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Approximating Sums of Consecutive Integral Roots

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Abstract

We present an alternative proof of Saltzman and Yuan's result on the sums of consecutive integral roots. We use the AM-GM-HM inequality to prove the main result. Moreover, the lower bound for which the result holds is greatly improved.

1 Introduction

For each real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. The sums of consecutive integral roots have been studied by many mathematicians (see [1]-[5]). For instance, the following identities hold for every positive integer n:

(1) $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$

(2) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} \rfloor = \lfloor \sqrt{9n+8} \rfloor$

(3) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \rfloor = \lfloor \sqrt{16n+20} \rfloor$

(4) $\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} \rfloor = \lfloor \sqrt{25n+49} \rfloor$

(5) $\lfloor \sqrt[3]{n} + \sqrt[3]{n+1} \rfloor = \lfloor \sqrt[3]{8n+3} \rfloor$

(6) $\lfloor \sqrt[3]{n} + \sqrt[3]{n+1} + \sqrt[3]{n+2} \rfloor = \lfloor \sqrt[3]{27n+26} \rfloor$.

On the contrary, Zhan [5] showed that for any real number c, there is a positive integer n such that

$$\left\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} + \sqrt{n+5} \right\rfloor \neq \left\lfloor \sqrt{36n+c} \right\rfloor.$$

Saltzman and Yuan [3] presented the following similar formula for $n \geq 4$.

$$\left\lfloor \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} + \sqrt{n+4} + \sqrt{n+5} \right\rfloor = \left\lfloor \sqrt{36n+89} \right\rfloor,$$

and gave the general formula: for all integers $p, m \geq 2$,

$$\left\lfloor \sqrt[p]{n} + \sqrt[p]{n+1} + \sqrt[p]{n+2} + \cdots + \sqrt[p]{n+m-1} \right\rfloor = \left\lfloor \sqrt[p]{m^p n + \frac{m^p(m-1)}{2} - 1} \right\rfloor \tag{1}$$

holds for all positive integers

$$n > \frac{m^p(m-1)(2m-1)(p-1)}{12p}. \tag{2}$$

They utilized properties of concave functions to approximate the sums of consecutive integral roots.

In this paper, we give an alternative proof of (1) using the AM-GM-HM inequality and obtain a new lower bound on n, which is approximately half of (2).

2 Results

In the sequel, let m and p be two positive integers greater than 1, let $M = \frac{(p-1)m^p(m^2-1)}{12p}$, and let $i_1, i_2, \ldots, i_p \in \{0, 1, 2, \ldots, m-1\}$. Denoted by A_p, G_p and H_p the arithmetic mean, the geometric mean, and the harmonic mean of the positive integers $n+i_1, n+i_2, \ldots, n+i_p$, respectively, i.e.,

$$\begin{aligned}
A_p &= \frac{1}{p}\sum_{j=1}^{p}(n+i_j),\\
G_p &= \sqrt[p]{(n+i_1)\ldots(n+i_p)},\\
H_p &= \frac{p}{\sum_{j=1}^{p}\frac{1}{n+i_j}}.
\end{aligned}$$

By the well-known AM-GM-HM inequality,

$$H_p \le G_p \le A_p$$

where equality holds if and only if $n+i_1 = n+i_2 = \cdots = n+i_p$. To prove the main theorem, we first present two technical lemmas.

Lemma 1. *For every positive integer n,*

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} (A_p - H_p) < \frac{M}{n^{p-1}} \left(n + \frac{m-1}{2} \right)^{p-2}.$$

Proof. Observe that

$$\begin{aligned} A_p - H_p &= \frac{1}{p} \sum_{j=1}^{p} (n+i_j) - \frac{p}{\sum_{j=1}^{p} \frac{1}{n+i_j}} \\ &= \frac{\left(\left(n + \frac{1}{p} \sum_{j=1}^{p} i_j \right) \left(\frac{1}{p} \sum_{k=1}^{p} \frac{\prod_{j=1}^{p} (n+i_j)}{n+i_k} \right) - \prod_{j=1}^{p} (n+i_j) \right)}{\frac{1}{p} \sum_{k=1}^{p} \frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}}. \end{aligned}$$

Hence

$$\frac{A_p - H_p}{p} \sum_{k=1}^{p} \frac{\prod_{j=1}^{p} (n+i_j)}{n+i_k} = \left(n + \frac{1}{p} \sum_{j=1}^{p} i_j \right) \left(\frac{1}{p} \sum_{k=1}^{p} \frac{\prod_{j=1}^{p} (n+i_j)}{n+i_k} \right) - \prod_{j=1}^{p} (n+i_j).$$

Next, we determine the coefficients of

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\left(n + \frac{1}{p} \sum_{j=1}^{p} i_j \right) \left(\frac{1}{p} \sum_{k=1}^{p} \frac{\prod_{j=1}^{p} (n+i_j)}{n+i_k} \right) - \prod_{j=1}^{p} (n+i_j) \right) \tag{3}$$

as a polynomial in variable n.

It is easy to see that the coefficients of n^p and n^{p-1} in (3) are zero. Moreover, we claim that, for each $t = 2, 3, \ldots, p$, the coefficient of n^{p-t} in (3) is equal to $M\binom{p-2}{t-2}\left(\frac{m-1}{2}\right)^{t-2}$ implying that the polynomial is $M\left(n + \frac{m-1}{2}\right)^{p-2}$.

The coefficient of n^{p-t} in (3), for $t = 2, 3, \ldots, p$, is equal to the sums of the following:

(i) $$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\frac{1}{p^2} \sum_{j=1}^{p} i_j \left(\sum_{S} \sum_{\substack{j_1 < \cdots < j_{t-1} \\ j_1, \ldots, j_{t-1} \in S}} i_{j_1} \cdots i_{j_{t-1}} \right) \right)$$

(ii) $$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\frac{1}{p} \sum_{S} \sum_{\substack{j_1 < \cdots < j_t \\ j_1, \ldots, j_t \in S}} i_{j_1} \cdots i_{j_t} \right)$$

$$\text{(iii)} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(- \sum_{\substack{j_1 < \cdots < j_t \\ j_1, \ldots, j_t \in \mathbb{N}_p}} i_{j_1} \cdots i_{j_t} \right)$$

where $\sum_S$ represents the sum over all $(p-1)$-subsets S of $\mathbb{N}_p = \{1, 2, \ldots, p\}$. (Note that for the case $t = p$, (ii) is an empty sum which is conventionally zero. This coincides with the fact that (ii) does not occur in the case $t = p$.)

Distributing $\sum_{j=1}^{p} i_j$ inside the double sum, (i) becomes

$$\frac{1}{p^2} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\sum_S \sum_{\substack{j_1 < \cdots < j_{t-1} \\ j_1, \ldots, j_{t-1} \in S}} \left(\sum_{r=1}^{t-1} i_{j_r} (i_{j_1} \cdots i_{j_{t-1}}) + \sum_{r \notin \{1, \ldots, t-1\}} i_{j_r} (i_{j_1} \cdots i_{j_{t-1}}) \right) \right).$$

Since the double sum consists of exactly $p\binom{p-1}{t-1}$ terms, using the symmetry of the i_j and combining the like terms, (i) amounts to

$$\frac{1}{p} \binom{p-1}{t-1} \left((t-1) \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1^2 \cdots i_{t-1} + (p-t+1) \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1 \cdots i_t \right). \tag{4}$$

Similarly, (ii) amounts to

$$\binom{p-1}{t} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1 \cdots i_t, \tag{5}$$

and (iii) amounts to

$$-\binom{p}{t} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} i_1 \cdots i_t. \tag{6}$$

Note that

$$\frac{p-t+1}{p} \binom{p-1}{t-1} + \binom{p-1}{t} - \binom{p}{t} = -\frac{t-1}{p} \binom{p-1}{t-1}$$

for $t = 2, \ldots, p$. Hence adding (4), (5) and (6) yields

$$\begin{aligned}
&\frac{t-1}{p}\binom{p-1}{t-1}\left(\sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1} i_1^2 \cdots i_{t-1} - \sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1} i_1 \cdots i_t\right) \\
&\qquad = \frac{p-1}{p}\binom{p-2}{t-2}\left(m^{p-t+1}\sum_{i=0}^{m-1} i^2 \left(\sum_{i=0}^{m-1} i\right)^{t-2} - m^{p-t}\left(\sum_{i=0}^{m-1} i\right)^{t}\right) \\
&\qquad = \frac{p-1}{p}\binom{p-2}{t-2} m^{p-t}\left(\sum_{i=0}^{m-1} i\right)^{t-2}\left(m\sum_{i=0}^{m-1} i^2 - \left(\sum_{i=0}^{m-1} i\right)^2\right) \\
&\qquad = M\binom{p-2}{t-2}\left(\frac{m-1}{2}\right)^{t-2},
\end{aligned}$$

which is the coefficient of n^{p-t} in $M\left(n + \frac{m-1}{2}\right)^{p-2}$. Therefore,

$$\begin{aligned}
\sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1}(A_p - H_p) &< \frac{1}{n^{p-1}}\sum_{i_1=0}^{m-1}\cdots\sum_{i_p=0}^{m-1}\left(\frac{A_p - H_p}{p}\sum_{k=1}^{p}\frac{\prod_{j=1}^{p}(n+i_j)}{n+i_k}\right) \\
&< \frac{M}{n^{p-1}}\left(n + \frac{m-1}{2}\right)^{p-2}.
\end{aligned}$$

□

Lemma 2. *For every positive integer* $n \geq M + \frac{(p-2)(m-1)}{2}$,

$$n^{p-1} \geq M\left(n + \frac{m-1}{2}\right)^{p-2}.$$

Proof. For $n \geq M + \frac{(p-2)(m-1)}{2}$,

$$\begin{aligned}
n^{p-1} &\geq Mn^{p-2} + \frac{(p-2)(m-1)}{2}n^{p-2} \\
&\geq Mn^{p-2} + \frac{(p-2)(m-1)}{2}Mn^{p-3} + \left(\frac{(p-2)(m-1)}{2}\right)^2 n^{p-3} \\
&\vdots \\
&\geq M\sum_{j=0}^{p-3}\left(\frac{(p-2)(m-1)}{2}\right)^j n^{p-2-j} + \left(\frac{(p-2)(m-1)}{2}\right)^{p-2} n \\
&\geq M\sum_{j=0}^{p-2}\left(\frac{(p-2)(m-1)}{2}\right)^j n^{p-2-j}.
\end{aligned}$$

Since $(p-2)^j \geq \binom{p-2}{j}$ for all $0 \leq j \leq p-2$,

$$n^{p-1} \geq M \sum_{j=0}^{p-2} \binom{p-2}{j} \left(\frac{m-1}{2}\right)^j n^{p-2-j} = M\left(n + \frac{m-1}{2}\right)^{p-2}$$

as desired. $\square$

Now we are ready to state and prove the main result.

Theorem 3. *For every positive integer $n \geq M + \frac{(p-2)(m-1)}{2}$,*

$$\lfloor \sqrt[p]{n} + \sqrt[p]{n+1} + \cdots + \sqrt[p]{n+m-1} \rfloor = \left\lfloor \sqrt[p]{m^p n + \frac{m^p(m-1)}{2} - 1} \right\rfloor.$$

Proof. Let

$$S = \sqrt[p]{n} + \sqrt[p]{n+1} + \sqrt[p]{n+2} + \cdots + \sqrt[p]{n+m-1}.$$

One obtains

$$S^p = \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \sqrt[p]{(n+i_1)\cdots(n+i_p)} = \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} G_p.$$

By the AM-GM-HM inequality,

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} A_p - \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} (A_p - H_p) < S^p < \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} A_p.$$

Observe that

$$\begin{aligned} \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} A_p &= \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} n + \sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} \left(\frac{1}{p}\sum_{j=1}^{p} i_j\right) \\ &= m^p n + \frac{m^p(m-1)}{2}. \end{aligned}$$

Moreover, by Lemma 1 and Lemma 2, we have

$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_p=0}^{m-1} (A_p - H_p) < \frac{M}{n^{p-1}} \left(n + \frac{m-1}{2}\right)^{p-2} \leq 1.$$

Thus

$$m^p n + \frac{m^p(m-1)}{2} - 1 < S^p < m^p n + \frac{m^p(m-1)}{2}$$

holds for all positive integers $n \geq M + \frac{(p-2)(m-1)}{2}$. Equivalently,

$$\lfloor \sqrt[p]{n} + \sqrt[p]{n+1} + \cdots + \sqrt[p]{n+m-1} \rfloor = \left\lfloor \sqrt[p]{m^p n + \frac{m^p(m-1)}{2} - 1} \right\rfloor$$

holds for all positive integers $n \geq M + \frac{(p-2)(m-1)}{2}$. $\square$

The table below compares the lower bounds of n between Saltzman and Yuan's work, $N_{SY} = \frac{m^p(m-1)(2m-1)(p-1)}{12p} + 1$, and our work, $N = \frac{(p-1)m^p(m^2-1)}{12p} + \frac{(p-2)(m-1)}{2}$, for small values of p and m.

	$p=3$		$p=4$		$p=5$		$p=6$	
m	N_{SY}	N	N_{SY}	N	N_{SY}	N	N_{SY}	N
2	2	2	4	4	7	8	14	16
3	16	13	51	43	163	133	507	409
4	75	55	337	243	1434	1029	5974	4273
5	251	169	1407	942	7501	5006	39063	29050
6	661	423	4456	2840	28513	18152	178201	113410
7	1487	918	11705	7209	87397	53792	637266	392176
8	2587	1796	26881	16135	229377	137637	1911467	1146894
9	5509	3244	55769	32813	535378	314940	5019166	2952466
10	9501	5505	106876	61884	1140001	660014	11875001	6875018

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