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Fitting decomposition of Casimir operators of quadratic Lie superalgebras.

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Abstract. Semisimple Lie superalgebras are Lie superalgebras with non-degenerate Killing forms [1]. In this paper we show that a quadratic Lie superalgebra $(\mathfrak{g},B)$ is semisimple if and only if its Casimir operator, Ω_B, associated to B is invertible. This result anables us to characterize quadratic Lie superalgebras with nilpotent Casimir operators: Ω_B is nilpotent if and only if $\mathfrak{g}$ is a Lie superalgebra without any nonzero semisimple ideal.

1. Introduction

Lie superalgebras considered in this paper are finite dimensional over a commutative field $\mathbb{K}$ with characteristic zero.

In our paper [1] we have choose a definition of semisimplicity for Lie superalgebras: a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is semisimple if and only if its Killing form $\mathcal{K}$ is non-degenerate. Next, we have study relation between semisimplicity and invertibility of the Casimir operator in the case of a quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$. In particular, we have shown that if the representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible then semisimplicity is equivalent to the invertibility of the Casimir operator. The first main result of the present paper gives equivalence between invertibility of the Casimir operators and semisimplicity of an arbitrary quadratic Lie superalgebra. Recall that if $(\mathfrak{g}, B)$ is a quadratic Lie algebra, the previous result has been obtained by the second author [3]. The second main result gives the characterization of quadratic Lie superalgebras with nilpotent Casimir operators. Finally, we obtain the Fitting decomposition of Casimir operators of quadratic Lie superalgebras.

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2. Semisimplicity and Casimir operators

Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra: in the sens that B is a supersymmetric even non-degenerate and $\mathfrak{g}$-invariant bilinear form. m Let $(e_i)_{1*i*n}$ and $(e'_i)_{1*i*n}$ two basis of the Lie algebra $\mathfrak{g}_{\bar{0}}$ such that

$$B(e_i, e'_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker's symbol. Let $(f_i)_{1*i*m}$ and $(f'_i)_{1*i*m}$ two basis of $\mathfrak{g}_{\bar{1}}$ such that

$$B(f_i, f'_j) = \delta_{ij}.$$

Let ω_B be the element of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ defined by

$$\omega_B = \sum_{i=1}^{n} e_i e'_i - \sum_{i=1}^{m} f_i f'_i.$$

The element ω_B is called the Casimir element of $(\mathfrak{g}, B)$; it is a central element in $\mathfrak{U}(\mathfrak{g})$. The Casimir element ω_B does not depend on the choice of the basis [6].

Let T denote the extension of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{U}(\mathfrak{g})$. The element $T(\omega_B)$ of $End(\mathfrak{g})$ is called the Casimir operator of $(\mathfrak{g}, B)$ associated with B. We write $\Omega_B = T(\omega_B)$.

Recall that if $\mathfrak{g}$ is a simple Lie superalgebra then the following theorem holds

Theorem 2.1. ([5]). *Let $(\mathfrak{g}, B)$ be a simple quadratic Lie superalgebra and Ω_B its Casimir operator associated to B. Then $\mathfrak{g}$ is semisimple if and only if Ω_B is invertible.* ■

Let us now give some definitions.

Definition 2.1 A graded ideal I of a quadratic Lie superalgebra $(\mathfrak{g}, B)$ is called non-degenerate (resp. degenerate) if the restriction of B to $I \times I$ is a non-degenerate (resp. degenerate) bilinear form.

Definition 2.2 A quadratic Lie superalgebra $(\mathfrak{g}, B)$ is said to be B-irreducible if $\mathfrak{g}$ does not contain any non trivial non-degenerate graded ideal.

The following theorem gives the relation between invertibility of Casimir operators and semisimplicity of quadratic Lie superalgebras.

Theorem 2.2. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and Ω_B its Casimir operator associated to B. Then Ω_B is invertible if and only if $\mathfrak{g}$ is semisimple.*

Proof. First case: We suppose that $(\mathfrak{g}, B)$ is B-irreducible. We show the theorem for this particular case.

1. Suppose that Ω_B is invertible. We claim that this implies that $\mathfrak{g}$ is a simple Lie superalgebra. Indeed:

Suppose that $\mathfrak{g}$ is not simple, then there exists I a minimal graded non trivial ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is B-irreducible and $I^\perp$ is a graded ideal of $\mathfrak{g}$, by minimality of I one has $I \cap I^\perp = I$. Now by invariance of B, we have

$$B([I, I^\perp], \mathfrak{g}) = B(I, [I^\perp, \mathfrak{g}]) = \{0\},$$

so the nondegeneracy of B implies that $[I, I^\perp] = \{0\}$ and, in particular, I is abelian. Denote by V the supplementary subspace of $I^\perp$ in $\mathfrak{g} : \mathfrak{g} = I^\perp \oplus V$. Note that the fact that B is non-degenerate and $I \neq \{0\}$ implies that $V \neq \{0\}$. Let $W = I \oplus V$. Then W is a subspace of $\mathfrak{g}$ such that $B_{|W\times W}$ is non-degenerate. More precisely, if $x \neq 0$ an element of I (resp. V), then there exists an element y of V (resp. I) such that $B(x, y) \neq 0$. Consequently, the map $\phi : I \to V^*$ (resp. $\varphi : V \to I^*$) defined by $\phi(x) = B(x, .)$ (resp. $\varphi(x) = B(x, .)$ for all $x \in I$ (resp. V) is an injective linear map. Hence, $\dim I = \dim V$. Because $B_{|W\times W}$ is non-degenerate, then $\mathfrak{g} = I \oplus W^\perp \oplus V$ and $B_{|W^\perp \times W^\perp}$ is non-degenerate.

Let $(e_i)_{1*i*n}$ be a basis of $\mathfrak{g}$ consisting of homogeneous elements such that: $(e_i)_{1*i*p}$ be a basis of I, $(e_i)_{p+1*i*2p}$ be a basis of V and $(e_i)_{2p+1*i*n}$ be a basis of $W^\perp$. Let ε_i be the degree of e_i. We introduce a second basis $(f_i)_{1*i*n}$ of $\mathfrak{g}$ such that

$$B(f_j, e_i) = \delta_{ij},$$

that $(f_i)_{p+1*i*2p}$ is a basis of I, $(f_i)_{1*i*p}$ is a basis of V and that $(f_i)_{2p+1*i*n}$ is a basis of $W^\perp$. The Casimir operator associated to B is

$$\Omega_B = \sum_{i=1}^{n} (-1)^{\varepsilon_i} \operatorname{ad} f_i \circ \operatorname{ad} e_i.$$

Then for all $x \neq 0$ in I, one has $\Omega_B(x) = 0$ because $[I, I^\perp] = \{0\}$, $I^\perp = I \oplus W^\perp$ and I is a graded ideal of $\mathfrak{g}$, which contradicts the fact that Ω_B is invertible.

So $(\mathfrak{g}, B)$ is a simple Lie superalgebra such that its Casimir operator is invertible, by Theorem 2.1, we conclude that $(\mathfrak{g}, B)$ is semisimple.

2. Conversely: If the quadratic B-irreducible Lie superalgebra $(\mathfrak{g}, B)$ is semisimple then $\mathfrak{g}$ is a direct sum of its simple graded ideals whose Killing forms are non-degenerate [6]. Let I be a non-zero simple graded ideal, then, by invariance and nondegeneracy of B, $I \cap I^\perp$ is an abelian graded ideal of I. Consequently, $I \cap I^\perp = \{0\}$, so I is a non-degenerate graded ideal of $\mathfrak{g}$, it follows that $\mathfrak{g} = I$ because $\mathfrak{g}$ is a B-irreducible Lie superalgebra. We conclude that $\mathfrak{g}$ is simple. Hence Ω_B is invertible by Theorem 2.1.

<u>**Second case:**</u> Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. By Proposition 2.6 of [2] one has

$$\mathfrak{g} = \overset{r}{\underset{i=1}{\oplus}} \mathfrak{g}_i$$

such that:

(i) $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$, for all $1 * i * n$;

(ii) $\mathfrak{g}_i$ is B_i-irreducible, where B_i is the restriction of B to $\mathfrak{g}_i \times \mathfrak{g}_i$, for all $1 * i * n$;

(iii) for all $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal.

Let $\omega^i_{B_i}$ be the Casimir element of $(\mathfrak{g}_i, B_i)$. Let T_i be the representation of $\mathfrak{U}(\mathfrak{g}_i)$, the unique extension of the adjoint representation of $\mathfrak{g}_i$. Denote by $\Omega^i{}_{B_i} = T_i(\omega^i_{B_i})$ the Casimir operator of $(\mathfrak{g}_i, B_i)$. Now $\omega_B = \omega^1{}_{B_1} + \cdots + \omega^r{}_{B_r}$. Then it is easy to see that

- $T(\omega^i{}_{B_i})(\mathfrak{g}_j) = 0$ if $i \neq j$,
- $T(\omega^i{}_{B_i})_{|\mathfrak{g}_i} = T_i(\omega^i{}_{B_i})$,
- $T(\omega^i{}_{B_i}) \circ T(\omega^j{}_{B_j}) = T(\omega^j{}_{B_j}) \circ T(\omega^i{}_{B_i})$,
- $\Omega_B = T(\omega_B) = \sum_{i=1}^{r} T(\omega^i{}_{B_i})$.

1. Suppose that the Casimir operator Ω_B of $(\mathfrak{g}, B)$ is invertible. Then, for all $i \in \{1, \cdots r\}$, $\Omega^i_{B_i}$ is invertible. The first case implies that $\mathfrak{g}_i$ is simple and semisimple. We conclude that $\mathfrak{g}$ is semisimple.

2. Conversely: If $\mathfrak{g}$ is a semisimple Lie superalgebra, then for all $i \in \{1, 2, \cdots, r\}$, $\mathfrak{g}_i$ is a simple semisimple Lie superalgebra [6]. Now by Theorem 2.1, for all $i \in \{1, 2, \cdots, r\}$, the Casimir operator $\Omega^i{}_{B_i}$ of $(\mathfrak{g}_i, B_i)$ is invertible. So the Casimir operator Ω_B of $(\mathfrak{g}, B)$ is invertible. ∎

Remark. It is well known that the invertibility of the Casimir operators is very useful in the cohomological theory of semisimple Lie superalgebras. Theorem 2.2 shows that, in the case of quadratic Lie superalgebras which are not semisimple, others methods would be necessary for the cohomological theory.

3. Fitting decomposition of the Casimir operator Ω_B.

Lemma 3.1. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and Ω_B its Casimir operator. Then*
1. *For all $X \in \mathfrak{g}$, $\Omega_B \circ \mathrm{ad}_{\mathfrak{g}}(X) = \mathrm{ad}_{\mathfrak{g}}(X) \circ \Omega_B$.*
2. *For all $X, Y \in \mathfrak{g}$, $B(\Omega_B(X), Y) = B(X, \Omega_B(Y))$.*
3. *For all integer $n * 1$, $B(\ker(\Omega_B)^n, \mathrm{Im}(\Omega_B)^n) = \{0\}$.*
4. *For all integer $n * 1$, $\ker(\Omega_B)^n$ is an ideal of $\mathfrak{g}$.*

Proof. The proof is an easy computation. ∎

Proposition 3.1. *If $(\mathfrak{g}, B)$ is a quadratic, B-irreducible Lie superalgebra and Ω_B its Casimir operator, then Ω_B is nilpotent or invertible.*

Proof. There exists an integer $n * 1$ such that

$$\mathfrak{g} = \ker(\Omega_B)^n \oplus \mathrm{Im}(\Omega_B)^n.$$

By Lemma 3.1, we deduce that $\ker(\Omega_B)^n$ is a non-degenerate graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is B-irreducible, then $\ker(\Omega_B)^n = \mathfrak{g}$ or $\ker(\Omega_B)^n = \{0\}$. So Ω_B is nilpotent or invertible. ∎

New we are in position to state the main result of this section. This result gives the characterization of quadratic Lie superalgebras with nilpotent Casimir operators.

Theorem 3.1. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and Ω_B its Casimir operator associated to B. Then Ω_B is nilpotent if and only if $\mathfrak{g}$ is a Lie superalgebra without nonzero semisimple graded ideal.*

Proof. By Proposition 2.6 of [2],

$$\mathfrak{g} = \overset{r}{\underset{i=1}{\oplus}} \mathfrak{g}_i$$

such that:
(i) $\mathfrak{g}_i$ is a non-degenerate graded ideal of $\mathfrak{g}$, for all $1 * i * n$;
(ii) $\mathfrak{g}_i$ is B_i-irreducible, where B_i is the restriction of B to $\mathfrak{g}_i \times \mathfrak{g}_i$, for all $1 * i * n$;
(iii) for all $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal.

1. Suppose that Ω_B is nilpotent. Then there exists a non negative integer n such that ${\Omega_B}^n = 0$. Then for all $1 * i * r$, the Casimir operator ${\Omega^i}_{B_i}$ of $(\mathfrak{g}_i, B_i)$ is nilpotent. By Theorem 2.2, for all $1 * i * r$, $\mathfrak{g}_i$ are not semisimple. Let I be a semisimple ideal of $\mathfrak{g}$. By Proposition 2.6 of [1] we have $I = [I, I]$, consequently $I = [I, \mathfrak{g}]$. So

$$I = [I, \mathfrak{g}_1] \oplus \cdots \oplus [I, \mathfrak{g}_r], \text{ and} I \cap \mathfrak{g}_i = [I, \mathfrak{g}_i] \text{ for all } i = 1, 2, \cdots, r.$$

Let i be in $\{1, 2, \cdots, r\}$. Proposition 2.14 of [1] implies that $I_i = I \cap \mathfrak{g}_i$ is a semisimple ideal of $\mathfrak{g}_i$. By the invariance and the nondegeneracy of B, $I_i \cap (I_i)^\perp$ is a graded abelian of I_i, consequently $I_i \cap (I_i)^\perp = \{0\}$. Hence, I_i is a non-degenerate ideal of $\mathfrak{g}_i$. Now, since $(\mathfrak{g}_i, B_i)$ is B_i-irreducible and $\mathfrak{g}_i$ is not semisimple, we deduce that for all i, $I_i = \{0\}$. Thus $I = \{0\}$.

2. Conversely: If $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra without nonzero semisimple graded ideal, then, for all $1 * i * r$, $\mathfrak{g}_i$ is not semisimple. By Proposition 3.1, ${\Omega^i}_{B_i}$ is nilpotent. Using the fact that:

- $T({\omega^i}_{B_i})(\mathfrak{g}_j) = 0$ if $i \neq j$,
- $T({\omega^i}_{B_i})_{|\mathfrak{g}_i} = T_i({\omega^i}_{B_i})$,
- $T({\omega^i}_{B_i}) \circ T({\omega^j}_{B_j}) = T({\omega^j}_{B_j}) \circ T({\omega^i}_{B_i})$,
- $\Omega_B = T(\omega_B) = \sum_{i=1}^{r} T({\omega^i}_{B_i})$,

we then deduce that Ω_B is nilpotent. ∎

Remark. This theorem generalizes Theorem 1 of [4].

We close this paper by given the Fitting decomposition of Casimir operators of quadratic Lie superalgebras.

Corollary 3.1. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and Ω_B its Casimir operator associated to B. Let S be a largest semisimple ideal of $\mathfrak{g}$. Then there exists an ideal J of $\mathfrak{g}$ such that $\mathfrak{g} = I \oplus J$, this decomposition is the Fitting decomposition of $\mathfrak{g}$ relative to Ω_B.*

Proof. By Corollary 2.21 of [1], $\mathfrak{g}$ contains a largest graded semisimple ideal S. Since B is non-degenerate and invariant, then $S \cap S^\perp$ is an abelian graded ideal of $\mathfrak{g}$, it follows that $S \cap S^{\perp} = \{0\}$ because S is a graded semisimple ideal. Consequently, S is a non-degenerate ideal of $\mathfrak{g}$, so $\mathfrak{g} = S \oplus J$ where $J = S^{\perp}$ which is a non-degerate graded ideal of $\mathfrak{g}$. Hence $(S, B_{S\times S})$ and $(J, B_{J\times J})$ are quadratic Lie superalgebras. Consequently,

$$(\Omega_B)_{|S} = \Omega_{(B_{S\times S})} \text{ and } a(\Omega_B)_{|J} = \Omega_{(B_{J\times J})}.$$

By Theorem 2.2, $(\Omega_B)_{|S}$ is invertible. Now, if we consider I a graded semisimple ideal of J, then I is a semisimple graded ideal of $\mathfrak{g}$ because $[S, J] = \{0\}$, it follows that $I \subset S$, so $I = \{0\}$. Hence, J is a Lie superalgebra without nonzero semisimple graded ideal, it follows, from Theorem 3.1, that $(\Omega_B)_{|J}$ is nilpotent. We conclude that $\mathfrak{g} = S \oplus J$ is the Fitting decomposition of $\mathfrak{g}$. ■

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