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Combinatorial Results for Semigroups of Order-Decreasing Partial Transformations

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Abstract

Let $\mathcal{PC}_n$ be the semigroup of all decreasing and order-preserving partial transformations of a finite chain. It is shown that $|\mathcal{PC}_n| = r_n$, where r_n is the large (or double) Schröder number. Moreover, the total number of idempotents of $\mathcal{PC}_n$ is shown to be $(3^n + 1)/2$.

1 Introduction and Preliminaries

Consider a finite chain, say $X_n = \{1, 2, \ldots, n\}$ under the natural ordering and let T_n and P_n be the full transformation semigroup and the semigroup of all partial transformations on X_n, under the usual composition, respectively. We shall call a partial transformation $\alpha : X_n \to X_n$, *order-decreasing (order-increasing)* or simply *decreasing (increasing)* if $x\alpha \leq x$ $(x\alpha \geq x)$ for all x in Dom α, and α is *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for x, y in Dom α. The semigroup of all decreasing full transformations is denoted by D_n, while the semigroup of all order-preserving full transformations is denoted by $\mathcal{O}_n$, and $D_n \cap \mathcal{O}_n$ is denoted by $\mathcal{C}_n$.

Various enumerative problems of an essentially combinatorial nature have been considered for certain classes of semigroups of transformations. For example, Howie [9] showed that the order and number of idempotents of $\mathcal{O}_n$ are, respectively,

$$|\mathcal{O}_n| = \binom{2n-1}{n-1} \text{ and } |E(\mathcal{O}_n)| = F_{2n},$$

where F_{2n} is the alternate Fibonacci number given by $F_1 = F_2 = 1$. More recently, Higgins [6] showed in particular, that $|\mathcal{C}_n|$ is the n-th Catalan number given by

$$|\mathcal{C}_n| = \frac{1}{n+1}\binom{2n}{n}$$

and $|E(\mathcal{C}_n)| = 2^{n-1}$. Further combinatorial properties for $\mathcal{C}_n$ were investigated by Laradji and Umar [10], where they showed that the number of maps in $\mathcal{C}_n$ such that $|\text{Im } \alpha| = r$ is the triangle of Narayana [17, A001263] numbers given by

$$|\{\alpha \in \mathcal{C}_n : |\text{Im } \alpha| = r\}| = \frac{1}{n-r+1} \binom{n-1}{r-1} \binom{n}{r}.$$

This paper investigates combinatorial properties of $\mathcal{PC}_n$, the semigroup of all decreasing and order-preserving partial transformations, along the lines of [10]. An alternative approach to finding the order and number of idempotents in $\mathcal{PC}_n$ is given in [11], however, the advantage of the approach given in this paper is that we get along the way some known triangular arrays of integers as well as some new ones, which are not yet listed in [17]. Ironically, it is this paper that motivated [11] and [12]. The following is a list (which is by no means exhaustive) of papers and books [1, 2, 3, 4, 5, 6, 8, 9, 20, 21, 22] each of which contains some interesting combinatorial results pertaining to semigroups of transformations. Initially, the only reference we could find about $\mathcal{PC}_n$ is Higgins [7, theorem 4.2], where it is shown that any finite $\mathcal{R}$ -trivial semigroup S divides some monoid $\mathcal{PC}_n$. However, the referee drew our attention to [15] and [18] where presentations of $\mathcal{PC}_n$ on a chain and trees, respectively, were studied and also to [14] where $\mathcal{PC}_n$ is studied in connection with theoretical computer science.

In Section 2, we give the necessary definitions that we need in the paper. In Section 3, we obtain the order of $\mathcal{PC}_n$ as the large or double Schröder number [13], via some natural equivalences on $\mathcal{PC}_n$. In Section 4, we show that the set of all idempotents of $\mathcal{PC}_n$ is of cardinality $(3^n + 1)/2$, again, via some natural equivalences on $E(\mathcal{PC}_n)$.

For standard terms and concepts in transformation semigroup theory see [5] or [8]. We now recall some definitions and notations to be used in the paper. Consider $X_n = \{1, 2, \ldots, n\}$ and let $\alpha : X_n \to X_n$ be a partial transformation. We shall denote by Dom α and Im α, the *domain* and *image set* of α, respectively. The semigroup P_n, of all partial transformations contains two important subsemigroups which have been studied recently. They are PD_n and $\mathcal{PO}_n$ the semigroups of all order-decreasing and order-preserving partial transformations, respectively (see [23] and [3, 4]). Now let

$$\mathcal{PC}_n = PD_n \cap \mathcal{PO}_n \tag{1.1}$$

be the semigroup of all decreasing and order-preserving partial transformations of X_n.

2 The order of $\mathcal{PC}_n$

Our main objective in this section is to obtain a formula for $|\mathcal{PC}_n|$. We initiate our investigation by considering two natural equivalences on $\mathcal{PC}_n$. The first equivalence is defined by equality of widths (*width* of $\alpha := |\text{Dom } \alpha|$), while the second equivalence is defined by equality of waists (*waist* of $\alpha := \max(\text{Im } \alpha)$). Taking the intersection of these two equivalences leads to the following definition of $f(n, r, k)$ as

$$f(n, r, k) = |\{\alpha \in \mathcal{PC}_n : |\text{Dom } \alpha| = r \wedge \max(\text{Im } \alpha) = k\}|. \tag{2.1}$$

Then clearly we have

$$f(n,0,0)=1, f(n,n,1)=1$$

and

$$f(n,r,0)=0 \text{ (if } r>0), f(n,0,k)=0 \text{ (if } k>0).$$

Slightly less clearly, we have

$$f(n,1,k)=n-(k-1)=n-k+1 \tag{2.2}$$

and

$$f(n,r,1)=\binom{n}{r} \tag{2.3}$$

In fact, $f(n,1,k)$ corresponds to the number of maps α (in $\mathcal{PC}_n$) with singleton domain and hence $\mathrm{Im}\ \alpha=\{k\}$. Since by the order-decreasing property, $x\in \mathrm{Dom}\ \alpha$ implies $x\in \{k,k+1,\ldots,n\}$, the result now follows. As for $f(n,r,1)$, it corresponds to all subsets of X_n of size r. A more general result is

Lemma 2.1 *For all $n\geq r, k\geq 0$, we have*

$$f(n,r,k)=f(n-1,r,k)+\sum_{t=0}^{k} f(n-1,r-1,t).$$

Proof. Essentially there are two cases to consider: $n\notin \mathrm{Dom}\ \alpha$ and $n\in \mathrm{Dom}\ \alpha$. In the former case there are clearly $f(n-1,r,k)$ maps of this type. In the latter case, since $n\alpha=k$, it is not difficult to see that there are

$$\sum_{t=0}^{k} f(n-1,r-1,t)$$

maps of this type. Hence the result follows.

A closed formula for $f(n,r,k)$ is possible, but before we propose this formula we would like to state this lemma from [10, lemma 3.3] which is obtained by combining equations (3) and (3b) from [16, p. 8].

Lemma 2.2 *For any $c\in\mathbb{R}$, and $q,m\in\mathbb{N}\cup\{0\}$, we have*

$$\sum_{j=0}^{m}(c-j)\binom{q+j}{j}=(c-m-1)\binom{m+q+1}{m}+\binom{m+q+2}{m}.$$

Proposition 2.3 *Let $f(n,r,k)$ be as defined in (2.1). Then for $n\geq r, k>0$, we have*

$$f(n,r,k)=\frac{n-k+1}{r}\binom{n-1}{r-1}\binom{k+r-2}{r-1}=\frac{n-k+1}{n}\binom{n}{r}\binom{k+r-2}{r-1}.$$

Proof. The proof is by induction, and by virtue of (2.2) and (2.3) which agree both with the assertion we may suppose that the result is true for all $1 \le r, k \le n$. We now prove that it is true for all $1 \le r, k \le n+1$. By Lemma 2.1 and the induction hypothesis successively, we have

$$\begin{aligned}
f(n+1,r,k) &= f(n,r,k) + \sum_{t=0}^{k} f(n,r-1,t) \\
&= \frac{n-k+1}{n}\binom{n}{r}\binom{k+r-2}{r-1} + \sum_{t=0}^{k}\frac{n-t+1}{n}\binom{n}{r-1}\binom{r+t-3}{r-2} \\
&= \frac{1}{n}\left\{\frac{n!(n-k+1)}{(n-r)!r(r-1)!}\binom{k+r-2}{r-1} + (n+1)\binom{n}{r-1}\sum_{t=1}^{k}\binom{r+t-3}{r-2}\right. \\
&\quad \left. -\binom{n}{r-1}\sum_{t=1}^{k} t\binom{r+t-3}{r-2}\right\} \\
&= \frac{1}{n}\binom{n}{r-1}\left\{\frac{n-r+1}{r}(n-k+1)\binom{k+r-2}{r-1} + (n+1)\sum_{t=0}^{k-1}\binom{r-2+t}{r-2}\right. \\
&\quad \left. -\sum_{t=0}^{k-1}(t+1)\binom{r-2+t}{r-2}\right\}. \\
&= \frac{1}{n}\binom{n}{r-1}\left\{\frac{(n-r+1)(n-k+1)}{r}\binom{k+r-2}{r-2}\right. \\
&\quad \left. +\sum_{t=0}^{k-1}(n-t)\binom{r-2+t}{r-2}\right\}.
\end{aligned}$$

However, by Lemma 2.2

$$\sum_{t=0}^{k-1}(n-t)\binom{r-2+t}{r-2} = (n-k)\binom{k+r-2}{k-1} + \binom{k+r-1}{k-1}$$

and so

$$\begin{aligned}
f(n+1,r,k) &= \frac{1}{n}\binom{n}{r-1}\left\{\frac{(n-r+1)(n-k+1)}{r}\binom{k+r-2}{r-1}\right. \\
&\quad \left. +(n-k)\binom{k+r-2}{r-1} + \binom{k+r-1}{r}\right\} \\
&= \frac{1}{n}\binom{n}{r-1}\binom{k+r-2}{r-1}\left\{\frac{(n-r+1)(n-k+1)}{r}\right. \\
&\quad \left. +(n-k) + \frac{k+r-1}{r}\right\} \\
&= \frac{1}{r}\binom{n}{r-1}\binom{k+r-2}{r-1}(n+2-k)
\end{aligned}$$

as required. To complete the induction step we still need to verify the result for $f(n+1,n+1,k)$ and $f(n+1,r,n+1)$. By using Lemmas 2.1 and 2.2 and the induction hypothesis these could be routinely verified. Thus the proof of Proposition 2.3 is complete.

Immediately, we have

Corollary 2.4 [10, proposition 3.10]. *Let $\mathcal{C}_n$ be the semigroup of all decreasing and order-preserving full transformations of X_n. Then*

$$|\{\alpha \in \mathcal{C}_n : \max(Im\ \alpha) = k\}| = f(n,n,k) = \frac{n-k+1}{n}\begin{pmatrix} n+k-2 \\ n-1 \end{pmatrix}.$$

Corollary 2.5 *For $n \geq r \geq 1$, we have*

$$f(n,r,r) = \frac{n-r+1}{n}\begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} 2r-2 \\ r-1 \end{pmatrix}.$$

Lemma 2.6 *Let $G(n,k) = \sum_{r=0}^{n} f(n,r,k)$. Then*

$$G(n,k) = \frac{n-k+1}{n}\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} k+r-2 \\ r-1 \end{pmatrix}.$$

Proposition 2.7 *Let $G(n,k) = \displaystyle\sum_{r=0}^{n} f(n,r,k)$. Then $G(n,0) = 1$, $G(n,1) = 2^n - 1$, $G(n,n) = \frac{1}{n}\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} n+r-2 \\ r-1 \end{pmatrix}$, and for $2 \leq k \leq n$, we have*

$$G(n,k) = 2G(n-1,k) - G(n-1,k-1) + G(n,k-1).$$

Proof. Since the initial and boundary conditions are clear it remains to show the recurrence:

$$\begin{aligned}
G(n,k) &= \sum_{r=0}^{n} f(n,r,k) = \sum_{r=0}^{n}\left\{f(n-1,r,k) + \sum_{t=1}^{k} f(n-1,r-1,t)\right\} \\
&= \sum_{r=0}^{n-1} f(n-1,r,k) + \sum_{t=0}^{k}\sum_{r=0}^{n} f(n-1,r-1,t) \\
&= G(n-1,k) + \sum_{t=0}^{k} G(n-1,t) \qquad (2.4) \\
&= 2G(n-1,k) + \sum_{t=0}^{k-1} G(n-1,t).
\end{aligned}$$

Thus from (2.4) we have

$$G(n,k-1) = G(n-1,k-1) + \sum_{t=0}^{k-1} G(n-1,t)$$

and so

$$G(n,k) - G(n,k-1) = 2G(n-1,k) - G(n-1,k-1)$$

from which the result follows.

Proposition 2.8 *Let* $F(n,r) = \sum_{k=0}^{n} f(n,r,k)$. *Then*

$$F(n,r) = \frac{1}{n}\binom{n}{r}\binom{n+r}{n-1}.$$

Proof. The proof is direct by using Lemma 2.2 and Proposition 2.3. Thus we have

$$\begin{aligned}
F(n,r) &= \sum_{k=0}^{n} f(n,r,k) = \sum_{k=0}^{n} \frac{n-k+1}{n}\binom{n}{r}\binom{k+r-2}{r-1} \\
&= \frac{1}{n}\binom{n}{r}\sum_{k=0}^{n}[n-(k-1)]\binom{k+r-2}{k-1} \\
&= \frac{1}{n}\binom{n}{r}\sum_{k=0}^{n}[n-(k-1)]\binom{(r-1)+(k-1)}{k-1} \\
&= \frac{1}{n}\binom{n}{r}\sum_{t=0}^{n-1}(n-t)\binom{(r-1)+t}{t} = \frac{1}{n}\binom{n}{r}\binom{n+r}{r-1}
\end{aligned}$$

as required.

Corollary 2.9 [6, theorem 3.1]. *Let* $\mathcal{C}_n$ *be the semigroup of all decreasing and order-preserving full transformations of* X_n. *Then*

$$|\mathcal{C}_n| = F(n,n) = \frac{1}{n}\binom{2n}{n-1}.$$

Remark 2.1 The triangular array of numbers $G(n,k)$, $f(n,r,r)$ and $F(n,r)$ are not yet listed in Sloane's encyclopaedia of integer sequences [17]. For some selected values of these numbers, see Tables 1-3.

From [13] and [19] we deduce that the large (or double) Schröder number denoted by r_n could be defined as

$$r_n = \frac{1}{n+1}\sum_{r=0}^{n}\binom{n+1}{n-r}\binom{n+r}{r}.$$

Moreover, r_n satisfies the recurrence:

$$(n+2)r_{n+1} = 3(2n+1)r_n - (n-1)r_{n-1} \tag{2.5}$$

for $n \geq 1$, with initial conditions $r_0 = 1$ and $r_1 = 2$. The (small) Schröder number is usually denoted by s_n and defined as $s_0 = 1, s_n = r_n/2\,(n \geq 1)$ and so it satisfies the same recurrence as r_n.

n \ k	0	1	2	3	4	5	6	7	$\sum G(n,k)$
0	1								1
1	1	1							2
2	1	3	2						6
3	1	7	8	6					22
4	1	15	24	28	22				90
5	1	31	64	96	112	90			394
6	1	63	160	288	416	484	394		1806
7	1	127	384	800	1344	1896	2200	1806	8558

Table 1. *G(n, k)*

n \ r	0	1	2	3	4	5	6	7	$\sum f(n,r,r)$
0	1								1
1	1	1							2
2	1	2	1						4
3	1	3	4	2					10
4	1	4	9	12	5				31
5	1	5	16	36	40	14			112
6	1	6	25	80	150	140	42		444
7	1	7	36	150	400	630	504	132	1860

Table 2. *f(n, r, r)*

n \ r	0	1	2	3	4	5	6	7	$\sum F(n,r)$
0	1								1
1	1	1							2
2	1	3	2						6
3	1	6	10	5					22
4	1	10	30	35	14				90
5	1	15	70	140	126	42			394
6	1	21	140	420	630	462	132		1806
7	1	28	252	1050	2310	2772	1716	429	8558

Table 3. *F(n, r)*

Remark 2.2 The double Schröder number is the number of all lattice paths in the Cartesian plane that start at $(0,0)$, end at (n,n), contain no points above the line $y=x$, and are composed only of steps $(1,0),(0,1)$ and $(1,1)$, i.e., $\rightarrow,\uparrow$ and $\nearrow$. The authors [11] established a bijection between the set of all such paths and $\mathcal{PC}_n$, and hence the order of $\mathcal{PC}_n$ was deduced.

We now have the main result of this section:

Theorem 2.10 *Let $\mathcal{PC}_n$ be as defined in (1.1). Then $|\mathcal{PC}_n| = r_n$, the double Schröder number.*

Proof. It is clear from Proposition 2.8 that

$$|\mathcal{PC}_n| = \sum_{r=0}^{n} F(n,r) = \sum_{r=0}^{n} \frac{1}{n}\binom{n}{r}\binom{n+r}{n-1} = \frac{1}{n+1}\sum_{r=0}^{n}\binom{n+1}{n-r}\binom{n+r}{r} = r_n.$$

We conclude the section with the following congruence result.

Proposition 2.11 *If n is prime then $r_n \equiv 4 \pmod{n}$.*

Proof. Since $r_n = \sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r}\binom{n+r}{n}$ and if n is prime then $n|\binom{n}{r}$, it follows that the only values of r that may not produce terms divisible by n in the sum are: $0, n-1$ and n. Hence

$$\begin{aligned} r_n &\equiv 1 + \frac{1}{n}\cdot n\binom{2n-1}{n} + \frac{1}{n+1}\binom{2n}{n} \pmod{n} \\ &= 1 + \binom{2n-1}{n} + \frac{1}{n+1}\binom{2n}{n} \pmod{n}. \end{aligned}$$

Now let $A = \binom{2n-1}{n}$, then $n!(n-1)!A = (2n-1)!$, that is

$$(n-1)!A = (n+1)(n+2)\cdot\dots\cdot[n+(n-1)] \equiv (n-1)! \pmod{n}.$$

Thus since $(n,(n-1)!) = 1$, it follows that

$$A \equiv 1 \pmod{n}.$$

Clearly, $2A = \binom{2n}{n}$ so that

$$\frac{1}{n+1}\binom{2n}{n} \equiv 2A \pmod{n}$$

and hence

$$r_n \equiv 4 \pmod{\mathrm{n}}.$$

3 The number of idempotents

As stated in the introduction the number of idempotents of various classes of semigroups of transformations has been computed. For further results see [1, 12, 20, 21, 22]. Our main task in this section is to compute the number of all idempotents in $\mathcal{PC}_n$. As in the previous section, we consider

$$e(n,r,k) = |\{\alpha \in \mathcal{PC}_n : \alpha^2 = \alpha, |\mathrm{Dom}\ \alpha| = r \wedge \max(\mathrm{Im}\ \alpha) = k\}|. \tag{3.1}$$

Then clearly we have

$$e(n,r,0) = \begin{cases} 1 & (r=0); \\ 0 & (r>0); \end{cases}, e(n,0,k) = \begin{cases} 1 & (k=0); \\ 0 & (k>0); \end{cases}$$

and

$$e(n,r,1) = \binom{n-1}{r-1}.$$

The latter corresponds to the number of all idempotents α in $\mathcal{PC}_n$ of width r and $\mathrm{Im}\ \alpha = \{1\}$, that is the number of all subsets of X_n each containing the element 1 and of size r. More generally, we have

Lemma 3.1 *For all $n \geq r, k \geq 1$ and $n > k$, we have*

$$e(n,r,k) = e(n-1,r,k) + e(n-1,r-1,k).$$

Proof. If $n \notin \mathrm{Dom}\ \alpha$ then $n \notin \mathrm{Im}\ \alpha$, by idempotency and so there are $e(n-1,r,k)$ idempotents of this type. If on the other hand $n \in \mathrm{Dom}\ \alpha$ then $n\alpha = k < n$ and of course $k\alpha = k$. It is now not difficult to see that the number of such idempotents is $e(n-1,r-1,k)$. Hence the result follows.

Lemma 3.2 *For $n \geq r \geq 1$, $e(n,r,n) = \displaystyle\sum_{t=0}^{n-1} e(n-1,r-1,t)$.*

Proof. Since $n = \max(\mathrm{Im}\ \alpha)$, it follows by the order-decreasing property that $n\alpha^{-1} = \{n\}$ and so there is no interference with the elements of $X_n \setminus \{n\}$ of which there are $\displaystyle\sum_{t=0}^{n-1} e(n-1,r-1,t)$ possible idempotents.

Proposition 3.3 *Let $e(n,r) = \displaystyle\sum_{k=0}^{n} e(n,r,k)$. For $n \geq r > 0$, we have*

$$e(n,r) = 2^{r-1}\binom{n}{r}.$$

Proof. First note that $e(n,1)$ is the number of all idempotents of width 1, that is of the form Dom $\alpha = \{x\}$ of which there are n of them and this agrees with the assertion of the proposition. Suppose now by way of induction $e(n,r)$ is true for all $n > r > 0$. Then using Lemmas 3.1 and 3.2 and the induction hypothesis successively, we have

$$\begin{aligned}
e(n,r) &= \sum_{k=0}^{n} e(n,r,k) = e(n,r,n) + \sum_{k=0}^{n-1} e(n,r,k) \\
&= \sum_{t=0}^{n-1} e(n-1,r-1,t) + \sum_{k=0}^{n-1} \{e(n-1,r,k) + e(n-1,r-1,k)\} \\
&= 2e(n-1,r-1) + e(n-1,r) \qquad (r \geq 2) \\
&= 2 \cdot 2^{r-2} \binom{n-1}{r-1} + 2^{r-1} \binom{n-1}{r} = 2^{r-1} \binom{n}{r}
\end{aligned}$$

as required.

Corollary 3.4 [6, theorem 3.19]. *Let $\mathcal{C}_n$ be the semigroup of all decreasing and order-preserving full transformations of X_n. Then*

$$|E(\mathcal{C}_n)| = e(n,n) = 2^{n-1}.$$

We now have the main result of this section:

Proposition 3.5 *Let $\mathcal{PC}_n$ be as defined in (1.1). Then $|E(\mathcal{PC}_n)| = \frac{1}{2}(3^n + 1)$.*

Proof.

$$\begin{aligned}
|E(\mathcal{PC}_n)| &= \sum_{r=0}^{n} e(n,r) = 1 + \sum_{r=1}^{n} e(n,r) = 1 + \sum_{r=1}^{n} 2^{r-1} \binom{n}{r} \\
&= 1 + \frac{1}{2} \sum_{r=1}^{n} 2^r \binom{n}{r} = 1 + \frac{1}{2}(3^n - 1) = \frac{1}{2}(3^n + 1).
\end{aligned}$$

Let $g(n,k)$ be the number of maps in $\mathcal{PC}_n$ of waist k. Then $g(n,k) = \sum_{r=0}^{n} e(n,r,k)$, and a closed formula for $g(n,k)$ is now possible. First we show the following lemma:

Lemma 3.6 *For all $n \geq k > 0$, $g(n,k) = 2^{n-k} g(k,k)$.*

Proof.

$$g(n,k) = \sum_{r=0}^{n} e(n,r,k) = \sum_{r=0}^{n} \{e(n-1,r,k) + e(n-1,r-1,k)\} = 2g(n-1,k).$$

By iteration we have

$$g(n,k) = 2^{n-k} g(k,k)$$

as required.

Now let $e_n = \sum_{k=0}^{n} g(n,k)$. Then we have

Lemma 3.7 $g(n,n) = e_{n-1}$.

Proof.

$$\begin{aligned} g(n,n) &= \sum_{r=0}^{n} e(n,r,n) = \sum_{r=0}^{n}\sum_{t=0}^{n} e(n-1,r-1,t) \\ &= \sum_{t=0}^{n}\sum_{r=0}^{n} e(n-1,r-1,t) = \sum_{t=0}^{n} g(n-1,t) = e_{n-1}. \end{aligned}$$

Proposition 3.8 *Let* $g(n,k) = \sum_{r=0}^{n} e(n,r,k)$*. For* $n \geq k > 0$*, we have*

$$g(n,k) = 2^{n-k-1}(3^{k-1}+1).$$

Proof. By Lemmas 3.6 and 3.7 and Proposition 3.5 successively we have

$$g(n,k) = 2^{n-k}g(k,k) = 2^{n-k}\cdot 2^{-1}(3^{k-1}+1) = 2^{n-k-1}(3^{k-1}+1) \qquad (k<n),$$

as required. Moreover, by Lemma 3.7 and Proposition 3.5

$$g(n,n) = e_{n-1} = \frac{1}{2}(3^{n-1}+1) = 2^{-1}(3^{n-1}+1)$$

as required. Hence the proof is complete.

Remark 3.1 The triangular array of numbers $e(n,r)$ is referred to in [17, A082137] as square arrays of transforms of binomial coefficients, read by anti-diagonals. But the triangular array of numbers $g(n,k)$ is not yet listed in [17]. For some selected values of these numbers, see Tables 4 and 5.

$n \backslash r$	0	1	2	3	4	5	6	7	$\sum e(n,r)$
0	1								1
1	1	1							2
2	1	2	2						5
3	1	3	6	4					14
4	1	4	12	16	8				41
5	1	5	20	40	40	16			122
6	1	6	30	80	120	96	32		365
7	1	7	42	140	280	336	224	64	1094

Table 4. *e(n, r)*

$n \backslash k$	0	1	2	3	4	5	6	7	$\sum g(n,k)$
0	1								1
1	1	1							2
2	1	2	2						5
3	1	4	4	5					14
4	1	8	8	10	14				41
5	1	16	16	20	28	41			122
6	1	32	32	40	56	82	122		365
7	1	64	64	80	112	164	244	365	1094

Table 5. *g(n, k)*

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