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14-term Arithmetic Progressions on Quartic Elliptic Curves

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Abstract

Let $P_4(x)$ be a rational quartic polynomial which is not the square of a quadratic. Both Campbell and Ulas considered the problem of finding an rational arithmetic progression $x_1, x_2, \ldots, x_n$, with $P_4(x_i)$ a rational square for $1 \le i \le n$. They found examples with $n = 10$ and $n = 12$. By simplifying Ulas' approach, we can derive more general parametric solutions for $n = 10$, which give a large number of examples with $n = 12$ and a few with $n = 14$.

1 Introduction

Let $P_4(x) = ax^4 + bx^3 + cx^2 + dx + e$ be a rational polynomial which is not the square of a quadratic. If $P_4(s) = t^2$ for rational (s, t) then P_4 is birationally equivalent to an elliptic curve. Campbell [1] investigated the possibility of such curves having a rational arithmetic progression $x_1, x_2, \ldots, x_n$ of arguments such that $P_4(x_i) = y_i^2, i = 1, \ldots, n$, and provided an example with $n = 12$.

Ulas [3] approached the problem in a different manner and succeeded in deriving a parametric solution for $n = 10$, and used a specific elliptic curve to derive a family of solutions for $n = 12$. He also introduced the name *quartic elliptic curve* for such curves.

In this short note, we show that Ulas' approach can be considerably simplified. This allows us to find a more general parametric solution for $n = 10$, which generates a large number of solutions for $n = 12$ and a few for $n = 14$, thus answering the open question at the end of Ulas' paper.

2 Algebraic Formulation

Ulas considers the arithmetic progression (AP for short) $\{1, 2, 3, \ldots, 9, 10\}$, and fits P_4 at $\{1, 2, 3, 4, 5\}$ to the set of values $\{p^2, q^2, r^2, s^2, t^2\}$. He then forces P_4 at $\{6, 7, 8, 9, 10\}$ to fit the set $\{t^2, s^2, r^2, q^2, p^2\}$.

It is a simple observation that this is equivalent to enforcing P_4 to be symmetric about $x = 5.5$. But we can easily translate the x-arguments so that the quartic is symmetric about $x = 0$, which makes P_4 an even function. Thus, the new approach is to assume $P_4(x) = ax^4 + bx^2 + c$.

Set $P_4(1) = P_4(-1) = p^2$, $P_4(3) = P_4(-3) = q^2$, and $P_4(5) = P_4(-5) = r^2$. It is standard linear algebra to find

$$a = \frac{2p^2 - 3q^2 + r^2}{384}$$

$$b = -\frac{34p^2 - 39q^2 + 5r^2}{192}$$

$$c = \frac{150p^2 - 25q^2 + 3r^2}{128}$$

Enforcing $P_4(7) = P_4(-7) = s^2$ implies that we must have

$$s^2 = 5p^2 - 9q^2 + 5r^2$$

Since $(1, 1, 1, 1)$ is an obvious solution, we can parameterize as follows

$$p = -5u^2 - 9v^2 + 18uv + 5w^2 - 10uw$$

$$q = 5u^2 - 10uv + 9v^2 - 10vw + 5w^2$$

$$r = 5u^2 - 10uw - 9v^2 + 18vw - 5w^2$$

Now, setting $P_4(9) = P_4(-9) = t^2$, gives

$$t^2 = 25u^4 + Eu^3 + Fu^2 + Gu + H \tag{1}$$

with

(i) $E = 40(15w - 7v)$,

(ii) $F = 2(347v^2 - 680vw + 25w^2)$,

(iii) $G = -8(63v^3 + 65v^2w - 235vw^2 + 75w^3)$,

(iv) $H = 81v^4 + 1440v^3w - 2330v^2w^2 + 800vw^3 + 25w^4$

We consider equation (1) as a quartic in u, with v, w as free parameters. Investigations show simple rational values of u which give rational t. One such is $u = 9v/5 - w$ which gives $t = 2(3v - 5w)(6v - 5w)/5$. The existence of this point shows that the quartic is birationally equivalent to an elliptic curve.

Using the method described in Mordell [2], we find that the elliptic curve can be written in the form

$$J^2 = K^3 - (137v^2 - 680vw + 475w^2)K^2 \tag{2}$$
$$+84(54v^4 - 495v^3w + 1425v^2w^2 - 1625vw^3 + 625w^4)K$$

with the transformation

$$u = \frac{2K(7v - 15w) + J - 42(27v^3 - 195v^2w + 325vw^2 - 125w^3)}{5K - 210(3v^2 - 20vw + 25w^2)} \tag{3}$$

This elliptic curve has an obvious point of order 2, namely $(0, 0)$. Numerical investigations suggest this is the only torsion point, but this would appear to be difficult to prove.

These numerical investigations, using small integer values of v and w, also indicate that the curve has rank at least 2, and often 3 or higher, with many integer points of small height.

Investigations found several algebraic formulae for points on the curves. Two of these are

$$(\ 2(3v - 5w)(6v - 5w)\ ,\ \pm 10(v - 5w)(3v - 5w)(6v - 5w)\ )$$

and

$$(\ 6(3v - w)^2\ ,\ \pm 6(3v - w)(3v^2 - 106vw + 91w^2)\ )$$

It is possible to use these and the previous algebraic expressions to derive parametric formulae for quartic elliptic curves with 10-term APs, but these expressions will only cover a small fraction of possible curves. In the next section we employ a simple search procedure for 12-term and 14-term curves.

3 Curve Searching

As was stated before, the elliptic curves (2) contain a large number of integer points of small height. For a selected pair of (v, w) values, we searched for rational points with $|K| < 10^7$. From these points, we generated u, then (p, q, r) and finally (a, b, c).

For the quartic produced, we tested if $P_4(11)$ was a square, giving a 12-term AP. We found several hundred such curves with (v, w) in the range $|v| + |w| < 100$.

For the successful curves, $P_4(13)$ was tested to be square. This discovered a total of 4 quartics giving a 14-term AP. These are

(a) $-17x^4 + 3130x^2 + 8551$,

(b) $2002x^4 - 226820x^2 + 18168514$,

(c) $3026x^4 - 222836x^2 + 3709234$,

(d) $34255x^4 - 1436006x^2 + 447963175$

Quartic (a) was derived from 3 different (v, w) pairs while (c) came from 2 different pairs. Given the paucity of 14-term curves compared to the abundance of 12-term curves, we would be very unlikely to find a 16-term curve with this method.

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References

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