, D_4, and U_7 to $M(A)$, and we remove the forward arrows $(3, 9)$ and $(4, 7)$. We also replace $(3, 8)$ with $(9, 8)$ and $(4, 5)$ with $(7, 5)$. We obtain the set of backward arrows shown in Figure 3. Note that this set of arrows does not need to be facial anymore: in our example, 9 is the head of $(10, 9)$ and $(11, 9)$ and it is also the tail of $(9, 8)$. Finally, in step three we add the letters U_1, D_3, U_5, D_6, U_6, D_7, U_8, D_9, U_9, D_{10}, U_{10}, and D_{11} to $M(A)$. We end up with the multiset

$$M(A) = \{U_1, H_2, D_3, D_3, D_4, U_5, D_6, U_6, D_7, U_7, H_7, U_8, D_9, U_9, U_9, D_{10}, U_{10}, D_{11}\}.$$

Define a linear order on the indexed letters by the inequalities $D_x < U_x < H_x < D_{x+1}$ for all positive integers x. We obtain the lattice path as follows.

Proposition 11. *The balanced Delannoy word* $\mathrm{DP}(A)$ *is obtained from the multiset* $M(A)$ *by reading the indexed letters in order and then omitting the subscripts.*

For the set of arrows shown in Figure 2, we obtain the word

$$UHDDDUDUDUHUDUUDUD.$$

The lattice path encoded by this word is shown in Figure 4.

Recall that a *weakly connected component* of a digraph is a connected component in the graph obtained by disregarding the direction of the arrows. In our facial digraphs, the weakly connected components are trees.

A quick outline of a proof of Proposition 11 is as follows. First prove the statement for facial digraphs without forward arrows. In this case, each nonmaximal element x of

Tail(y) contributes two consecutive letters $D_x U_y$. Next, observe that a digraph and its twisted digraph have the same weakly connected components. Furthermore, the arrows that moved under the twisted operation still have the same set of heads. Hence, after recording the horizontal steps in step (1) we may perform all twisting operations simultaneously in step (2). We leave the remaining details to the reader.

5 Concluding remarks

The faces of the type B associahedron Q_n^B have a natural order by inclusion. What is the image of this order under our bijection? Is there a different ordering of the balanced Delannoy paths that would yield a more natural order preserving bijection?

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