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On Kazhdan's property (T) and Kazhdan constants associated to a Laplacian for SL(3,R)

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Abstract. The first purpose of this paper is to give a very elementary proof of Property (T) for $\mathrm{SL}(3, \mathbf{k})$ over any local field $\mathbf{k}$. Here we use a modification of an argument due to Burger. The second approach to Property (T) is based on spectral properties of a Laplacian in the enveloping algebra. It is shown that for a connected Lie group G Property (T) can be characterized by a spectral property of a Laplacian on the space of smooth K-finite vectors, where K is a compact subgroup of G.

1. Introduction

A locally compact group has *Kazhdan's Property* (T) if the following holds: whenever a (strongly continuous) unitary representation of G has almost invariant vectors, then actually it has a nonzero fixed vector. Recall that a representation $(\pi, \mathcal{H})$ has *almost invariant vectors* if, for any $\varepsilon > 0$ and for any compact set K of G, there exists a unit vector $\xi \in \mathcal{H}$ with $\|\pi(g)\xi - \xi\| < \varepsilon$ for all $g \in K$.

Property (T), discovered in 1967 by D. Kazhdan [11], is a powerful tool, with applications, for instance, in rigidity, geometry, graph theory and operator algebras (see [14, 20, 12, 7]). Most semisimple Lie groups have Property (T). More precisely, all simple Lie groups, except those which are locally isomorphic to $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$, have Property (T). It is an important fact that Property (T) is inherited by lattices. So, for instance, $\mathrm{SL}(n,\mathbb{Z})$ has Property (T) for $n \geq 3$.

The main step in establishing Property (T) for simple Lie groups of $\mathbb{R}$–rank ≥ 2 is the proof for $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ of this property. The usual proofs for $\mathrm{SL}(3,\mathbb{R})$ (and for $\mathrm{Sp}(2,\mathbb{R})$) are based on the study - by means of Mackey's theory - of the irreducible unitary representations of a copy of the semi-direct product $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ inside $\mathrm{SL}(3,\mathbb{R})$ (or the semi-direct product of $\mathrm{SL}(2,\mathbb{R})$ by the space of symmetric 2×2–matrices), see [14, 20, 12]. Another argument, due to M. Burger, based on the so-called Furstenberg lemma, appears in [9]. In [10], an alternative proof is given using estimates of independent interest for matrix coefficients of unitary representations.

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The first purpose of this paper is to give a very elementary proof of Property (T) for $\mathrm{SL}(3,\mathbf{k})$ over any local field $\mathbf{k}$. It is a modification of Burger's argument mentioned above. Instead of Furstenberg lemma, it uses the fact that there is no $\mathrm{SL}(2,\mathbf{k})$–invariant mean on the Borel sets in $\mathbf{k}^2 \setminus \{0\}$ (see 2.2 below). In case $\mathbf{k} = \mathbb{R}$, a new proof is given in terms of the Lie algebra for the following well–known but crucial fact (Lemma 2.4): If a vector in a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ is invariant under the upper triangular unipotent matrices then it is invariant under $\mathrm{SL}(2,\mathbb{R})$. In fact, our proof shows that the same result is true for the universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Our second approach to Property (T) for $\mathrm{SL}(3,\mathbb{R})$ is of a quantitative nature and is based on spectral properties of a *Laplacian* in the enveloping algebra $\mathcal{U}(\mathfrak{sl}(3,\mathbb{R}))$. In [3], Kazhdan's Property (T) for a connected Lie group G is characterized as follows: Let $\mathfrak{g}$ be the Lie algebra of G, let $X_1, X_2, \ldots, X_n$ be a basis of $\mathfrak{g}$, and let $\Delta := -(X_1^2 + X_2^2 + \ldots + X_n^2)$ be the associated Laplacian in $\mathcal{U}(\mathfrak{g})$. Then G has Property (T) if and only if there exists $\varepsilon > 0$ such that $\inf \mathrm{sp}(d\pi(\Delta)) \geq \varepsilon$ for any unitary representation $(\pi, \mathcal{H})$ of G without nonzero fixed vector, where $d\pi$ denotes the derived representation of π in the space $\mathcal{H}^\infty$ of C^∞–vectors in $\mathcal{H}$ and sp the spectrum. As this is more convenient for computations, we first show that one may equally consider the smaller space of K–finite vectors for a compact subgroup K of G. Recall that a vector $\xi \in \mathcal{H}$ is K–finite if the linear span of $\pi(K)\xi$ has finite dimension.

Theorem 1.1. *The connected Lie group G has Property (T) if and only if there exists a constant $\varepsilon > 0$ such that*

$$\inf\{\langle d\pi(\Delta)\xi, \xi\rangle,\ \xi \in \mathcal{H}^{\infty,K}, \|\xi\| = 1\} \geq \varepsilon$$

for any unitary representation $(\pi, \mathcal{H})$ of G without nonzero fixed vector, where $\mathcal{H}^{\infty,K}$ is the space of all K–finite C^∞–vectors in $\mathcal{H}$ for a compact subgroup K of G.

Our main result gives a bound for the constant ε appearing above.

Theorem 1.2. *Let $K := \mathrm{SO}(3,\mathbb{R})$, and let $X_1, X_2, \ldots, X_8$ be the following basis of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$:*

$$X_1 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_5 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X_6 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$X_7 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_8 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and let

$$\Delta := -(X_1^2 + X_2^2 + \ldots + X_8^2).$$

Then, for any unitary representation $(\pi, \mathcal{H})$ *of* $\mathrm{SL}(3,\mathbb{R})$ *without nonzero fixed vector,*

$$\inf\{\langle d\pi(\Delta)\xi, \xi\rangle;\ \xi \in \mathcal{H}^{\infty,K},\ \|\xi\| = 1\} \ \geq\ \alpha \approx 0.4613\ ,$$

where α *is the maximal value of the function* $\frac{2\sin^2\theta}{\pi\theta}$ *on* $\mathbb{R}$.

The reason for the choice of the above basis is that $\Delta = -\mathcal{C} + 2\mathcal{C}_K$, where $\mathcal{C}$ and $\mathcal{C}_K$ are the Casimir operators of $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SO}(3,\mathbb{R})$, respectively. The proof of the above estimate is inspired by some ideas due to Howe and Tan [10], Chapter V.3.3.

Kazhdan's constants depending on a generating set may also be defined at the group level (see [9]). Such constants have been studied, for instance, in [1, 4, 5, 6, 8, 15, 17]. The paper is organized as follows. In Section 2, we give the proof of Kazhdan's Property (T) for $\mathrm{SL}(3,\mathbf{k})$. Section 3 is devoted to the proof of Theorem 1.1, and Theorem 1.2 is proved in Section 4.

2. Kazhdan Property (T) for $\mathrm{SL}(3,\mathbf{k})$

The proof depends on the following three lemmas. The first lemma says that the representation $(\pi,\mathcal{H})$ is amenable in the sense of [2], where more general results are proved. We thank S. Popa for the following direct and simple proof.

Lemma 2.1. *Let* G *be a locally compact group, and let* $(\pi,\mathcal{H})$ *be a unitary representation of* G *with almost invariant vectors. Then there is an* $\mathrm{Ad}(G)$*–invariant state* φ *on the* C^**–algebra* $\mathcal{L}(\mathcal{H})$ *of all bounded operators on* $\mathcal{H}$*, that is, a positive linear form* $\varphi : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ *with* $\varphi(\mathrm{I}) = 1$ *and* $\varphi(\pi(x)T\pi(x)^{-1}) = \varphi(T)$ *for all* $x \in G$, $T \in \mathcal{L}(\mathcal{H})$.

Proof. Let $\{\xi_n\}_n \subseteq \mathcal{H}, \|\xi_n\| = 1$, with $\lim_{n\to\infty} \|\pi(x)\xi_n - \xi_n\| = 0$ for all $x \in G$. Define states φ_n on $\mathcal{L}(\mathcal{H})$ by

$$\varphi_n(T) := \langle T\xi_n, \xi_n\rangle, \quad T \in \mathcal{L}(\mathcal{H})\ .$$

Since the set of states on $\mathcal{L}(\mathcal{H})$ is a weak–$*$–compact subset of the unit ball of $\mathcal{L}(\mathcal{H})^*$, we may assume (upon passing to a subnet) that there exists a state φ on $\mathcal{L}(\mathcal{H})$ with

$$\lim_{n\to\infty} \varphi_n(T) = \varphi(T) \quad \forall T \in \mathcal{L}(\mathcal{H})\ .$$

Then φ is $\mathrm{Ad}(G)$–invariant as, for any $x \in G$, $T \in \mathcal{L}(\mathcal{H})$,

$$\begin{aligned} |\varphi_n(\pi(x)T\pi(x)^{-1}) - \varphi_n(T)| &= |\langle T\pi(x)\xi_n, \pi(x)\xi_n\rangle - \langle T\xi_n, \xi_n\rangle| \\ &\leq 2\|T\|\|\pi(x)\xi_n - \xi_n\| \end{aligned}$$

and hence

$$|\varphi(\pi(x)T\pi(x)^{-1}) - \varphi(T)| = \lim_{n\to\infty} |\varphi_n(\pi(x)T\pi(x)^{-1}) - \varphi_n(T)| = 0\ . \qquad \blacksquare$$

Recall that a local field is a locally compact, nondiscrete field (archimedean or nonarchimedean), and that the topology of such a field is defined by an absolute value. In fact, any such field is isomorphic either to $\mathbb{R}$, to $\mathbb{C}$, to a finite extension of the p–adic numbers or to the Laurent series in one variable over a finite field (see [19, Chap. I,A73]).

Lemma 2.2. *Let* $\mathbf{k}$ *be a local field . Let* $\mathrm{SL}(2,\mathbf{k})$ *act on* $\mathbf{k}^2$ *in the natural way. Then the Dirac measure at* $\{0\}$ *is the only finitely additive* $\mathrm{SL}(2,\mathbf{k})$*–invariant probability measure on the Borel sets of* $\mathbf{k}^2$*. Equivalently, there is no finitely additive,* $\mathrm{SL}(2,\mathbf{k})$*–invariant probability measure on the Borel sets of* $\mathbf{k}^2 \setminus \{0\}$.

Proof. Let

$$\mu : \mathcal{B}(\mathbf{k}^2) \to \mathbb{R}^+$$

be a finitely additive, $\mathrm{SL}(2,\mathbf{k})$–invariant probability measure on the Borel sets $\mathcal{B}(\mathbf{k}^2)$ of $\mathbf{k}^2$. Let $|\cdot|$ be an absolute value on $\mathbf{k}$. Let

$$\Omega := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \setminus \{0\};\ |y| \geq |x| \right\} .$$

Take a sequence $\{\lambda_n\}_n \subseteq \mathbf{k}$ with $|\lambda_{n+1}| > |\lambda_n| + 2$ for all $n \in \mathbb{N}$, and let

$$g_n := \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbf{k}) .$$

Then

$$\Omega_n := g_n\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \setminus \{0\};\ \frac{|x|}{|\lambda_n|+1} \leq |y| \leq \frac{|x|}{|\lambda_n|-1} \right\} .$$

Indeed, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega$,

$$g_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \lambda_n y \\ y \end{pmatrix}$$

and

$$\begin{aligned} |x+\lambda_n y| &\geq |\lambda_n y| - |x| \geq (|\lambda_n|-1)|y| , \\ |x+\lambda_n y| &\leq |x| + |\lambda_n y| \leq (|\lambda_n|+1)|y| . \end{aligned}$$

Clearly the sets Ω_n are pairwise disjoint, as

$$\frac{1}{|\lambda_n|-1} < \frac{1}{|\lambda_m|+1} \quad \text{for } n > m .$$

Hence $\sum_{i=1}^n \mu(\Omega_i) \leq \mu(\mathbf{k}^2 \setminus \{0\}) \leq 1$ for all $n \in \mathbb{N}$. Since $\mu(\Omega_i) = \mu(g_i\Omega) = \mu(\Omega)$, this shows that $\mu(\Omega) = 0$. Now, let

$$\Omega' := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \setminus \{0\};\ |x| \geq |y| \right\} .$$

Then $\mu(\Omega') = \mu(\Omega) = 0$. Since $\Omega \cup \Omega' = \mathbf{k}^2 \setminus \{0\}$, $\mu(\mathbf{k}^2 \setminus \{0\}) = 0$. So, μ is the Dirac measure at 0. ■

Remark 2.3. As the proof shows, the above lemma applies to other groups than $\mathrm{SL}(2,\mathbf{k})$, for instance to $\mathrm{SL}(2,\mathbb{Z})$ when $\mathrm{char}\,\mathbf{k}=0$. The conclusion of the lemma is certainly known to several people. A. Valette showed us a proof of the lemma, in the case $k=\mathbb{R}$, using only two matrices from $\mathrm{SL}(2,\mathbb{Z})$.

The last ingredient is the following well-known lemma (see [9, 14, 20, 12]) for which we give a new proof based on consideration of the Lie algebra, in the case where $\mathbf{k}=\mathbb{R}$.

Lemma 2.4. *Let $(\pi,\mathcal{H})$ be a unitary representation of $\mathrm{SL}(2,\mathbf{k})$. Assume that*

$$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \, x \in \mathbf{k} \right\}$$

has a nonzero fixed vector $\xi \in \mathcal{H}$. Then ξ is fixed by $\mathrm{SL}(2,\mathbf{k})$.

Proof. $(\mathbf{k}=\mathbb{R})$ Let

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}(2,\mathbb{R})$ with usual commutator relations

$$[X,Y]=H, \quad [H,X]=2X, \quad [H,Y]=-2Y \,.$$

The space $\mathcal{H}^N$ of the N–invariant vectors is invariant under the subgroup A generated by H . Hence, $\mathcal{H}^N$ contains a dense subspace $\mathcal{D}$ of C^∞–vectors under the action of AN . Clearly, $X=0$ on $\mathcal{D}$ and $\mathcal{D}$ is H–invariant. We first show that $H=0$ on $\mathcal{D}$. We write $W\xi$ instead of $d\pi(W)\xi$ for $W\in\mathfrak{sl}(2,\mathbb{R}), \xi\in\mathcal{H}$, whenever this makes sense. Recall that each W is skew-symmetric on $\mathcal{H}^\infty$. In fact, it is well-known that iW is essentially selfadjoint on $\mathcal{H}^\infty$ for any $W\in\mathfrak{sl}(2,\mathbb{R})$. Consider

$$\mathcal{C} := H^2+2(XY+YX) = H^2+4XY-2H \,,$$

the Casimir operator of $\mathfrak{sl}(2,\mathbb{R})$. We have, for any $\xi\in\mathcal{D}, \eta\in\mathcal{H}^\infty$:

$$\begin{aligned} \langle \xi, \mathcal{C}\eta\rangle &= \langle \xi, H^2\eta\rangle + 4\langle \xi, XY\eta\rangle - 2\langle \xi, H\eta\rangle \\ &= \langle \xi, (H^2-2H)\eta\rangle = \langle \left(H^2+2H\right)\xi, \eta\rangle \,. \end{aligned}$$

Thus, $\mathcal{D}$ is contained in the domain of $\mathcal{C}^*$ and $\mathcal{C}^* = H^2+2H$ on $\mathcal{D}$.
As is well-known (see [18, p.269, Ex.(3)], for instance), $\mathcal{C}$ is essentially selfadjoint on $\mathcal{H}^\infty$. Whence $\mathcal{C}\big|_{\mathcal{D}} = \mathcal{C}^*\big|_{\mathcal{D}}$ or $\left(H^2+2H\right)^* = H^2-2H$ on $\mathcal{D}$. Thus $H=0$ on $\mathcal{D}$. Now, fix $\xi\in\mathcal{D}$. Then , for any $\eta\in\mathcal{H}^\infty$,

$$\langle \xi, YH\eta\rangle = 2\langle \xi, Y\eta\rangle \,,$$

as $[H,Y]=-2Y$ and $H\xi=0$.
On the other hand, the range of $H-2\mathrm{I}$ as an operator on $\mathcal{H}^\infty$ is dense in $\mathcal{H}$. Indeed, this follows from the fact that iH is essentially selfadjoint on $\mathcal{H}$.
Thus,

$$\langle \xi, Y\eta\rangle = 0 \quad \text{for all } \eta\in\mathcal{D}' \,,$$

where $\mathcal{D}' = (H - 2\mathrm{I})\mathcal{H}^\infty$. This shows that ξ is in the domain of $Y^* = -\overline{Y}$ and that $\overline{Y}\xi = 0$. Hence, by Stone's theorem,

$$\exp(tY)\xi = \exp(t\overline{Y})\xi = \xi$$

for all $t \in \mathbb{R}$. Thus ξ is fixed by the subgroup $\overline{N}$, generated by Y . Hence, any $\xi \in \mathcal{D}$ is fixed by $\mathrm{SL}(2,\mathbb{R})$. By density, this is true for any N–fixed vector in $\mathcal{H}$. ■

Remark 2.5. The above proof is somewhat involved because it is not a priori clear whether the space of N–fixed vectors contains any nonzero C^∞–vector. The arguments above become much shorter in the case of an N–invariant C^∞–vector ξ as the reader may wish to verify.
On the other hand, because it relies on Lie algebra considerations, the proof works for any covering group of $\mathrm{SL}(2,\mathbb{R})$ (where N has to be taken as the one–parameter subgroup generated by X).

Theorem 2.6. [11] $\mathrm{SL}(3,\mathbf{k})$ *has Kazhdan's Property (T).*

Proof. Let $(\pi,\mathcal{H})$ be a unitary representation of $\mathrm{SL}(3,\mathbf{k})$ with almost invariant vectors. Let

$$H := \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{k}), \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \right\} \simeq \mathrm{SL}(2,\mathbf{k}) \ltimes \mathbf{k}^2,$$

and let

$$V := \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{k}^2 \right\} \simeq \mathbf{k}^2 .$$

Let

$$P : \mathcal{B}(\hat{V}) \to \mathcal{L}(\mathcal{H}), \quad E \mapsto P(E)$$

be the projection valued measure associated with the unitary representation $\pi|_V$ of the abelian group V , see e. g., [13]. Clearly

$$\pi(g)P(E)\pi(g)^{-1} = P(g\cdot E) \quad \forall g \in \mathrm{SL}(2,\mathbf{k}), E \in \mathcal{B}(\hat{V}) . \tag{1}$$

Here

$$g\cdot\gamma\begin{pmatrix} x \\ y \end{pmatrix} := \gamma\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right), \quad g \in \mathrm{SL}(2,\mathbf{k}), \gamma \in \hat{V}$$

is the dual action.
By Lemma 2.1, there exists an $\mathrm{Ad}(\mathrm{SL}(3,\mathbf{k}))$–invariant state φ on $\mathcal{L}(\mathcal{H})$. Define

$$m(E) := \varphi(P(E)) \quad \forall E \in \mathcal{B}(\mathbf{k}^2) .$$

Then, m is a finitely additive probability measure on $\mathcal{B}(\hat{V})$. Moreover, m is $\mathrm{SL}(2,\mathbf{k})$–invariant, by (1). Now $\hat{\mathbf{k}}$ identifies with $\mathbf{k}$ in such a way that the action of $\mathrm{SL}(2,\mathbf{k})$ corresponds to the transpose of the action of $\mathrm{SL}(2,\mathbf{k})$ on $\mathbf{k}^2$ (see [19]). In this way, m becomes an invariant finitely additive probability measure on the

Borel sets of $\mathbf{k}^2$. So, by Lemma 2.2, m is the Dirac measure at 0. In particular, $P(\{0\}) \neq 0$. This shows that $\pi|_V$ has a nonzero invariant vector ξ. By Lemma 2.4, ξ is invariant under the following two copies of $\mathrm{SL}(2,\mathbf{k})$

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix};$$

which generate together $\mathrm{SL}(3,\mathbf{k})$. This concludes the proof. ■

3. Kazhdan Constants and K-Finite Vectors

In this section, we prove Theorem 1.1. Let $(\pi,\mathcal{H})$ be a (strongly continuous) unitary representation of a connected Lie group G and denote by $d\pi$ the derived representation of the Lie algebra $\mathfrak{g}$. Let $(X_1, X_2, \ldots, X_n)$ be a basis of $\mathfrak{g}$ and $\Delta := -\sum_{j=1}^n X_i^2 \in \mathcal{U}(\mathfrak{g})$ be the associated Laplacian. Then $d\pi(\Delta)$ is defined on $\mathcal{H}^\infty$, positive and essentially selfadjoint. Let $\overline{d\pi(\Delta)}$ be its closure.

In [3] the operators $d\pi(\Delta)$ and $\overline{d\pi(\Delta)}$ are used in order to decide whether or not $(\pi,\mathcal{H})$ contains weakly the trivial representation. The main result of that paper is

Theorem 3.1. *Let $(\pi,\mathcal{H})$ be a unitary representation of a connected Lie group G . Then the following are equivalent:*

(i) $(\pi,\mathcal{H})$ *contains weakly the trivial representation.*

(ii) 0 *is an approximative eigenvalue of* $d\pi(\Delta)$.

(iii) 0 *is a spectral value of* $\overline{d\pi(\Delta)}$.

We are going to see that one may also consider the restriction of the Laplacian to smaller subspaces where the computations become easier. Recall that $\xi \in \mathcal{H}$ is called *analytic for* π if the mapping

$$g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is analytic on G for all $\eta \in \mathcal{H}$. The space of analytic vectors is denoted by $\mathcal{H}^\omega$. Finally, let T be an operator in $\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is called *analytic for* T if the series

$$\sum_{j=1}^{\infty} \frac{\|T^j\xi\|}{j!} t^j$$

has positive convergence radius. By the spectral theorem, the analytic vectors of a selfadjoint operator are dense in $\mathcal{H}$. The following result of R. Goodman relates these notions of analyticity.
Define

$$B_\pi := \left(\mathrm{I} + \overline{d\pi(\Delta)}\right)^{\frac{1}{2}} .$$

B_π is a selfadjoint, positive operator, $B_\pi \geq \mathrm{I}$. Then ξ is an analytic vector for B_π if and only if ξ is an analytic vector for π and this is the case if and only if $\|d\pi(\Delta)^m\xi\| \leq (2m)!M^m\|\xi\|$, for all $m \in \mathbb{N}_0$ and a suitable $M > 0$. For all these results, see [18, Chapter 4.4].

Proposition 3.2. *With the above notations, let $\mathcal{D}$ be a subspace of $\mathcal{H}$. Assume that $\mathcal{D} \cap \mathcal{H}^\omega$ is dense in $\mathcal{H}$. Then the following are equivalent:*

(i) $(\pi, \mathcal{H})$ *contains weakly the trivial representation.*

(ii) $\inf\{\langle d\pi(\Delta)\xi, \xi\rangle;\ \xi \in \mathcal{D}, \|\xi\| = 1\} = 0$.

Proof. By 3.1 it is enough to prove that (i) implies (ii).
Since $\mathcal{D} \cap \mathcal{H}^\omega$ is dense in $\mathcal{H}$, Goodman's result implies that $\mathcal{D}$ contains a dense set of vectors analytic for $B_\pi|_\mathcal{D}$, where B_π is defined as above. Since $B_\pi|_\mathcal{D}$ is symmetric, Nelson's theorem (see e.g., [16, X.39]) shows that $B_\pi|_\mathcal{D}$ is essentially selfadjoint, whence $\overline{B_\pi|_\mathcal{D}} = B_\pi$. By functional calculus, there is a sequence $\{\xi_n\}_n$ of unit vectors in the domain of B_π , satisfying $\|B_\pi(\xi_n) - \xi_n\| < \frac{1}{2n}$. Let $\{\eta_n\}_n$ be sequence in $\mathcal{D}$ with $\|B_\pi\xi_n - B_\pi\eta_n\| + \|\xi_n - \eta_n\| < \frac{1}{2n}$. Then $\|B_\pi(\eta_n) - \eta_n\| < \frac{1}{n}$. Let $\psi_n := \frac{\eta_n}{\|\eta_n\|} \in \mathcal{D}$. Then $\lim_{n\to\infty}\langle(B_\pi - \mathrm{I})\psi_n, \psi_n\rangle = 0$, by Cauchy–Schwarz inequality and, hence,

$$\lim_{n\to\infty}\langle d\pi(\Delta)\psi_n, \psi_n\rangle = 0 = \lim_{n\to\infty}\langle(B_\pi - \mathrm{I})^2\psi_n, \psi_n\rangle + 2\lim_{n\to\infty}\langle(B_\pi - \mathrm{I})\psi_n, \psi_n\rangle = 0$$

as desired. ■

An often useful choice of $\mathcal{D}$ is the following: Let K be a compact subgroup of G. By the Peter–Weyl theorem, we may decompose $(\pi|_K, \mathcal{H})$ in K-isotypic components $\mathcal{H}(\sigma)$

$$\mathcal{H} = \sum_{\sigma\in\hat{K}} \mathcal{H}(\sigma) .$$

Then the subspaces (algebraic sum)

$$\mathcal{H}^{\infty,K} := \sum_{\sigma\in\hat{K}} (\mathcal{H}(\sigma) \cap \mathcal{H}^\infty) \quad \text{and} \quad \mathcal{H}^{\omega,K} := \sum_{\sigma\in\hat{K}} (\mathcal{H}(\sigma) \cap \mathcal{H}^\omega)$$

are dense in $\mathcal{H}$ [18, 4.4.3.1,4.4.5.16].

Corollary 3.3. *With the above notations, the following are equivalent:*

(i) $(\pi, \mathcal{H})$ *contains weakly the trivial representation.*

(ii) $\kappa_K(d\pi(\Delta), G) := \inf\{\langle d\pi(\Delta)\xi, \xi\rangle,\ \xi \in \mathcal{H}^\infty_K, \|\xi\| = 1\} = 0$.

We define the infinitesimal Kazhdan constant $\kappa_K(\Delta, G)$ to be the least upper bound of all$\{\kappa_K(d\pi(\Delta), G)$ where π ranges through all unitary representations which do not contain the trivial representation$\}$.

Corollary 3.4. *Let G be a connected Lie group. Then G has property (T) if and only if there exists $\varepsilon > 0$ such that*

$$\kappa_K(\Delta, G) > \varepsilon .$$

Proof. Using 3.2, this is similar to the proof of [3, 3.10]. ■

4. Kazhdan Constants for $\mathrm{SL}(3,\mathbb{R})$

We now give a bound for the above Kazhdan's constant in the case of $\mathrm{SL}(3,\mathbb{R})$. Consider the subgroup

$$H := \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\ x, y \in \mathbb{R} \right\} \simeq \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2 .$$

Observe that if $(\pi, \mathcal{H})$ is a unitary representation of $\mathrm{SL}(3,\mathbb{R})$ and if ξ is fixed under the action of the subgroup

$$\mathbb{R}^2 \simeq V := \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y \in \mathbb{R} \right\},$$

then ξ is a fixed vector under $\mathrm{SL}(3,\mathbb{R})$ (see Lemma 2.5 above, and the proof of 2.6).
So, we may assume that $(\pi, \mathcal{H})$ is a unitary representation of $\mathrm{SL}(3,\mathbb{R})$ without V–fixed vectors. As in the proof of 2.6, there exists a projection valued measure P on the Borel sets of $\hat{V} \simeq V$ such that

$$\pi|_V = \int_{\hat{V}} \gamma dP(\gamma) \quad \text{and} \quad P(\{0\}) = 0 ,$$

and we have for all Borel sets $E \subseteq \hat{V}$:

$$P(g \cdot E) = \pi(g) P(E) \pi(g)^{-1}$$

(compare with (1)) for all $g \in \mathrm{SL}(2,\mathbb{R})$. Next, choose as a basis of $\mathfrak{sl}(2,\mathbb{R})$:

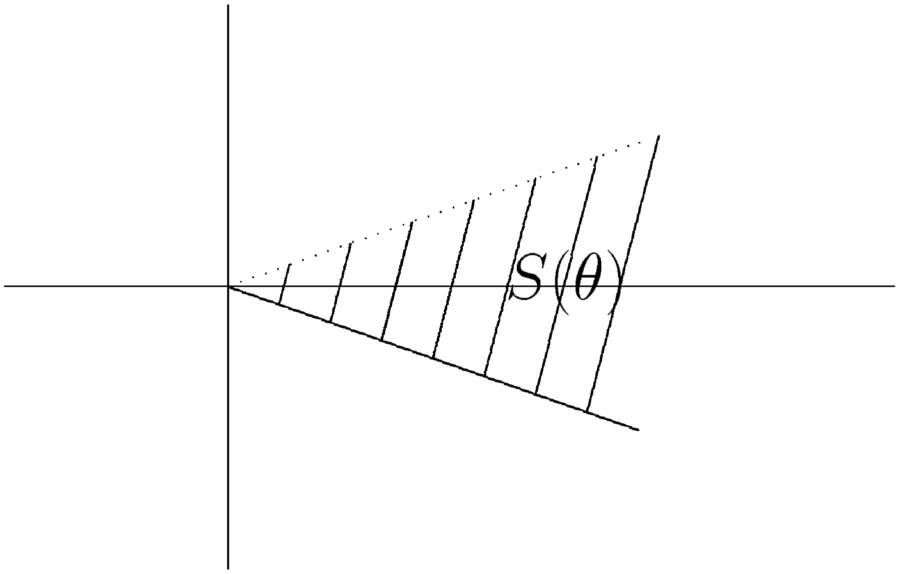
$$K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

If we embed $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(3,\mathbb{R})$ as above, K, A, B correspond to X_1, X_2, and X_3 in Theorem 1.2.
We have the commutator relations

$$[K, A] = -2B,\ [K, B] = 2A,\ [A, B] = 2K .$$

For an angle $0 \leq \theta \leq \pi$, let $S(\theta)$ be the sector

$$S(\theta) := \left\{ v \in \hat{V},\ \arg v \in \left[-\frac{\theta}{2}, \frac{\theta}{2} \right[\right\} .$$



The first crucial fact is that, for any unit K–eigenvector v, we have

$$\|P(S(\theta))v\|^2 = \frac{\theta}{2\pi} \; . \qquad (2)$$

This is an easy computation (see [10, p.223]). Indeed, since V has no nonzero fixed vector, $P(\mathbb{R}^2 \setminus \{0\}) = \mathrm{I}$. For an arc E on the unit circle, set $\mu(E) = \|P(S(E))v\|^2$, where $S(E)$ is the sector defined by E. The usual properties of a spectral measure show that μ is a measure on the circle. Further, μ is rotation invariant by the above relation (1). Hence, μ is the normalized arc lengh on the unit circle proving (2).

Set $a_t := \exp(tA)$, and let θ_t be the angle corresponding to the sector $a_t \cdot S(\theta)$. One easily computes that

$$\theta_t = 2\arctan(e^{2t} \tan \frac{\theta}{2}) \; . \qquad (3)$$

For a C^∞– vector v , we have

$$\|P(S(\theta_t))v\|^2 = \|\pi(a_t)P(S(\theta))\pi(a_{-t})v\|^2 = \|P(S(\theta))\pi(a_{-t})v\|^2 \; ,$$

by (1), and hence, differentiating at $t = 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \|P(S(\theta_t))v\|^2 &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \|P(S(\theta))\pi(a_{-t})v\|^2 \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \langle P(S(\theta))\pi(a_{-t})v, \pi(a_{-t})v\rangle \\ &= -\langle d\pi(A)v, P(S(\theta))v\rangle - \langle P(S(\theta))v, d\pi(A)v\rangle \; . \end{aligned}$$

Let v be a C^∞– vector which is a unit K–eigenvector. Then, (2) and (3) show that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \|P(S(\theta_t))v\|^2 &= \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \arctan(e^{2t} \tan \frac{\theta}{2}) = \frac{1}{\pi}\frac{2\tan\frac{\theta}{2}}{1 + (\tan\frac{\theta}{2})^2} \\ &= \frac{1}{\pi} \sin\theta \; . \end{aligned}$$

Hence,

$$-\langle d\pi(A)v, P(S(\theta))v\rangle - \langle P(S(\theta))v, d\pi(A)v\rangle \;=\; \frac{1}{\pi}\sin\theta\ .$$

Using Cauchy-Schwarz inequality, we find

$$2\|d\pi(A)v\|\sqrt{\frac{\theta}{2\pi}} \;=\; 2\|d\pi(A)v\|\|P(S(\theta))v\| \;\geq\; \frac{1}{\pi}\sin\theta\ ,$$

and

$$\|d\pi(A)v\| \;\geq\; \frac{1}{\sqrt{2\pi}}\frac{\sin\theta}{\sqrt{\theta}}\ ,$$

for all smooth unit K–eigenvectors v .
Observe that the same inequality holds for B , since B is conjugate to A under $\mathrm{SO}(2,\mathbb{R})$.

Now, we consider a second copy of $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2$ in $\mathrm{SL}(3,\mathbb{R})$, namely

$$H' := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & a & b \\ y & c & d \end{pmatrix},\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),\, x,y\in\mathbb{R} \right\}.$$

The same computation as above shows that, for $j\in\{2,3\}$ (respectively $j\in\{5,6\}$) :

$$\|d\pi(X_j)v\| \;\geq\; \frac{1}{\sqrt{2\pi}}\frac{\sin\theta}{\sqrt{\theta}}\ , \tag{4}$$

if v is a smooth unit X_1–eigenvector (respectively, if v is a smooth unit X_4–eigenvector).

Consider the Cartan decomposition $\mathfrak{sl}(3,\mathbb{R}) = \mathfrak{so}(3,\mathbb{R})\oplus\mathfrak{p}$, where $\mathfrak{p}$ are the symmetric matrices in $\mathfrak{sl}(3,\mathbb{R})$. Then X_1, X_4, X_7 is an orthonormal basis of $\mathfrak{so}(3,\mathbb{R})$ and X_2, X_3, X_5, X_6, X_8 is one of $\mathfrak{p}$, with respect to $\frac{1}{6}K(X,Y)$, where $K(X,Y)$ is the Killing form on $\mathrm{SL}(3,\mathbb{R})$. Hence, the Casimir operator $\mathcal{C}$ on $\mathrm{SL}(3,\mathbb{R})$ is

$$\mathcal{C} = (X_2^2 + X_3^2 + X_5^2 + X_6^2 + X_8^2) \,-\, (X_1^2 + X_4^2 + X_7^2),$$

the Casimir operator on $\mathrm{SO}(3,\mathbb{R})$ is $-(X_1^2 + X_4^2 + X_7^2)$, and

$$\Delta \;=\; -2(X_1^2 + X_4^2 + X_7^2) \,-\, \mathcal{C}\ , \tag{5}$$

Now, let w be a smooth $\mathrm{SO}(3,\mathbb{R})$–finite unit vector. Then $w=\sum_{j=1}^k v_j$, where the v_j belong to the isotypic components of pairwise inequivalent representations (in particular, they are pairwise orthogonal). By (5), $d\pi(\Delta)$ preserves the isotypic components. Hence,

$$\begin{aligned}\langle\Delta w, w\rangle \;&=\; \sum_{i,j=1}^k \langle d\pi(\Delta)v_i, v_j\rangle \;=\; \sum_{j=1}^k \langle d\pi(\Delta)v_j, v_j\rangle \\ &\geq\; \sum_{j=1}^k \left(\langle d\pi(\Delta_1)v_j, v_j\rangle + \langle d\pi(\Delta_2)v_j, v_j\rangle\right)\ ,\end{aligned}$$

where $\Delta_1 := -\sum_{\nu=1}^{3} X_\nu^2$ and $\Delta_2 := -\sum_{\nu=4}^{6} X_\nu^2$. Moreover, observe that $\Delta_1 = -\Box_1 - 2X_1^2$ and $\Delta_2 = -\Box_2 - 2X_4^2$, where $\Box_i$, $i = 1,2$, are the respective Casimir operators in the copies of $\mathfrak{sl}(2,\mathbb{R})$. Hence, if $v_j = \sum_{\ell=1}^{r_j} u_\ell^j$ is the orthogonal decomposition of the X_1–finite vector v_j in X_1–eigenvectors, equation (4) yields

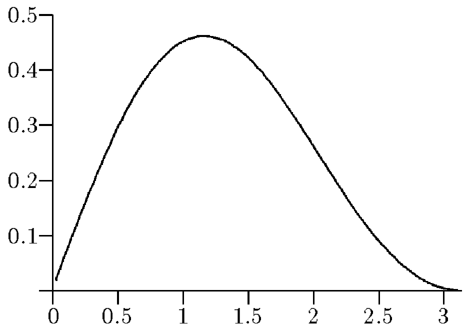
$$\langle d\pi(\Delta_1)v_j, v_j\rangle \geq \sum_{\ell=1}^{r_j} 2\left(\frac{1}{\sqrt{2\pi}}\frac{\sin\theta}{\sqrt{\theta}}\right)^2 \|u_\ell^j\|^2 = \frac{\sin^2\theta}{\pi\theta}\|v_j\|^2$$

Decomposing v_j into X_4–eigenvectors yields the same inequality, with Δ_1 replaced by Δ_2. Thus,

$$\langle \Delta w, w\rangle \geq \frac{2\sin^2\theta}{\pi\theta} \tag{6}$$

for any smooth $\mathrm{SO}(3,\mathbb{R})$–finite unit vector w .

Numerical computations show that the function $f(\theta) := \frac{2\sin^2\theta}{\pi\theta}$ assumes its maximal value of ≈ 0.4613 at $\theta \approx 1.1656$.



Thus

$$\langle \Delta w, w\rangle \geq 0.4613 ,$$

for all unit $\mathrm{SO}(3,\mathbb{R})$–finite vectors w .

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