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Log-Concavity of Recursively Defined Polynomials

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Abstract

Fourier coefficients of powers of the Dedekind eta function can be studied by polynomials introduced by M. Newman. We generalize the defining recurrence relations in this paper. From this we derive new families of polynomials, which approximate these polynomials from below and above. Although these families are recursively defined, we are able to determine explicit closed formulas for both approximating polynomials. (For the original polynomials closed formulas are not yet known.) Furthermore, we obtain that both approximating families and the coefficients involved are log-concave and unimodal.

1 Introduction

Let $\eta(\tau)$ be the Dedekind eta function [7]. Put $q := e^{2\pi i \tau}$, where τ is in the upper half-space $\mathbb{H}$. Let further $z \in \mathbb{C}$. We consider the Fourier expansion

$$\left(q^{-\frac{1}{24}}\eta(\tau)\right)^{-z} = \prod_{n=1}^{\infty} (1-q^n)^{-z} = \sum_{n=0}^{\infty} P_n(z)\, q^n. \tag{1}$$

We recover several famous sequences [2, Introduction]. For example, the partition numbers $p(n)$ and the Ramanujan numbers $\tau(n)$ are directly linked to $z=1$ and $z=-24$,

$$\begin{aligned}(P_n(1))_{n=0}^{\infty} = (p(n))_{n=0}^{\infty} &: \quad 1,1,2,3,5,\ldots \quad (\text{A000041});\\ (P_{n-1}(-24))_{n=1}^{\infty} = (\tau(n))_{n=1}^{\infty} &: \quad 1,-24,252,\ldots \quad (\text{A000594}).\end{aligned}$$

The integer-valued polynomials $P_n(X)$ have degree n and positive integral coefficients A_k^n for $0 \leq k \leq n-1$:

$$P_n(X) = \frac{X}{n!} \sum_{k=0}^{n-1} A_k^n \; X^k. \tag{2}$$

Newman [5] calculated the coefficients recursively for $0 \leq k \leq n-1$. For example $A_0^n = (n-1)!\sigma(n)$, $A_{n-1}^n = 1$, and $A_{n-2}^n = 3n(n-1)/2$. Here $\sigma(n)$ is the sum of the divisors of n. Further Newman determined the first ten polynomials and their integral zeros and found a recursion formula for the involved polynomials. Apparently there is yet no closed explicit formula for these polynomials defined by

$$P_n(X) = \frac{X}{n} \sum_{k=1}^{n} \sigma(k) P_{n-k}(X) \tag{3}$$

and the involved coefficients.

Numerical calculations indicate that the sequence of the coefficients A_k^n seems to be close to unimodal and stable (Hurwitz polynomial). Here we normalize $P_n(X)$ by $n!/X$. A direct approach seems to be out of reach at the moment, since there is still a lack of understanding the properties of the arithmetic function $\sigma(n)$ completely. Nevertheless the value distribution of the polynomials $P_n(X)$, especially the zeros, have significant applications. For r even and positive, the sequences $\left(P_n(-r)\right)_n$ are lacunary if and only if $r = 2,4,6,8,10,14,26$. This result is due to Serre [8] and is in relation to modular forms with complex multiplication (CM-forms). The non-existence of -24 as a zero is equivalent to the Lehmer conjecture [4].

In this paper we generalize the recurrence relation (3), studying closed formulas for the polynomials associated with the functions $g(n) = n$ and $g(n) = n^2$. They are integer valued and approximate $\sigma(n)$ from below and above.

Definition 1. Let $g(n)$ be an arithmetic function. We define a family of polynomials $P_n^g(X)$ associated with g. Let $P_0^g(X) := 1$ and

$$\begin{aligned} P_n^g(X) \; &:= \; \frac{X}{n} \sum_{k=1}^{n} g(k) \; P_{n-k}^g(X), & (4)\\ &= \; \frac{X}{n!} \sum_{k=0}^{n-1} A_k^n(g) \, X^k. & (5)\end{aligned}$$

Let $\varphi_1(n) = n$ and $\varphi_2(n) = n^2$. In what follows, we study the properties of the associated polynomials $P_n^{\varphi_1}(X)$ and $P_n^{\varphi_2}(X)$. Their properties are related to $P_n(X) = P_n^{\sigma}(X)$, since

$\varphi_1(n) < \sigma(n) < \varphi_2(n)$ for $n > 1$. We obtain $A_{n-1}^n(\varphi_1) = A_{n-1}^n = A_{n-1}^n(\varphi_2) = 1$. Further, let $0 \leq k \leq n-2$. Then

$$A_k^n(\varphi_1) < A_k^n < A_k^n(\varphi_2). \tag{6}$$

Theorem 2. *Let $\varphi_1(n) = n$. Then the coefficients of $P_n^{\varphi_1}(X)$ are given by*

$$A_k^n(\varphi_1) = \frac{n!}{(k+1)!}\binom{n-1}{k}. \tag{7}$$

Although the binomial coefficients are twisted by $1/(k+1)!$, they remain log-concave.

Corollary 3. *The sequence of the coefficients of $P_n^{\varphi_1}(X)$*

$$\left(A_k^n(\varphi_1)\right)_{k=0}^{n-1}$$

is strongly log-concave and hence unimodal.

Further we can determine the index, and therefore the size of the largest coefficient.

Corollary 4. *The index K_1 of the maximal coefficient $A_k^n(\varphi_1)$ is given by*

$$\sqrt{n+1} - 2 \leq K_1 < \sqrt{n+1} - 1. \tag{8}$$

In general, K_1 is not always unique.

Numerical calculations indicate that the zeros of $P_n^{\varphi_1}(X)$ are simple and that the polynomials are stable in the sense of Hurwitz (neglecting $X = 0$).

It is also possible to get similar results for $P_n^{\varphi_2}(X)$, although the involved coefficients $A_k^n(\varphi_2)$ are more complicated. We also studied the polynomials $P_n^{\varphi}(X)$ associated with $\varphi(n) := n\ln(n)$ and $\varphi(n) := n\sqrt{n}$. But a closed formula for the coefficients is very difficult to obtain.

Theorem 5. *Let $\varphi_2(n) = n^2$. Then the coefficients of $P_n^{\varphi_2}(X)$ are given by*

$$A_k^n(\varphi_2) = \frac{n!}{(k+1)!}\binom{n+k}{2k+1}. \tag{9}$$

Corollary 6. *The sequence of the coefficients of $P_n^{\varphi_2}(X)$*

$$\left(A_k^n(\varphi_2)\right)_{k=0}^{n-1}$$

is strongly log-concave and hence unimodal.

Corollary 7. *The coefficients $A_k^n(\varphi_2)$ assume their maximum at the index K_2, such that $K_2 \leq k < K_2 + 1$ is the real solution of the cubic equation $4k^3 + 19k^2 + 28k + 13 = n^2$. Let*

$$D := \frac{1}{8}n^2 - \frac{91}{1728} + \frac{1}{144}\sqrt{324n^4 - 273n^2 - 51},$$

then $\sqrt[3]{D} + \frac{25}{144\sqrt[3]{D}} \leq K_2 < \sqrt[3]{D} + \frac{25}{144\sqrt[3]{D}} + 1$. For large n we obtain $K_2 \approx (n/2)^{2/3}$.

Numerical calculations indicate that the zeros of $P_n^{\varphi_1}(X)$ are simple and that the polynomials are stable.

Let x_0 be a positive real number. Let g_1, g_2 be two arithmetic functions, satisfying $1 \leq g_1(n) \leq g_2(n)$. Then $P_n^{g_1}(x_0) \leq P_n^{g_2}(x_0)$. We let $p(n)$ denote the partition numbers [1]. As an application we obtain an approximation of the partition numbers from below. Let $N = 200$. Then

$$\frac{p(n)}{P_n^{\varphi_1}(1)} < 647.71 \qquad \text{for } n \leq N. \tag{10}$$

Finally we obtain for $x = -1$ the following result.

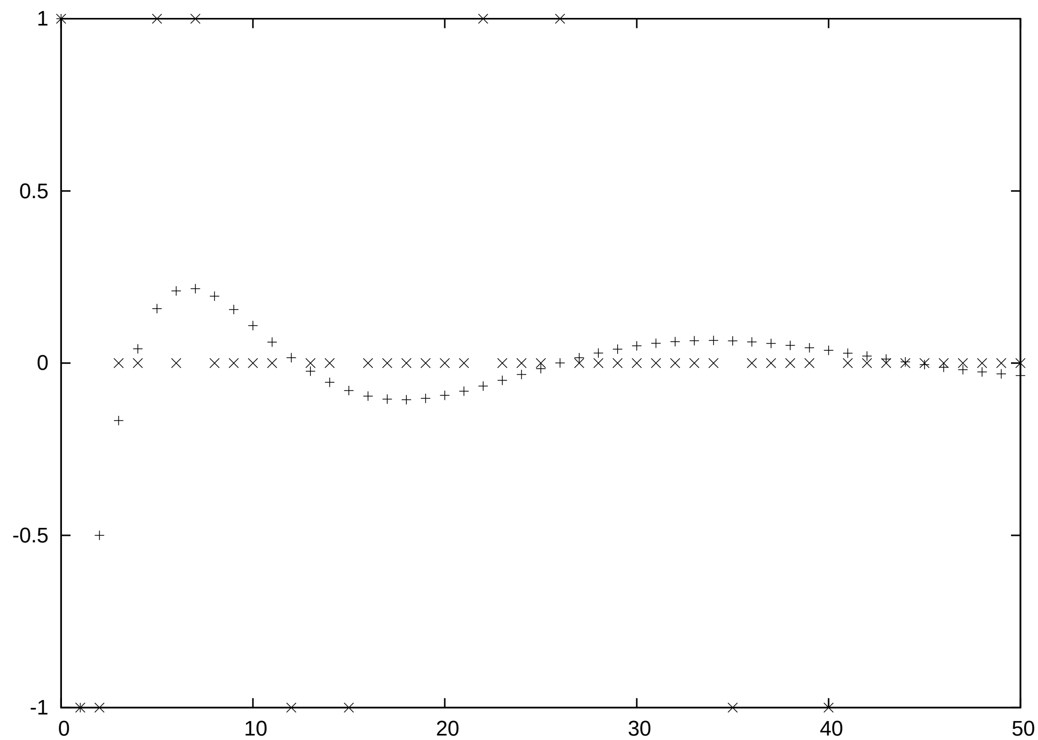

Figure 1: '+' marks the value of $P_n^{\varphi_1}(-1)$, '×' marks the value of $P_n^{\sigma}(-1)$ depending on n.

Euler showed that $P_n(-1)$ takes only the values $-1, 0, 1$ and has high vanishing rate, called superlacunary [6] in the language of modular forms. This is reflected by the asymptotic behavior of the values of the sequence $P_n^{\varphi_1}(-1)$, although $x_0 < 0$.

2 Log-Concavity and maximal coefficients

A sequence $a_0, a_1, a_2, \ldots, a_n$ of real numbers is called unimodal if the sequence increases steadily at first and then decreases steadily [9]. The sequence of binomial coefficients and

Stirling numbers are unimodal. In the case where

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_K \geq a_{K+1} \geq a_{K+2} \geq \cdots \geq a_m, \tag{11}$$

the K is called the *index* of the sequence. The index does not have to be unique. In general, it is not clear how to determine the index.

Another important property of some sequences is (strong) log-concavity. A sequence is called *log-concave* if for all $k = 1, 2, \ldots, m-1$:

$$a_k^2 \geq a_{k+1}\, a_{k-1}. \tag{12}$$

Note that log-concavity implies unimodality.

Due to Newton's inequality, the sequence of coefficients of a polynomial with (real coefficients and) only real zeros is strongly log-concave. It is very likely that $P_n^{\varphi_1}(X)$ and $P_n^{\varphi_2}(X)$ have only real zeros. But $P_n(X)$ definitely does not have only real zeros. Heim et al. [3] showed that, for example, in the case $n = 10$ non-real zeros appear. Nevertheless, directly applying the log-concave criterion works for φ_1, φ_2, due to the explicit formulas proved in this paper.

2.1 Proof of Corollary 3

Let $n \geq 3$ be given. We consider the sequence

$$a_k := A_k^n(\varphi_1) = \frac{n!}{(k+1)!} \binom{n-1}{k}$$

for $k = 0, \ldots, n-1$. We show that for each pair (n, k) with $1 \leq k \leq n-2$ an $\alpha > 1$ exists, such that

$$\frac{A_k^n(\varphi_1)^2}{A_{k+1}^n(\varphi_1)\, A_{k-1}^n(\varphi_1)} \geq \alpha. \tag{13}$$

This quotient results in

$$\begin{aligned}
\frac{\left(\frac{1}{(k+1)!}\binom{n-1}{k}\right)^2}{\frac{1}{k!}\binom{n-1}{k-1}\frac{1}{(k+2)!}\binom{n-1}{(k+1)}} \quad &= \quad \frac{\frac{1}{k+1}\left(\frac{(n-1)!}{k!(n-k-1)!}\right)^2}{\frac{(n-1)!}{(k-1)!(n-k)!}\frac{1}{k+2}\frac{(n-1)!}{(k+1)!(n-k-2)!}} \\
&= \quad \frac{\frac{1}{k+1}\frac{1}{k(n-k-1)}}{\frac{1}{n-k}\frac{1}{k+2}\frac{1}{k+1}} \\
&= \quad \frac{(n-k)\,(k+2)}{k\,(n-k-1)} \geq \alpha > 1.
\end{aligned}$$

Hence the sequence $A_k^n(\varphi_1)$ is strongly log-concave. This implies unimodality.

We obtain $\alpha = \alpha_n \geq 1 + \left(3n + 1 + 2\sqrt{2n\,(n+1)}\right)(n-1)^{-2}$. Depending on n α decreases to 1.

To show that α decreases to 1, consider the denominator of the derivative of $\frac{(n-k)(k+2)}{k(n-1-k)}$ with respect to k. The denominator yields $2n\,(n+1)-(k-2n)^2$. Hence the minimum of the denominator for $1 \leq k \leq n-2$ is obtained at $k = 2n - \sqrt{2n\,(n+1)}$. This yields the claimed value.

2.2 Proof of Corollary 6

Let $n \geq 2$ be given. We consider the sequence

$$a_k := A_k^n(\varphi_2) = \frac{n!}{(k+1)!}\binom{n+k}{2k+1}$$

for $k = 0, \ldots, n-1$. Hence $a_0, a_1, \ldots, a_m$ with $m = n-1$. We show that for each pair (n,k) with $0 \leq k \leq n-2$ a $\beta > 1$ exists such that

$$\frac{A_k^n(\varphi_2)^2}{A_{k+1}^n(\varphi_2)\,A_{k-1}^n(\varphi_2)} > \beta. \tag{14}$$

This quotient results in

$$\begin{aligned}
\frac{\left(\frac{1}{(k+1)!}\binom{n+k}{2k+1}\right)^2}{\frac{1}{k!}\binom{n+k-1}{2k-1}\frac{1}{(k+2)!}\binom{n+k+1}{2k+3}} &= \frac{\frac{1}{k+1}\left(\frac{(n+k)!}{(2k+1)!(n-k-1)!}\right)^2}{\frac{(n+k-1)!}{(2k-1)!(n-k)!}\frac{1}{k+2}\frac{(n+k+1)!}{(2k+3)!(n-k-2)!}} \\
&= \frac{\frac{1}{k+1}\frac{n+k}{(2k+1)2k(n-k-1)}}{\frac{1}{n-k}\frac{1}{k+2}\frac{n+k+1}{(2k+3)(2k+2)}} \\
&= \frac{(n+k)\,(n-k)\,(k+2)\,(2k+3)\,(2k+2)}{(k+1)\,(n+k+1)\,(2k+1)\,2k\,(n-k-1)} \\
&> \frac{n^2-k^2}{n^2-(k+1)^2} \geq \beta > 1.
\end{aligned}$$

Hence the sequence $A_k^n(\varphi_2)$ is strongly log-concave. This implies unimodality.

We have $\frac{n^2-k^2}{n^2-(k+1)^2} = 1 + \frac{2k+1}{n^2(k+1)^2}$. This is increasing as a function of k for $1 \leq k \leq n-2$. So the minimum is attained at $k=1$ and results in $1+\frac{3}{n^2-4}$. For $n \to \infty$ $1+\frac{3}{n^2-4}$ again decreases to 1. Even if we could determine the minimum of the expression before the 'strictly less' sign, we would obtain the same limiting behavior.

2.3 Proof of Corollary 4

The idea is simple. Let $n \geq 2$. Let $a_k = A_k^n(\varphi_1)$, $0 \leq k \leq n-1$. We analyze the possible sign changes of

$$\Delta_k(\varphi_1) := a_{k+1} - a_k = A_{k+1}^n(\varphi_1) - A_k^n(\varphi_1), \quad 0 \leq k \leq n-2. \tag{15}$$

We obtain

$$\Delta_k(\varphi_1) = \gamma(k,n) \left(\frac{1}{(k+2)\,(k+1)} - \frac{1}{n-k-1} \right), \tag{16}$$

where $\gamma(k,n)$ is a positive rational number. Hence $\Delta_k(\varphi_1)$ has the same sign as

$$n + 1 - (k+2)^2.$$

Hence the coefficients increase for $k < \sqrt{n+1} - 2$, and decrease for $k > \sqrt{n+1} - 2$. In the case of equality we have two maxima.

2.4 Proof of Corollary 7

Let $n \geq 2$. Let $a_k = A_k^n(\varphi_2)$, $0 \leq k \leq n-1$. We analyze the possible sign changes of

$$\Delta_k(\varphi_2) := a_{k+1} - a_k = A_{k+1}^n(\varphi_2) - A_k^n(\varphi_2), \quad 0 \leq k \leq n-2. \tag{17}$$

We obtain

$$\Delta_k(\varphi_2) = \gamma'(k,n) \left(\frac{n+k+1}{(k+2)\,(2k+3)\,(2k+2)} - \frac{1}{n-k-1} \right). \tag{18}$$

where $\gamma'(k,n)$ is a positive rational number. Hence $\Delta_k(\varphi_2)$ has the same sign as

$$-4k^3 - 19k^2 - 28k - 13 + n^2.$$

As a function of $k \geq 0$, the expression $\Delta_k(\varphi_2)$ is decreasing. Hence there exists exactly one $0 \leq K \leq n-1$ such that

$$A_{K-1}^n(\varphi_2) < A_K^n(\varphi_2) \geq A_{K+1}^n(\varphi_2).$$

We used the computer algebra system `Maple` to calculate the algebraic expression defining K. Nevertheless, it is obvious that $K \approx 2^{-2/3}\, n^{\frac{2}{3}}$ for large n.

3 Explicit formulas for $P_n^{\varphi_1}(X)$

Let us start with a list of the polynomials for $n = 1, 2, \ldots, 7$. Note that they are no longer integer-valued polynomials, although the modified coefficients are integral.

$$\begin{aligned}
P_1^{\varphi_1}(X) &= X \\
P_2^{\varphi_1}(X) &= \frac{1}{2} X\,(X+2) \\
P_3^{\varphi_1}(X) &= \frac{1}{6} X\,\left(X^2 + 6X + 6\right) \\
P_4^{\varphi_1}(X) &= \frac{1}{24} X\,\left(X^3 + 12X^2 + 36X + 24\right) \\
P_5^{\varphi_1}(X) &= \frac{1}{120} X\,\left(X^4 + 20X^3 + 120X^2 + 240X + 120\right)
\end{aligned}$$

$$
\begin{aligned}
P_6^{\varphi_1}(X) &= \frac{1}{720} X \left(X^5 + 30X^4 + 300X^3 + 1200X^2 + 1800X + 720\right) \\
P_7^{\varphi_1}(X) &= \frac{1}{5040} X \left(X^6 + 42X^5 + 630X^4 + 4200X^3 + 12600X^2 + 15120X + 5040\right).
\end{aligned}
$$

Remark 8. The first polynomials $\frac{n!}{X} P_n^{\varphi_1}(X)$ are irreducible.

Proof of Theorem 2. We start with the identity

$$
\sum_{n=0}^{\infty} P_n^{\varphi_1}(X) \; q^n = \exp\left(X \sum_{n=1}^{\infty} q^n \right). \tag{19}
$$

It is useful to substitute $\left(\sum_{n=1}^{\infty} q^n\right)^k$ by the multi-index sum

$$
\sum_{m_1,\ldots,m_k=1}^{\infty} q^{m_1+\cdots+m_k}.
$$

Comparing the coefficients of q^n in (19) leads to

$$
\begin{aligned}
P_n(X) &= \sum_{k=1}^{n} \frac{1}{k!} \left(\sum_{m_1+\cdots+m_k=n} 1 \right) X^k \\
&= \frac{X}{n!} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!} \left(\sum_{m_1+\cdots+m_{k+1}=n} 1 \right) X^k.
\end{aligned}
$$

Note that

$$
\sum_{m_1+\cdots+m_k=n} 1 = \binom{n-1}{k-1}. \tag{20}
$$

We prove (20) by induction. For $k=1$ we have $1 = \binom{n-1}{0} = \binom{n-1}{k-1}$. Suppose the formula holds for k then

$$
\begin{aligned}
\sum_{m_1+\cdots+m_{k+1}=n} 1 &= \sum_{m_{k+1}=1}^{n-k} \sum_{m_1+\cdots+m_k=n-m_{k+1}} 1 = \sum_{m_{k+1}=1}^{n-k} \binom{n-m_{k+1}-1}{k-1} \\
&= \sum_{m_{k+1}=k}^{n-1} \binom{m_{k+1}-1}{k-1} = \binom{n-1}{k}.
\end{aligned}
$$

The proof (again by induction) of the identity $\sum_{m=k}^{n-1} \binom{m-1}{k-1} = \binom{n-1}{k}$ we leave to the reader. $\square$

4 Explicit formulas for $P_n^{\varphi_2}(X)$

Let us start with a list of the polynomials for $n = 1, 2, \ldots 7$. Note that they are no longer integer-valued polynomials, although the modified coefficients are integral.

$$\begin{aligned}
P_1^{\varphi_2}(X) &= X \\
P_2^{\varphi_2}(X) &= 1/2\, X\,(4+X) \\
P_3^{\varphi_2}(X) &= 1/6\, X\left(X^2+12\,X+18\right) \\
P_4^{\varphi_2}(X) &= 1/24\, X\left(X^3+24\,X^2+120\,X+96\right) \\
P_5^{\varphi_2}(X) &= \frac{1}{120} X\left(X^4+40\,X^3+420\,X^2+1200\,X+600\right) \\
P_6^{\varphi_2}(X) &= \frac{1}{720} X\left(X^5+60\,X^4+1080\,X^3+6720\,X^2+12600\,X+4320\right) \\
P_7^{\varphi_2}(X) &= \frac{1}{5040} X\left(X^6+84\,X^5+2310\,X^4+25200\,X^3+105840\,X^2+141120\,X+35280\right)
\end{aligned}$$

Before we prove Theorem 5 we show the following useful property.

Lemma 9. *For $n \geq 1$ and $1 \leq k \leq n$ we obtain*

$$\sum_{m_1+\cdots+m_k=n} m_1 m_2 \cdots m_k = \binom{n+k-1}{2k-1}. \tag{21}$$

Proof. The proof is by induction on n and k. For $k=1$ we have $n = \binom{n}{1} = \binom{n+k-1}{2k-1}$. Let now $N \geq K \geq 2$ be fixed. We assume that formula (21) holds for $k < K$. For $k = K$ we also assume that the formula (21) holds for $n < N$. Then we will prove formula (21) for $n = N$ and $k = K$. We obtain for

$$\sum_{m_1+\cdots+m_K=N} m_1 m_2 \cdots m_K$$

the expressions

$$\begin{aligned}
& \sum_{m_K=1}^{N+1-K} m_K \sum_{m_1+\cdots+m_{K-1}=N-m_K} m_1 m_2 \cdots m_{K-1} \\
= & \sum_{m_K=1}^{N+1-K} \sum_{\substack{m_1+\cdots+m_{K-1}\\=N-m_K}} m_1 \cdots m_{K-1} + \sum_{m_K=1}^{N-K} m_K \sum_{\substack{m_1+\cdots+m_{K-1}\\=N-1-m_K}} m_1 m_2 \cdots m_{K-1} \\
= & \sum_{m_K=1}^{N+1-K} \binom{N-m_K+K-2}{2K-3} + \sum_{m_1+\cdots+m_K=N-1} m_1 m_2 \cdots m_K \\
= & \sum_{m_K=2K-2}^{N+K-2} \binom{m_K-1}{2K-3} + \binom{N+K-2}{2K-1} \\
= & \binom{N+K-2}{2K-2} + \binom{N+K-2}{2K-1} = \binom{N+K-1}{2K-1}.
\end{aligned}$$

Note that here we again used the identity

$$\sum_{m=k}^{n-1} \binom{m-1}{k-1} = \binom{n-1}{k}.$$

□

Proof of Theorem 5. We obtain

$$\begin{aligned}
\exp\left(X \sum_{n=1}^{\infty} \frac{n^2}{n} q^n\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} X^k \left(\sum_{n=1}^{\infty} n q^n\right)^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} X^k \left(\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} m_1 \cdots m_k q^{m_1+\cdots+m_k}\right) \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k!} X^k \left(\sum_{m_1+\cdots+m_k=n} m_1 m_2 \cdots m_k\right) q^n.
\end{aligned}$$

In the next step we apply Lemma 9 and obtain

$$\begin{aligned}
& 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k!} \binom{n+k-1}{2k-1} X^k q^n \\
= \; & 1 + \sum_{n=1}^{\infty} \frac{X}{n!} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!} \binom{n+k}{2k+1} X^k q^n,
\end{aligned}$$

which yields the desired result. □

5 Data

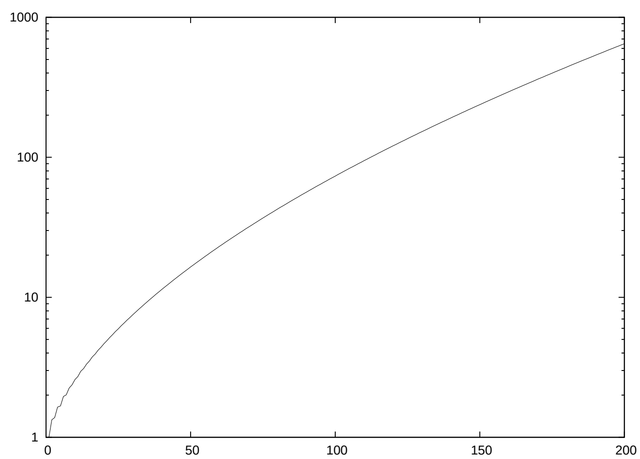

Figure 2: Quotient $P_n^{\sigma}(1) / P_n^{\varphi_1}(1)$ depending on n.

n	$P_n^1(1)$	$P_n^\sigma(1)$	$P_n^2(1)$
0	1	1	1
1	1	1	1
2	1.5	2	2.5
3	2.16667	3	5.16667
4	3.04167	5	10.0417
5	4.175	7	18.8417
6	5.62639	11	34.4181
7	7.46687	15	61.4752
8	9.78058	22	107.694
9	12.6669	30	185.485
10	16.2426	42	314.694
11	20.6448	56	526.768
12	26.0337	77	871.113
13	32.5961	101	1424.73
14	40.5493	135	2306.78
15	50.1454	176	3700.32
16	61.676	231	5884.91
17	75.4781	297	9284.78
18	91.9399	385	14540.1
19	111.508	490	22612
20	134.694	627	34935.4
21	162.087	792	53643.4
22	194.356	1002	81891.7
23	232.27	1255	124329
24	276.702	1575	187773
25	328.648	1958	282189

Table 1: Values of $P_n^{\varphi_1}(1)$, the partition numbers $P_n^\sigma(1)$, and $P_n^{\varphi_2}(1)$.

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