Journal of Lie Theory
Volume **10** (2000) 127–146

Noncompact, almost simple groups operating on locally compact, connected translation planes

Harald Löwe

Communicated by J. Hilgert

Abstract. Let $\mathbb{E}$ be a locally compact, connected translation plane. The aim of this paper is a detailed investigation of noncompact, connected, almost simple subgroups of the point stabilizer $\mathbb{G}_0$ of $\mathbb{E}$. It turns out that the only possible groups Δ of this kind are 2–fold covering groups of $\mathrm{PSO}_m(\mathbb{R},1)$ for $3 \leq m \leq 10$. Moreover, the nontrivial central element of Δ is the reflection at 0. Furthermore, we will show the existence of an orbit $\mathcal{S}$ (called the *weight sphere* of Δ) homeomorphic to an $(m-2)$–sphere on which the action of Δ is equivalent to the natural action of $\mathrm{PSO}_m(\mathbb{R},1)$ on $\mathbb{S}_{m-2}$. This weight sphere $\mathcal{S}$ is characterized as the set of those lines in $\mathcal{L}_0$ whose stabilizer in Δ is a minimal parabolic subgroup of Δ.

As a by–product we prove that a semisimple subgroup of $\mathbb{G}_0$ always has real rank 0 or 1.

1. Introduction

Locally Compact, Connected Translation Planes. A projective plane is called topological if its point space and its line space are endowed with Hausdorff topologies such that the operations of joining points and intersecting lines are continuous. As in the theory of topological groups one obtains the nicest results under the additional assumption that these topologies are locally compact and connected. Examples are the projective planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions), the so–called "classical planes". For a detailed introduction we refer the reader to [10].

We are mainly interested in a special kind of locally compact, connected projective planes: *Translation planes* are distinguished by the existence of a line L_∞ such that the automorphisms fixing exactly the points on L_∞ form a group which is transitive on the set of points outside L_∞. This group is called the *translation group* with *translation axis* L_∞. The point space of a locally compact, connected translation plane is a manifold of dimension $n \in \{2, 4, 8, 16\}$; moreover, the projective lines (regarded as subspaces of the point set) are spheres of dimension $l = n/2$ (cf. [10, 64.1]). In the sequel we refer to n as the dimension of the plane.

ISSN 0949–5932 / \$2.50

For a non–classical locally compact, connected translation plane one knows that every (continuous) automorphism leaves the translation axis L_∞ invariant ([10, 64.4(c)]). Therefore, we can pass to the corresponding affine plane without losing automorphisms. The advantage of this procedure is a linearization: The affine point space P is in a natural way a right vector space of even dimension over the kernel $\mathbb{K}$ of a coordinatizing quasifield and, moreover, $\mathbb{K}$ is a topological field isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ ([10, 42.6]). The translation group coincides with the group of vector translations $x \mapsto x+v$ of P. The line pencil $\mathcal{L}_0 = \{L \in \mathcal{L} \mid 0 \in L\}$ (where $\mathcal{L}$ denotes the set of affine lines) consists of vector subspaces of P of dimension $(\dim_{\mathbb{K}} P)/2$. The other lines are exactly the affine cosets of the elements of $\mathcal{L}_0$. It follows immediately that a locally compact, connected translation plane of dimension 2 is isomorphic to the real plane.

The group $\mathbb{G}$ of all (continuous) automorphisms of $(P, \mathcal{L})$ is a Lie group with respect to the compact–open topology ([10, 44.6]). Clearly, $\mathbb{G}$ is a semidirect product of the stabilizer $\mathbb{G}_0$ of the origin 0 and the translation group. All elements of $\mathbb{G}_0$ are semilinear maps of the $\mathbb{K}$–vector space P. The kernel $\mathbb{K}$ is encoded in $\mathbb{G}_0$: The normal subgroup $\mathbb{G}_{[0,L_\infty]}$ (consisting of all those elements of $\mathbb{G}_0$ which leave the translation axis pointwise fixed) is precisely the group $\{x \mapsto xa \mid a \in \mathbb{K},\, a \neq 0\}$. The connected component of $\mathbb{G}_0$ is an almost direct product of the connected component of the so–called *reduced stabilizer* [1]

$$\mathrm{S}\mathbb{G}_0 = \{\gamma \in \mathbb{G}_0 \mid \gamma \text{ is } \mathbb{K}\text{–linear and } \det_{\mathbb{K}}\gamma = 1\}$$

and the connected component of $\mathbb{G}_{[0,L_\infty]}$, see the considerations in [10, 81c].

Remark 1.1. According to [10, 42.6], the group $\mathbb{G}_{[0,L_\infty]}$ is isomorphic to $\mathbb{R}^\times$, $\mathbb{C}^\times$ or $\mathbb{H}^\times = \mathbb{R}_{\mathrm{pos}} \times \mathrm{Spin}_3\mathbb{R}$. We conclude that a noncompact, connected, almost simple subgroup of $\mathbb{G}_0$ is contained in the reduced stabilizer $\mathrm{S}\mathbb{G}_0$.

We shall interpret a locally compact, connected affine translation plane $\mathbb{E} = (P, \mathcal{L})$ as follows: Since $\mathbb{K}$ can be considered as a real vector space, P is a real vector space, too. The set $\mathcal{L}_0$ consists of vector subspaces of dimension $l = (\dim P)/2$. It is easy to see that $\mathcal{L}_0$ has to be a spread, i.e. $\mathcal{L}_0$ covers P and any two distinct elements $K, L \in \mathcal{L}_0$ satisfy $P = K \oplus L$. The topology of $\mathcal{L}_0$ as a subspace of $\mathcal{L}$ and the topology of $\mathcal{L}_0$ as a subspace of the Grassmannian manifold $G_l(P)$ of all l–dimensional vector subspaces of P coincide ([10, 64.4 (a)])[2]. The reduced stabilizer $\mathrm{S}\mathbb{G}_0$ is a closed subgroup of $\mathrm{SL}(P)$ ([10, 44.6]).

The results given in this paper rest on a deep theorem due to Hähl which describes the structure of closed subgroups of $\mathrm{S}\mathbb{G}_0$ fixing two different lines in $\mathcal{L}_0$, see [10, 81.8]:

Theorem 1.2. *Let $\Gamma \leq \mathrm{S}\mathbb{G}_0$ be a closed, connected subgroup which fixes two distinct lines $W, S \in \mathcal{L}_0$. Consider the normal subgroup*

$$K = \{\gamma \in \Gamma \mid |\det(\gamma|_W)| = |\det(\gamma|_S)| = 1\},$$

[1] If $\mathbb{K} = \mathbb{H}$, then the plane is isomorphic to the classical plane over $\mathbb{H}$ ([2, Thm.1]); in this case the determinant $\det_{\mathbb{H}}\gamma$ is understood to be the real determinant of γ and hence the reduced stabilizer equals $\mathrm{SL}_2\mathbb{H}$ in its usual operation on the affine quaternion plane.

[2] Conversely, if $\mathcal{S} \subseteq G_l(P)$ is a compact spread, then $\mathcal{S}$ defines a locally compact, connected translation plane ([10, 64.4(d)]).

where det *denotes the real determinant.*

Then K *is the largest compact subgroup of* Γ. *If* Γ *is not compact, then* Γ *is a direct product* $\Gamma = K \times Y$ *of* K *and a closed one–dimensional subgroup* Y *isomorphic to* $\mathbb{R}$. *We observe that* K *is connected.*

Moreover, for any closed one–dimensional subgroup Y *isomorphic to* $\mathbb{R}$, *there is an isomorphism* $\rho : \mathbb{R} \to Y$ *having the following property: For* $t \to -\infty$, *the maps* $\rho(t)$ *converge to the constant map* $\mathcal{L}_0 \setminus \{S\} \to \{W\}$ *uniformly on each compact subset of* $\mathcal{L}_0 \setminus \{S\}$. *For* $t \to \infty$, *an analogous convergence property holds with the rôles of* W *and* S *interchanged.*

Remark. (a) Let $\Gamma \le \mathbb{SG}_0$ as above. Then a closed one–dimensional subgroup $Y \le \Gamma$ isomorphic to $\mathbb{R}$ will be called a *compression subgroup* of Γ.

(b) If $Y \le \Gamma$ is a compression subgroup, then for every isomorphism $\rho : \mathbb{R} \to Y$ we have either

$$\lim_{t\to\infty} \rho(t)(L) = W \text{ and } \lim_{t\to-\infty} \rho(t)(L) = S \text{ for all } L \in \mathcal{L}_0 \setminus \{W, S\}, \text{ or}$$

$$\lim_{t\to\infty} \rho(t)(L) = S \text{ and } \lim_{t\to-\infty} \rho(t)(L) = W \text{ for all } L \in \mathcal{L}_0 \setminus \{W, S\}.$$

Notation. Throughout this paper, let $\mathbb{E} = (P, \mathcal{L})$ be a locally compact, connected affine translation plane of dimension $n = 2l$ (where $l \in \{1, 2, 4, 8\}$) with reduced stabilizer $\mathbb{SG}_0$.

For a Lie group G, we denote its neutral element by e. Moreover, G^e is the connected component of e in G and T_eG is the Lie algebra of G. If $\mathfrak{h} \le T_eG$ is a subalgebra, then we write $\exp \mathfrak{h}$ for the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$.

If V is a real vector space of finite dimension, then $\mathrm{GL}(V)$ will denote the Lie group of invertible linear maps of V. As usual, $\mathrm{SL}(V)$ is the group of linear maps of V with determinant 1. Notice that the Lie algebra $\mathfrak{gl}(V)$ of $\mathrm{GL}(V)$ consists of all linear maps of V. The Lie algebra $\mathfrak{sl}(V)$ of $\mathrm{SL}(V)$ consists of all linear maps with vanishing trace.

A diagonal matrix with diagonal entries $\xi_1, \dots, \xi_n$ will be abbreviated by $\mathrm{diag}(\xi_1, \dots, \xi_n)$.

2. Results

The aim of this and two subsequent papers[3] is to investigate the structure of a noncompact, connected subgroup Δ of the reduced stabilizer of a locally compact, connected translation plane. The first step in this direction deals with almost simple groups and shows that the possibilities are very restricted in this case:

Theorem A. *Consider a connected, almost simple subgroup* Δ *of* $\mathbb{G}_0$. *Let* $U \ne \{0\}$ *be a* Δ*–irreducible subspace of the point space* P. *If* Δ *is not compact, then only the following possibilities can occur:*

[3] Löwe, H.: *Noncompact subgroups of the reduced stabilizer of a locally compact, connected translation plane* and *Parabolic collineation groups of locally compact, connected translation planes*, in preparation

(1) $\Delta = \mathrm{SL}_2\mathbb{R}$ *and* $\dim U$ *is even. If* $n = 2$, *then* $\mathbb{E}$ *is isomorphic to the real affine plane. For* $n = 4$ *all possible planes are known, see [10, 73.22]*

(2) $\Delta = \mathrm{SL}_2\mathbb{C}$ *and* $\dim U$ *is divisible by* 4. *If* $n = 4$, *then* $\mathbb{E}$ *is isomorphic to the complex plane. If* $n = 8$, *then* $\dim U = 4$ *and all possible planes are known, see [3].*

(3) $\Delta = \mathrm{U}_2(\mathbb{H}, 1)$ *and* $\dim U = 8$. *If* $n = 8$, *then* $\mathbb{H}$ *is the quaternion plane.*

(4) $\Delta = \mathrm{SL}_2\mathbb{H}$ *and* $\dim U = 8$. *If* $n = 8$, *then* $\mathbb{H}$ *is the quaternion plane*[4].

(5) $\Delta = \mathrm{Spin}_k(\mathbb{R}, 1)$ *for some* $7 \leq k \leq 10$ *and* $\dim U = 16$. *In this case* $\mathbb{E}$ *is isomorphic to the octonion plane.*

In any case Δ *is a* 2*–fold covering group of the connected component* $\mathrm{PSO}^e_m(\mathbb{R}, 1)$ *of* $\mathrm{PSO}_m(\mathbb{R}, 1)$ *for some*[5] m, $3 \leq m \leq 10$ *and its nontrivial central element acts as the reflection* $x \mapsto -x$ *at the origin.*

The proof is surprisingly easy if one uses real Cartan subgroups of Δ. Roughly speaking a real Cartan subgroup C is a maximal Ad–diagonalizable subgroup of Δ. It turns out that C fixes exactly two lines $W, S \in \mathcal{L}_0$ and, moreover, is a compression subgroup of $\Delta_{W,S}$. According to (1.2) the dimension of C (called the real rank of Δ) equals 1. The assertions can now be obtained by studying representations of groups of real rank 1.

We point out that each group Δ occuring in Theorem A acts on a classical plane. If Δ is not $\mathrm{SL}_2\mathbb{R}$ or $\mathrm{SL}_2\mathbb{C}$, then the representation of Δ on the point space P of an arbitrary plane is equivalent to the representation of Δ on the classical plane of dimension $\dim P$. It remains an open problem whether this is true for $\Delta = \mathrm{SL}_2\mathbb{C}$. For $\Delta = \mathrm{SL}_2\mathbb{R}$, the analogous statement is *not* true: Betten constructed an example of a 4–dimensional translation plane on which $\mathrm{SL}_2\mathbb{R}$ acts in its 4–dimensional irreducible representation, cf. [1].

However, there is another analogy between the classical and the non–classical case: A noncompact, almost simple subgroup Δ of the stabilizer of a classical plane has exactly one closed orbit $\mathcal{S}$ on $\mathcal{L}_0$. This orbit is homeomorphic to a sphere and can be characterized as the set of lines in $\mathcal{L}_0$ which are fixed by some parabolic subgroup of Δ. From this point of view there is no difference between the classical and the non–classical case:

Theorem B. *Let* $\Delta \leq \mathbb{G}_0$ *be a connected, noncompact, almost simple group. Then* Δ *has precisely one closed orbit* $\mathcal{S}$ *in* $\mathcal{L}_0$. *Let* $3 \leq m \leq 10$ *be the integer such that* Δ *is a* 2*–fold covering group of the connected component* $\mathrm{PSO}^e_m(\mathbb{R}, 1)$ *of* $\mathrm{PSO}_m(\mathbb{R}, 1)$. *(The existence of* m *is guaranteed by Theorem A). Then* $\mathcal{S}$ *satisfies the following assertions:*

(1) $\mathcal{S}$ *is homeomorphic to a sphere of dimension* $m - 2$.

[4]For $n = 16$, the classification of all possible planes admitting a group isomorphic to $\mathrm{SL}_2\mathbb{H}$ will be given in another paper. It turns out that — except in the case of the octonion plane — the automorphism groups of these planes have dimension at most 35.

[5]In particular, $m = 3$ for $\Delta = \mathrm{SL}_2\mathbb{R}$, $m = 4$ for $\Delta = \mathrm{SL}_2\mathbb{C}$, $m = 5$ for $\Delta = \mathrm{U}_2(\mathbb{H}, 1)$, and $m = 6$ for $\Delta = \mathrm{SL}_2\mathbb{H}$.

(2) *The action of* Δ *on* $\mathcal{S}$ *is equivalent to the natural action of* $\mathrm{PSO}_m^e(\mathbb{R},1) = \Delta/\{\pm e\}$ *on the sphere*

$$\mathbb{S}_{m-2} = \{\mathbb{R}\cdot(x_1, x_2, \ldots, x_{m-1}, 1) \mid x_1^2 + x_2^2 + \ldots + x_{m-1}^2 = 1\}$$

contained in the real projective $(m-1)$*-space. In particular,* Δ *is doubly transitive on* $\mathcal{S}$ *for* $m = 3$ *and triply transitive for* $m \geq 4$.

(3) *If* $m \gneqq l/2$ *(where* l *is the dimension of a line), then* $\mathbb{E}$ *is isomorphic to the classical translation plane of dimension* $n = 2l$ *and thus* Δ *is a subgroup of* $\mathrm{Spin}_{l+2}(\mathbb{R}, 1)$.

Consider a line $L \in \mathcal{S}$. Then L is fixed by some parabolic subgroup Π of Δ, see (7.3), and thus is fixed by the real Cartan subgroup $C \leq \Pi$, too. It turns out that L can be considered as the sum of those real weight spaces of Δ belonging to positive real weights[6]. For this reason we call L a *weight line* and $\mathcal{S}$ — being the set of all weight lines — the *weight sphere* of Δ.

Organization of this Paper. We start with a short introduction to semisimple real Lie algebras.

In Section 4 we investigate arbitrary diagonalizable subgroups of $\mathrm{S}\mathbb{G}_0$. The aim is to show that these groups have dimension 1 and fix exactly two lines in $\mathcal{L}_0$. These lines can be recovered from the "real weight spaces" of the group. The latter is a consequence of a general result concerning arbitrary compression subgroups of $\mathrm{S}\mathbb{G}_0$, cf. (4.1).

Since a real Cartan subgroup of a noncompact, semisimple group $\Delta \leq \mathbb{G}_0$ is diagonalizable on P and contained in $\mathrm{S}\mathbb{G}_0$, we apply these results in Section 5 and obtain the main tools for the proof of Theorem A. The proof itself will be given at the end of Section 6, which is devoted to the investigation of groups of real rank 1.

The action of a noncompact, connected, almost simple group on $\mathcal{L}_0$ is the subject of the last chapter, where we will prove the assertion concerning the weight sphere stated in Theorem B.

3. Semisimple Real Lie Algebras

For basic facts concerning semisimple real Lie algebras we refer to Knapp [6]. Throughout this section let $\mathfrak{g}$ be a semisimple real Lie algebra.

A subalgebra $\mathfrak{a} \leq \mathfrak{g}$ is called a *real Cartan subalgebra* if $\mathfrak{a}$ is contained in the Cartan complement $\mathfrak{p}$ of some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ and if it is maximal among the subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}$.

Since $[\mathfrak{p};\mathfrak{p}]$ is a subspace of $\mathfrak{k}$, every real Cartan subalgebra of $\mathfrak{g}$ is abelian. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a fixed Cartan decomposition, then a real Cartan subalgebra $\mathfrak{a}$ contained in $\mathfrak{p}$ is often called a *maximal abelian subspace of* $\mathfrak{p}$ in the literature.

We remark that $\mathfrak{a}$ is a real Cartan subalgebra if, and only if, $\mathfrak{a}$ is the abelian part of some Iwasawa decomposition of $\mathfrak{g}$. (This follows directly from the

[6]For real weights and real weight spaces see the Sect. 3.

construction of the Iwasawa decomposition.) Moreover, real Cartan subalgebras are characterized as the maximal ad–diagonalizable subalgebras of $\mathfrak{g}$.

As a direct consequence of [6, 6.19] and [6, 6.51] we obtain the following result:

Proposition 3.1. *Any two real Cartan subalgebras of* $\mathfrak{g}$ *are conjugate by an inner automorphism of* $\mathfrak{g}$.

In particular, any two real Cartan subalgebras of $\mathfrak{g}$ have same dimension. We refer to this dimension as the *real rank* $\mathrm{rk}_{\mathbb{R}}\mathfrak{g}$ of $\mathfrak{g}$. Notice that the real rank of $\mathfrak{g}$ equals 0 if, and only if, $\mathfrak{g}$ is a compact (semisimple) Lie algebra.

Proposition 3.2. [9, Sec. 4.1] *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be semisimple Lie algebras. Then* $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mathrm{rk}_{\mathbb{R}}\mathfrak{g}_1 + \mathrm{rk}_{\mathbb{R}}\mathfrak{g}_2$.

We shall now study faithful representations $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ of a semisimple real Lie algebra $\mathfrak{g}$ on a finite dimensional real vector space V. Fix a real Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. A *real weight* of φ (with respect to $\mathfrak{a}$) is a linear map $\omega : \mathfrak{a} \to \mathbb{R}$ such that the corresponding *real weight space*

$$V_\omega := \{x \in V \mid \varphi(A)(x) = \omega(A) \cdot x \text{ for all } A \in \mathfrak{a}\}$$

contains nonzero elements.

Remark 3.3. Suppose (in addition to the preceding assumptions) that $\mathrm{rk}_{\mathbb{R}}\mathfrak{g} = 1$. Then $\mathfrak{a} = \mathbb{R} \cdot X$ holds for every $X \in \mathfrak{a} \setminus \{0\}$. Consequently, a real weight $\omega : \mathfrak{a} \to \mathbb{R}$ of the representation φ is uniquely determined by the eigenvalue $\omega(X)$ of $\varphi(X)$. Moreover, the real weight space V_ω and the eigenspace of $\varphi(X)$ belonging to $\omega(X)$ coincide. We will identify ω and $\omega(X)$ in this situation: A real weight of φ with respect to X is an eigenvalue of $\varphi(X)$.

The real weights of φ are intimately related to the weights of the complexification $\varphi_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}})$: According to [9, Chap.4, Sec.4.1], we find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{a}$. Then every weight $\lambda : \mathfrak{h} \to \mathbb{C}$ of $\varphi_{\mathbb{C}}$ is real on $\mathfrak{a}$, cp. the proof of Prop. 4.3 in [9, Chap.4]. Since $\varphi_{\mathbb{C}}(\mathfrak{h})$ is diagonalizable on $V_{\mathbb{C}}$, we obtain that $\varphi(\mathfrak{a})$ is diagonalizable on V, i.e. $\varphi(\mathfrak{a})$ consists of diagonal matrices with real entries with respect to a properly chosen basis of V. Thus, the vector space V is a direct sum of the real weight spaces of φ. One easily derives that

$$V_\omega = V \cap \bigoplus_{\lambda \in \Delta;\, \lambda|_{\mathfrak{a}} = \omega} (V_{\mathbb{C}})_\lambda$$

holds for every real weight ω, where Δ is the set of weights of $\varphi_{\mathbb{C}}$ and where $(V_{\mathbb{C}})_\lambda$ is the weight space corresponding to the weight $\lambda \in \Delta$. It follows that if ω is a real weight, then $-\omega$ is a real weight, too, and the dimensions of the corresponding real weight spaces V_ω and $V_{-\omega}$ coincide. We record these facts:

Proposition 3.4. *Let* $\mathfrak{g}$ *be a semisimple real Lie algebra and let* $\mathfrak{a} \leq \mathfrak{g}$ *be a real Cartan subalgebra. Consider a representation* φ *of* $\mathfrak{g}$ *on some finite dimensional real vector space* V. *Let* Λ *be the set of real weights of* φ *with respect to* $\mathfrak{a}$.

(a) $\varphi(\mathfrak{a})$ *is diagonalizable on* V. *Therefore,* $V = \bigoplus_{\omega \in \Lambda} V_\omega$ *is the direct sum of real weight spaces.*

(b) *If* ω *is a real weight, then* $-\omega$ *is a real weight, too. Moreover,* $\dim V_\omega = \dim V_{-\omega}$ *holds for every* $\omega \in \Lambda$.

We should finally introduce the global counterpart to real Cartan subalgebras: Let Δ be a semisimple Lie group (preferably connected). A connected subgroup $C \leq \Delta$ is called a *real Cartan subgroup* if, and only if, its Lie algebra T_eC is a real Cartan subalgebra of $T_e\Delta$. The *real rank* $\mathrm{rk}_{\mathbb{R}}\Delta$ of Δ is defined to be the real rank of $T_e\Delta$.

We mention here that a real Cartan subgroup $C \leq \Delta$ is always closed in Δ: Since $\mathrm{ad}\, T_eC$ is diagonalizable on $T_e\Delta$, the group $\mathrm{Ad}\, C$ is diagonalizable, too, and hence $\mathrm{Ad}\, C$ is closed in $\mathrm{SL}(T_e\Delta)$. This proves our claim.

4. Compression Subgroups and Diagonalizable Subgroups of the Reduced Stabilizer

Let $\mathbb{E} = (P, \mathcal{L})$ be a locally compact, connected affine translation plane of dimension $n = 2l$ and denote its reduced stabilizer by $\mathbb{SG}_0$.

Let $\Phi \leq \mathbb{SG}_{W,S}$ be a compression subgroup with respect to the lines $W, S \in \mathcal{L}_0$, $W \neq S$. Note that $\dim \Phi = 1$. Choose a generator $T \in \mathfrak{gl}(P)$ of the Lie algebra of Φ. Then $\rho : \mathbb{R} \to \Phi$; $r \mapsto \exp(rT)$ is a parametrization of Φ. In view of (1.2), we have that

$$\lim_{r\to\infty} \rho(r)(L) = S \quad \text{and} \quad \lim_{r\to-\infty} \rho(r)(L) = W$$

holds for every $L \in \mathcal{L}_0 \setminus \{W, S\}$. Moreover, since W and S are Φ–invariant, T fixes both W and S. Thus, T may be written as $T = T_W + T_S$, where $T_W = T|_W$ and $T_S = T|_S$ are endomorphisms of W and S, respectively.

Let $\alpha_k + \beta_k i$, $(k = 1, \dots, s)$ be the eigenvalues of T with $\beta_k \geq 0$ and let P_k be the generalized eigenspace of $\alpha_k + \beta_k i$. In other words, if d_k denotes the multiplicity of the eigenvalue $\alpha_k + \beta_k i$, then $P_k = \ker(T - \alpha_k \cdot \mathbb{1})^{d_k}$ for $\beta_k = 0$ and $P_k = \ker(T^2 - 2\alpha_k T + (\alpha_k^2 + \beta_k^2)\mathbb{1})^{d_k}$ for $\beta_k \neq 0$. Notice that $\alpha_k + \beta_k i$ is an eigenvalue of T_W [of T_S] if, and only if, $P_k \cap W \neq \{0\}$ $[P_k \cap S \neq \{0\}]$, and that every eigenvalue of T_W [of T_S] is also an eigenvalue of T.

Proposition 4.1. *We consider a compression subgroup* $\Phi \leq \mathbb{SG}_{W,S}$ *as above. Choose an index* k_0 *such that* $\alpha_{k_0} = \max\{\alpha_k \mid P_k \cap W \neq \{0\}\}$. *Then*

$$W = \bigoplus_{k;\, \alpha_k \leq \alpha_{k_0}} P_k \quad \text{und} \quad S = \bigoplus_{k;\, \alpha_k \gneqq \alpha_{k_0}} P_k.$$

In particular, the real part of any eigenvalue of T_W *is strictly smaller than the real part of every eigenvalue of* T_S. *Moreover, if* $\alpha_k + \beta_k i$ *is an eigenvalue of* T_W *[of* T_S*], then* P_k *is a subspace of* W *[of* S*].*

Remark. We point out that we can recover W and S from the compression subgroup Φ.

Proof. (1) Let $\alpha_k+\beta_k i$ and $\alpha_j+\beta_j i$ be eigenvalues of T_W and T_S, respectively. First, we will show that $\alpha_k \lneqq \alpha_j$.

(2) If $\beta_k \neq 0$, then there exists a 2–dimensional subspace E_k of W such that

$$T_{E_k} = \begin{pmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{pmatrix}.$$

Therefore, we obtain the following equation:

$$\rho(r)|_{E_k} = \exp(rT)|_{E_k} = \exp(rT_{E_k}) = e^{\alpha_k r} \cdot C_k(r),$$

where

$$C_k(r) = \begin{pmatrix} \cos(\beta_k r) & -\sin(\beta_k r) \\ \sin(\beta_k r) & \cos(\beta_k r) \end{pmatrix}$$

is an element of the orthogonal group $\mathrm{SO}(E_k)$. If $\beta_k = 0$, then there exists a one–dimensional subspace E_k of W with

$$\rho(r)|_{E_k} = e^{\alpha_k r} \cdot C_k(r), \text{ where } C_k(r) \in \mathrm{SO}(E_k) = \{\mathbb{1}\}.$$

Analogously, there exists a subspace E_j of S such that

$$\rho(r)|_{E_j} = e^{\alpha_j r} \cdot C_j(r), \text{ where } C_j(r) \in \mathrm{SO}(E_j).$$

We choose an increasing sequence r_ν of real numbers with $\lim_{\nu\to\infty} r_\nu = \infty$ such that the sequence $(C_k(r_\nu), C_j(r_\nu))$ possesses a limit (D_k, D_j) in the compact set $\mathrm{SO}(E_k) \times \mathrm{SO}(E_j)$.

(3) We claim that $\alpha_j \gneqq \alpha_k$. Aiming at a contradiction, we assume $\alpha_j - \alpha_k \leq 0$ and infer

$$\lim_{\nu\to\infty} e^{(\alpha_j-\alpha_k)r_\nu} = \varepsilon = \begin{cases} 0 & \text{if } \alpha_j - \alpha_k \lneqq 0 \\ 1 & \text{if } \alpha_k = \alpha_j \end{cases}.$$

Now, we choose $x \in E_k \setminus \{0\}$ and $y \in E_j \setminus \{0\}$. Then the line $L = 0 \vee (x+y)$ is different from W and S and therefore $\lim_{r\to\infty} \rho(r)(L) = S$. In contradiction to this fact, we have that the limit

$$\begin{aligned} \lim_{\nu\to\infty} \rho(r_\nu)(\mathbb{R}\cdot(x+y)) &= \lim_{\nu\to\infty} \mathbb{R}\cdot(e^{\alpha_k r_\nu} C_k(r_\nu)(x) + e^{\alpha_j r_\nu} C_j(r_\nu)(y)) \\ &= \lim_{\nu\to\infty} \mathbb{R}\cdot(C_k(r_\nu)(x) + e^{(\alpha_j-\alpha_k)r_\nu} C_j(r_\nu)(y)) \\ &= \mathbb{R}\cdot(D_k(x) + \varepsilon \cdot D_j(y)) \end{aligned}$$

of the one–dimensional subspaces $\rho(r_\nu)(\mathbb{R}\cdot(x+y))$ of $\rho(r)(L)$ is not contained in S since $y \neq 0$.

(4) In view of (3), the real part of any eigenvalue of T_W is strictly smaller than the real part of every eigenvalue of T_S. By the definition of k_0, we obtain that $\alpha_k + \beta_k i$ is an eigenvalue of T_W if, and only if, $\alpha_k \leq \alpha_{k_0}$. If W_k denotes the generalized eigenspace of T_W with respect to $\alpha_k + \beta_k i$, then

$$W = \bigoplus_{k;\,\alpha_k \leq \alpha_{k_0}} W_k \leq \bigoplus_{k;\,\alpha_k \leq \alpha_{k_0}} P_k,$$

since W_k is a subspace of P_k for every k. Analogously, we have

$$S = \bigoplus_{j;\,\alpha_j \gneqq \alpha_{k_0}} S_j \leq \bigoplus_{j;\,\alpha_j \gneqq \alpha_{k_0}} P_j,$$

where S_j denotes the generalized eigenspace of T_S with respect to the eigenvalue $\alpha_j + \beta_j i$ (with $\alpha_j \gneqq \alpha_{k_0}$). From

$$W \oplus S = P = \bigoplus_k P_k$$

one derives easily that $P_k = W_k$ holds for every k with $\alpha_k \le \alpha_{k_0}$ and that $S_j = P_j$ holds for every j with $\alpha_j \gneqq \alpha_{k_0}$. This finishes the proof of the proposition. ■

As a next step, we apply this proposition to diagonalizable subgroups of the reduced stabilizer. To this end, we have to show first that such a group is a compression subgroup:

Proposition 4.2. *Let $\Theta \le \mathrm{S}\mathbb{G}_0$ be a diagonalizable, connected subgroup. Then there exist exactly two distinct Θ–invariant lines $W, S \in \mathcal{L}_0$ and $\Theta \le \mathrm{S}\mathbb{G}_{W,S}$ is a compression subgroup. In particular,* $\dim \Theta = 1$ *and we find a parametrization $\rho : \mathbb{R} \to \Theta$ such that*

$$\lim_{t\to\infty} \rho(t)(L) = S \text{ and } \lim_{t\to-\infty} \rho(t)(L) = W$$

holds for every $L \in \mathcal{L}_0 \setminus \{W, S\}$.

Proof. Clearly, Θ is closed in $\mathrm{S}\mathbb{G}_0$ and isomorphic to $\mathbb{R}^m$ for some m. The point space $P = \mathbb{R}^n$ is a direct sum $P = X_1 \oplus \ldots \oplus X_n$ of one–dimensional Θ–invariant subspaces. The one–dimensional subspace X_1 is contained in some line $W \in \mathcal{L}_0$. For $\vartheta \in \Theta$, the lines W and $\vartheta(W)$ intersect in $X_1 = \vartheta(X_1)$ and hence are equal. Consequently, Θ leaves W invariant. Since $\dim W = l$, there exists an index $1 \le j \le n = 2l$ such that X_j is not a subspace of W. Let $S \in \mathcal{L}_0 \setminus \{W\}$ be the line containing X_j. By the same arguments as above, S is Θ–invariant, too.

We have proved that $\Theta \cong \mathbb{R}^m$ is a closed subgroup of $\mathrm{S}\mathbb{G}_{W,S}$. According to (1.2), the codimension of a maximal compact subgroup of Θ is at most 1. Consequently, $m = 1$ and $\Theta \cong \mathbb{R}$ is a compression subgroup of $\mathrm{S}\mathbb{G}_{W,S}$. The remaining assertions follow from (1.2). ■

Since Θ fixes the lines $W, S \in \mathcal{L}_0$, we infer that Θ is diagonalizable on both W and S. We choose bases $\{w_1, \ldots, w_l\}$ and $\{s_1, \ldots, s_l\}$ of W and S, respectively, such that the parametrization ρ described in (4.2) is given by

$$\rho(t) = \mathrm{diag}(e^{\lambda_1 t}, \ldots e^{\lambda_l t}, e^{\mu_1 t}, \ldots e^{\mu_l t}) \text{ for } t \in \mathbb{R}$$

with respect to the basis $\{w_1, \ldots, w_l, s_1, \ldots, s_l\}$ of P. From (4.1) we conclude that $\lambda_i \lneqq \mu_j$ holds for every pair i, j (with $1 \le i, j \le l$). In other words, we have just shown the following:

Proposition 4.3. *Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be a basis of P and let*

$$\xi_1 \le \cdots \le \xi_n$$

be real numbers. For $t \in \mathbb{R}$, define the linear map $\rho(t) \in \mathrm{GL}(P)$ by

$$\rho(t) = \mathrm{diag}(e^{\xi_1 t}, \ldots, e^{\xi_n t}),$$

where the matrix is given with respect to the basis $\mathcal{X}$.

Suppose that $\Theta = \{\rho(t) \mid t \in \mathbb{R}\}$ is a subgroup of the reduced stabilizer $\mathbb{S}\mathbb{G}_0$. Then the subspaces W and S spanned by $\{x_1, \dots, x_l\}$ and $\{x_{l+1}, \dots, x_n\}$, respectively, are Θ–invariant lines. Moreover, ξ_l is strictly smaller than ξ_{l+1}, i.e. $\xi_i \lneqq \xi_{l+j}$ holds for every pair i, j (with $1 \leq i, j \leq l$).

5. Semisimple Subgroups of $\mathbb{G}_0$

Throughout this section let Δ be a connected, semisimple Lie group and let $\varphi : \Delta \to \mathbb{G}_0$ be a Lie homomorphism with discrete kernel. In particular, φ is a representation of Δ on the vector space $P = \mathbb{R}^n$.

Let $\mathfrak{g} = T_e\Delta$ be the Lie algebra of Δ. By $\varphi_\star$ we denote the derivative of φ. Notice that $\varphi_\star : \mathfrak{g} \to \mathfrak{gl}(P)$ is a faithful representation of the semisimple Lie algebra $\mathfrak{g}$.

Proposition 5.1. *Let Δ be a connected, semisimple Lie group and let $\varphi : \Delta \to \mathbb{G}_0$ be a Lie homomorphism with discrete kernel. Then the real rank $\mathrm{rk}_{\mathbb{R}}\Delta$ equals 0 or 1. In particular, Δ is either a compact group or Δ is an almost direct product of an almost simple group of real rank 1 and a compact group.*

Moreover, if $C \leq \Delta$ is a real Cartan subgroup, then its image $\varphi(C)$ is contained in the reduced stabilizer $\mathbb{S}\mathbb{G}_0$.

Proof. Recall that Δ is a compact group if, and only if, $\mathrm{rk}_{\mathbb{R}}\Delta = 0$. Therefore, it sufficies to consider the case that $\mathrm{rk}_{\mathbb{R}}\Delta \geq 1$. Let $C \leq \Delta$ be a real Cartan subgroup. Then C is contained in some noncompact, almost simple normal subgroup Δ_1 of Δ. The image $\varphi(\Delta_1)$ is contained in the reduced stabilizer $\mathbb{S}\mathbb{G}_0$, see (1.1). Moreover, $\varphi(C)$ is diagonalizable on P, cf. (3.4). Using (4.2) we derive that $\mathrm{rk}_{\mathbb{R}}\Delta = \dim C$ is at most 1. According to (3.2), at most one of the almost simple factors of Δ has real rank 1 while the others have real rank 0 and thus are compact. This finishes the proof. ■

Corollary 5.2. *Let $\mathbb{P}$ be a locally compact, connected projective translation plane. If the group Γ of all continuous collineations of $\mathbb{P}$ contains a semisimple group of real rank at least 2, then $\mathbb{P}$ is a classical plane and Γ does not fix any point or line.*

Remark. The connected components $\mathrm{Aut}(\mathrm{P}_2\mathbb{F})^e = \mathrm{PSL}_3\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) and $\mathrm{Aut}(\mathrm{P}_2\mathbb{O})^e = \mathrm{E}_6(-26)$ of the automorphism groups of the classical projective planes are almost simple Lie groups of real rank 2.

Proof. Let $\Delta \leq \Gamma$ be a semisimple group. If $\mathbb{P}$ is not classical, then Γ leaves the translation axis invariant ([10, 64.4(c)]) and thus operates on the corresponding locally compact, connected affine translation plane. Being an affine group, Γ is a semidirect product $\Gamma = \Gamma_0 \ltimes \mathbb{R}^n$ of the stabilizer Γ_0 and the translation group. We conclude that every Levi complement of Γ is conjugate to a subgroup of Γ_0. Without loss of generality, we therefore may assume that the semisimple group Δ is a subgroup of Γ_0. Applying (5.1) shows that $\mathrm{rk}_{\mathbb{R}}\Delta$ equals 0 or 1. ■

We restrict our attention to the noncompact case:

Proposition 5.3. *Let Δ be a noncompact, connected, semisimple Lie group and let $\varphi : \Delta \to \mathbb{G}_0$ be a Lie homomorphism with discrete kernel. Notice that* $\mathrm{rk}_{\mathbb{R}}\Delta = 1$, *cf. (5.1). Fix a real Cartan subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{g} = T_e\Delta$ and an element $X \in \mathfrak{a} \setminus \{0\}$. Let Ω be the set of real weights of $\varphi_\star$ with respect to X (cp. (3.3)). As usual, P_ω denotes the real weight space of $\varphi_\star$ belonging to $\omega \in \Omega$. Then the following statements hold:*

(a) 0 *is not a real weight of* $\varphi_\star$.

(b) *Both vector subspaces*

$$L_+ = \bigoplus_{\omega\in\Omega;\,\omega>0} P_\omega \quad \text{and} \quad L_- = \bigoplus_{\omega\in\Omega;\,\omega<0} P_\omega$$

of P are lines.

Remark. Since $\dim \mathfrak{a} = 1$, the set $\{L_+, L_-\} \subseteq \mathcal{L}_0$ depends only on the choice of $\mathfrak{a}$, not on the choice of $X \in \mathfrak{a}$.

Proof. By (5.1), the real Cartan subgroup $\varphi(\exp \mathfrak{a})$ is contained in the reduced stabilizer $\mathbb{S}\mathbb{G}_0$. In view of (3.4) there exists a basis $\mathcal{X} = \{x_1, \ldots, x_n\}$ of P such that $\varphi(\exp(tX))$ is expressed as the matrix

$$\exp(tX) = \mathrm{diag}(e^{\omega_1 t}, \ldots, e^{\omega_n t})$$

with respect to $\mathcal{X}$, where $\omega_1 \leq \omega_2 \leq \ldots \leq \omega_n$ are the real weights of $\varphi_\star$ (i.e. the eigenvalues of $\varphi_\star(X)$). By (4.3), the vector subspaces

$$W = \mathrm{span}\{x_1, \ldots, x_l\} \text{ and } S = \mathrm{span}\{x_{l+1}, \ldots, x_n\}$$

are lines and, moreover, $\omega_1 \leq \ldots \leq \omega_l \lneqq \omega_{l+1} \leq \ldots \leq \omega_n$.

Assume that $\omega_l \geq 0$. We rewrite W and S, respectively, as a direct sum

$$W = P_{\omega_l} \oplus \bigoplus_{\omega\in\Omega;\,\omega\lneqq\omega_l} P_\omega \quad \text{and} \quad S = \bigoplus_{\omega\in\Omega;\,\omega\gneqq\omega_l} P_\omega$$

of real weight spaces. If $\omega \gneqq \omega_l \geq 0$ is a real weight, then $-\omega \lneqq 0 \leq \omega_l$ is a real weight, too, and $\dim P_{-\omega} = \dim P_\omega$, cf. (3.4). Contrary to $\dim W = \dim S = l$, this implies that

$$U = \bigoplus_{\omega\in\Omega;\,\omega\gneqq\omega_l} P_{-\omega}$$

would be proper subspace of W with $\dim U = \dim S$. We conclude that $\omega_l \lneqq 0$. Similary one proves $\omega_{l+1} \gneqq 0$. In particular, 0 is not a weight and $L_+ = S$ and $L_- = W$ are lines. ∎

Corollary 5.4. *Let $\Delta \leq \mathbb{G}_0$ be a noncompact, semisimple group. Then Δ cannot act trivially on a nonzero subspace of P.*

Proof. If Δ did operate trivially on a subspace $U \neq 0$, then 0 would be a weight of Δ, contrary to (5.3). ∎

Definition 5.5. Let $\Delta \leq \mathbb{G}_0$ be a (connected) semisimple Lie group of real rank 1. Then $L \in \mathcal{L}_0$ is called a *weight line* with respect to Δ if L is fixed by some real Cartan subgroup C of Δ. We refer to the set of all weight lines as the *weight sphere* of Δ.

6. Proof of Theorem A

Let Δ be a noncompact, connected, almost simple subgroup of the stabilizer $\mathbb{G}_0$. Then Δ is contained in the reduced stabilizer $\mathbb{S}\mathbb{G}_0$, cf. (1.1). Let $\mathfrak{g}$ be the Lie algebra of Δ and let $U \neq \{0\}$ be a Δ–irreducible subspace of P.

The real rank of Δ equals 1 by (5.1) and hence $\mathfrak{g}$ is isomorphic to an algebra of the following list, see for example [8, p. 312]:

$\mathfrak{so}_m(\mathbb{R}, 1)$, where $m \geq 3$,

$\mathfrak{su}_m(\mathbb{C}, 1)$, where $m \geq 3$,

$\mathfrak{u}_m(\mathbb{H}, 1)$, where $m \geq 3$,

the exceptional Lie algebra $\mathfrak{f}_4(-20)$.

Remark. We have isomorphisms $\mathfrak{so}_3(\mathbb{R}, 1) \cong \mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{so}_4(\mathbb{R}, 1) \cong \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{so}_5(\mathbb{R}, 1) \cong \mathfrak{u}_2(\mathbb{H}, 1)$ and $\mathfrak{so}_6(\mathbb{R}, 1) \cong \mathfrak{sl}_2(\mathbb{H})$.

Lemma 6.1. *The stabilizer $\mathbb{G}_0$ does not contain a subgroup which is locally isomorphic to* $\mathrm{SU}_3(\mathbb{C}, 1)$.

Proof. Assume that $\Delta \leq \mathbb{G}_0$ is locally isomorphic to $\mathrm{SU}_3(\mathbb{C}, 1)$. The almost simple group Δ acts completely reducible on P, i.e. $P = U_1 \oplus \ldots \oplus U_m$ is a direct sum of Δ–irreducible subspaces $U_i \leq P$.

Recall that $\dim U_i = 1$ is impossible by (5.4) and that $\dim P \in \{2, 4, 8, 16\}$. Checking the list of irreducible representations of $\mathrm{SU}_3(\mathbb{C}, 1)$, cp. [10, Sect. 95], yields $\dim U_i \in \{6, 8, 12\}$. From $\dim U_1 + \ldots + \dim U_m = \dim P \in \{2, 4, 8, 16\}$ we obtain $\dim U_i = 8$ as the only possibility.

On the other hand, $\dim U_i = 8$ implies that $\Delta|_{U_i}$ realizes the adjoint representation of $\mathrm{SU}_3(\mathbb{C}, 1)$, which has 0 as a real weight in contradiction to (5.3). ■

Together with some well known bounds for the dimension of automorphism groups of locally compact, connected translation planes, this lemma proves the following result:

Proposition 6.2. *Let $\Delta \leq \mathbb{G}_0$ be a connected, almost simple group. If Δ is not compact, then Δ is locally isomorphic to one of the following groups:*

1. $\mathrm{SL}_2\mathbb{F}$, *where* $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$, *or*
2. $\mathrm{U}_2(\mathbb{H}, 1)$, *or*
3. $\mathrm{Spin}_m(\mathbb{R}, 1)$ *with* $7 \leq m \leq 10$.

Remark. Each of the groups listed above operates on the octonion plane. Thus, our result is the best possible.

Proof. Recall that the subgroup Δ of $\mathbb{G}_0$ does not contain a subgroup locally isomorphic to $\mathrm{SU}_3(\mathbb{C},1)$ by (6.1). Consequently, Δ is not locally isomorphic to $\mathrm{SU}_m(\mathbb{C},1)$ or $\mathrm{U}_m(\mathbb{H},1)$ for any $m \geq 3$.

Moreover, the largest possible dimension of the stabilizer $\mathbb{G}_0$ of a locally compact, connected affine translation plane equals[7] $\dim(\mathbb{R}_{\mathrm{pos}} \times \mathrm{Spin}_{10}(\mathbb{R},1)) = 46$, cf. [10, 81.9, 82.27]. Thus, Δ is not locally isomorphic to $\mathrm{SO}_m(\mathbb{R},1)$ for any $m \geq 11$ nor is it locally isomorphic to the exceptional group $\mathrm{F}_4(-20)$.

Excluding these groups from the list of simple groups of real rank 1 (cp. the beginning of this section) finishes the proof. ■

We should note a corollary which is interesting in its own:

Corollary 6.3. *Let $\Gamma \leq \mathbb{G}_0$ be a closed subgroup and let $\Delta_1, \Delta_2 \leq \Gamma$ be connected, almost simple, noncompact subgroups. If $\dim \Delta_1 = \dim \Delta_2$, then Δ_1 and Δ_2 are conjugate in Γ.*

Proof. Up to conjugation, we may assume that both Δ_1 and Δ_2 are contained in the same Levi complement of Γ. Therefore, it suffices to consider the case that Γ is semisimple and connected. Since Γ cannot be a compact group, Γ is isomorphic to an almost direct product $\Gamma = \Delta \times C$, where Δ is a simple group of real rank 1 and where C is a compact group, cf. (5.1). Obviously, both Δ_1 and Δ_2 are subgroups of Δ. Moreover, by (6.2), the group Δ is locally isomorphic to $\mathrm{SO}_m(\mathbb{R},1)$ for some m. Now the assertion can be easily derived from the following fact:

If $\mathfrak{h}$ is a noncompact, simple subalgebra of the Lie algebra

$$\mathfrak{so}_m(\mathbb{R},1) = \{A \in \mathfrak{gl}_m\mathbb{R} \mid I_m A I_m = -A^{\mathrm{tr}}\}$$

(where I_m denotes the $(m \times m)$–diagonal matrix $\mathrm{diag}(1,\dots,1,-1)$), then $\mathfrak{h}$ is conjugate to the subalgebra

$$\mathfrak{u}_l = \left\{ \begin{pmatrix} 0 & \\ & B \end{pmatrix} \in \mathfrak{gl}_m\mathbb{R} \;\middle|\; I_{l+1} B I_{l+1} = -B^{\mathrm{tr}} \right\} \cong \mathfrak{so}_l(\mathbb{R},1)$$

of $\mathfrak{so}_m(\mathbb{R},1)$ for some l.

For the proof of this claim, let $\mathfrak{so}_m(\mathbb{R},1) = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{so}_m(\mathbb{R},1)$ (where $\mathfrak{k}$ is the compact part) such that $\mathfrak{h} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{h})$ is a Cartan decomposition of $\mathfrak{h}$. (The existence of such a decomposition is guaranteed by a theorem of Mostow, cp. [7, p.53].) Note that the dimension $l = \dim(\mathfrak{p} \cap \mathfrak{h}) \neq 0$ since $\mathfrak{h}$ is noncompact.

By [6, 6.19], the subspace $\mathfrak{p}$ is conjugate by an inner automorphism of $\mathfrak{so}_m(\mathbb{R},1)$ to the noncompact part

$$\mathfrak{q} = \left\{ \begin{pmatrix} 0 & x \\ x^{\mathrm{tr}} & 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^{m-1} \right\}$$

of the standard Cartan decomposition of $\mathfrak{so}_m(\mathbb{R},1)$. Moreover, the group

$$\exp \mathrm{ad} \left\{ \begin{pmatrix} C & \\ & 0 \end{pmatrix} \;\middle|\; C \in \mathfrak{so}_{m-1}\mathbb{R} \right\}$$

[7] $\mathbb{R}_{\mathrm{pos}} \times \mathrm{Spin}_{10}(\mathbb{R},1)$ is the stabilizer of the origin of the affine octonion plane.

operates transitively on the set of l–dimensional subspaces of $\mathfrak{q}$. Consequently, we may assume that

$$\mathfrak{p}_{\mathfrak{h}} = \mathfrak{p} \cap \mathfrak{h} = \left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & y^{\mathrm{tr}} & 0 \end{array} \right) \middle| \; y \in \mathbb{R}^l \right\}.$$

In particular, we obtain $\mathfrak{h} = [\mathfrak{p}_{\mathfrak{h}}; \mathfrak{p}_{\mathfrak{h}}] \oplus \mathfrak{p}_{\mathfrak{h}} = \mathfrak{u}_l$. ■

We will now investigate each of the possible groups appearing in (6.2) separately.

Convention 6.4. *We will use the following notation in* (6.5) – (6.8)*: Let Δ be a connected, almost simple Lie group of real rank* 1 *which operates on P as a group of automorphisms. To be more precise, let $\varphi : \Delta \to GL(P)$ be a representation with discrete kernel such that $\varphi(\Delta) \leq \mathbb{G}_0$ and thus $\varphi(\Delta) \leq \mathrm{S}\mathbb{G}_0$ by (1.1). We will study a nonzero φ–irreducible subspace $U \leq P$. By $\pi : \Delta \to GL(U)$ we denote the corresponding irreducible representation defined by $\pi(\delta) = \varphi(\delta)|_U$. The kernel of π then is the discrete normal subgroup $K(\pi)$.*

We start with the smallest group $\mathrm{SL}_2\mathbb{R}$:

Proposition 6.5. *Let $\Delta = \mathrm{SL}_2\mathbb{R}$ and retain the notation of* (6.4).

Then the dimension $\dim U$ is even and the kernel of π is trivial.

In particular, $\varphi(\Delta)$ is isomorphic to $\mathrm{SL}_2\mathbb{R}$ and the central involution σ of $\Delta = \mathrm{SL}_2\mathbb{R}$ acts as the reflection $\varphi(\sigma) : P \to P;\ x \mapsto -x$.

Proof. Let $m = \dim U$. Consider the real Cartan subgroup

$$C = \left\{ \left(\begin{array}{cc} e^r & 0 \\ 0 & e^{-r} \end{array} \right) \middle| \; r \in \mathbb{R} \right\}$$

of $\Delta = \mathrm{SL}_2\mathbb{R}$. According to [5, thm.12 on p.85], $\pi(C)$ is given by

$$\pi(C) = \{\mathrm{diag}(e^{(-m+1)r}, e^{(-m+3)r}, \dots, e^{(m-3)r}, e^{(m-1)r}) \mid r \in \mathbb{R}\}$$

with respect to some basis of U. Since 0 is not a real weight of π by (5.3), we conclude that $m = \dim U$ is even. An even–dimensional irreducible representation of $\mathrm{SL}_2\mathbb{R}$ always has trivial kernel and its centralizer is always $\mathbb{R}$, see the table in [10, Sect. 95]. Thus, for every φ–irreducible subspace U of P we have that $\varphi(\Delta)|_U \cong \mathrm{SL}_2\mathbb{R}$ and that $\varphi(\sigma)|_U = -\mathrm{id}$. ■

Proposition 6.6. *Let $\Delta = \mathrm{SL}_2\mathbb{C}$ and retain the notation of* (6.4).

Then $\dim U \in \{4, 8, 12, 16\}$, the kernel of π is always trivial and the representation π of Δ on U is complex linear. In particular, $\varphi(\Delta)$ is isomorphic to $\mathrm{SL}_2\mathbb{C}$.

Proof. The representations of $\mathrm{SL}_2\mathbb{C}$ in dimension 4, 8, 12 and 16 are exactly those representations of dimension at most 16 with trivial kernel, see [10, Sect. 95]. Thus, we have to exclude all representations with nontrivial kernel, i.e we have to prove that $\Delta|_U$ is not isomorphic to $\mathrm{PSL}_2\mathbb{C}$. But this is clear, since $\Delta|_U$ cannot contain a group isomorphic to $\mathrm{PSL}_2\mathbb{R}$ by (6.5). ■

Proposition 6.7. *Let $\Delta = \mathrm{U}_2(\mathbb{H}, 1)$ and retain the notation of* (6.4).

Then $\dim U = 8$ and the representation π of Δ on U is equivalent to the usual representation of $\mathrm{U}_2(\mathbb{H}, 1)$ on $\mathbb{H}^2$.

If $\dim P = 8$, then P is isomorphic to the quaternion plane.

Proof. Since Δ cannot act trivially on a subspace of P, the dimension $\dim P \in \{2, 4, 8, 16\}$ has to be a sum of dimensions of irreducible representations of $\mathrm{U}_2(\mathbb{H}, 1)$. Irreducible representations of $\mathrm{U}_2(\mathbb{H}, 1)$ in dimension at most 16 have dimension 5, 8, 10 or 14, see [10, Sect. 95]. By simple combinatorics, we infer that only the (unique) 8–dimensional representation is possible. This proves the first part of our assertion.

If $\dim P = 8$, then $\mathbb{E}$ is the quaternion plane, see [10, 82.25]. ■

Proposition 6.8. *Let $\Delta = \mathrm{SL}_2\mathbb{H}$ and retain the notation of* (6.4).

(a) For $\dim P \leq 8$, the only possibility is the natural action of $\Delta = \mathrm{SL}_2\mathbb{H}$ on the quaternion plane.

(b) If $\dim P = 16$, then $\varphi = \pi_1 \times \pi_2$, where π_1 is the natural representation of $\Delta = \mathrm{SL}_2\mathbb{H}$ on $\mathbb{H}^2$ and where π_2 is the contragredient representation on $\mathbb{H}^2$.

In particular, we can identify P and the 4–dimensional right quaternal vector space $\mathbb{H}^2 \times \mathbb{H}^2$ in such a way that $\varphi(\Delta)$ is given by

$$\varphi(\Delta) = \left\{ \begin{pmatrix} A & \\ & A^\star \end{pmatrix} \middle| A \in \mathrm{SL}_2\mathbb{H} \right\},$$

where $A^\star = (\bar{A}^{\mathrm{tr}})^{-1}$ for a matrix $A \in \mathrm{SL}_2\mathbb{H}$.

Proof. If $\dim P \leq 8$, then P is isomorphic to the quaternion plane and $\Delta = \mathrm{SL}_2\mathbb{H}$ acts in the usual way, cp. [10, 82.25]. It remains to treat the case $\dim P = 16$. By [10, Sect. 95], a φ–irreducible subspace U has dimension 6, 8, 10 or 15. If $\dim U \in \{5, 10, 15\}$, then the kernel of the representation π equals the center of Δ. But this implies that $\varphi(\Delta)|_U \cong \mathrm{PSL}_2\mathbb{H}$ would contain a subgroup isomorphic to $\mathrm{PSL}_2\mathbb{R}$ in contradiction to (6.5).

Consequently, P is a direct sum of two φ–irreducible 8–dimensional subspaces U_1 and U_2. Let π_1 and π_2 denote the representations of $\Delta = \mathrm{SL}_2\mathbb{H}$ on U_1 and U_2, respectively. Notice that π_i either is the natural representation of $\mathrm{SL}_2\mathbb{H}$ on $\mathbb{H}^2$, or is the contragredient one.

If π_1 and π_2 are equivalent, then we can identify $P = U_1 \oplus U_2$ with $\mathbb{H}^2 \times \mathbb{H}^2$ such that $\varphi(\Delta)$ becomes the group

$$\varphi(\Delta) = \left\{ \begin{pmatrix} A & \\ & A \end{pmatrix} \middle| A \in \mathrm{SL}_2\mathbb{H} \right\}.$$

In this case the weight lines belonging to the real Cartan subgroup

$$\varphi(C) = \{\mathrm{diag}(e^r, e^{-r}, e^r, e^{-r}) \mid r \in \mathbb{R}\}$$

are $W = \mathbb{H} \times 0 \times \mathbb{H} \times 0$ and $S = 0 \times \mathbb{H} \times 0 \times \mathbb{H}$. Obviously, the compact group

$$K = \{\mathrm{diag}(a, 1, a, 1) \mid a \in \mathbb{H},\ \|a\| = 1\} \cong \mathrm{Spin}_3\mathbb{R}$$

leaves W invariant and fixes S pointwise. But this is impossible by [2].

Thus, π_1 and π_2 are inequivalent , i.e. π_1 realizes (without loss of generality) the natural representation of $\mathrm{SL}_2\mathbb{H}$ and π_2 realizes the contragredient one. This proves part (b). ∎

Proposition 6.9. *Let $\Delta \leq \mathbb{G}_0$ be locally isomorphic to* $\mathrm{Spin}_m(\mathbb{R}, 1)$ *for some m, $7 \leq m \leq 10$. Then $(P, \mathcal{L})$ is isomorphic to the octonion plane and Δ is isomorphic to* $\mathrm{Spin}_m(\mathbb{R}, 1)$ *in its usual embedding in* $\mathrm{Spin}_{10}(\mathbb{R}, 1)$. *In particular, Δ operates irreducibly on P.*

Proof. Δ contains a group Ξ locally isomorphic to $\mathrm{Spin}_7(\mathbb{R}, 1)$. Thus, $(P, \mathcal{L})$ is isomorphic to the octonion plane by [4] and Δ is a subgroup of the reduced stabilizer $\mathrm{S}\mathbb{G}_0 = \mathrm{Spin}_{10}(\mathbb{R}, 1)$ of the octonion plane. By (6.3), we infer that Δ is conjugate to the group $\mathrm{Spin}_m(\mathbb{R}, 1)$ in its usual embedding in $\mathrm{Spin}_{10}(\mathbb{R}, 1)$ for some m, $7 \leq m \leq 10$. ∎

Remark. The proof of Theorem A can be pieced together from Propositions 6.2 – 6.9 and the remark at the beginning of Sect. 6.

7. Proof of Theorem B

Let Δ be a noncompact, connected, almost simple subgroup of the stabilizer $\mathbb{G}_0$ and let $\mathcal{S} \subseteq \mathcal{L}_0$ be the weight sphere of Δ. From Theorem A we know that Δ is a 2–fold covering group of $\mathrm{PSO}_m(\mathbb{R}, 1)$ for some m, $3 \leq m \leq 10$. The Lie algebra $\mathfrak{g} = T_e\Delta$ therefore is given by

$$\mathfrak{g} = \mathfrak{so}_m(\mathbb{R}, 1) = \{A \in \mathfrak{gl}_m\mathbb{R} \mid I \cdot A \cdot I = -A^{\mathrm{tr}}\},$$

where I denotes the matrix $I = \mathrm{diag}(1, \ldots, 1, -1)$. In order to avoid confusion we define the faithful representation $\varphi_\star : \mathfrak{g} \to \mathfrak{gl}(P)$ as the derivative of the natural embedding $\varphi = \mathrm{id} : \Delta \to \mathbb{G}_0$.

Lemma 7.1. *Δ leaves $\mathcal{S}$ invariant.*

Proof. A line $L \in \mathcal{L}_0$ is an element of $\mathcal{S}$ if, and only if, L is fixed by some real Cartan subgroup C of Δ. For $\delta \in \Delta$, the conjugate $\delta C \delta^{-1}$ is a real Cartan subgroup fixing $\delta(L)$ and thus $\delta(L)$ is an element of $\mathcal{L}$ again. ∎

Before discussing the action of Δ on $\mathcal{S}$, we have a closer look at the Lie algebra $\mathfrak{g} = \mathfrak{so}_m(\mathbb{R},1)$. For this purpose we fix the real Cartan subalgebra

$$\mathfrak{a} = \mathbb{R}\cdot X \leq \mathfrak{g} = \mathfrak{so}_m(\mathbb{R},1), \text{ where } X = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

of $\mathfrak{g}$. The corresponding Cartan involution is

$$\iota : \mathfrak{so}_m(\mathbb{R},1) \to \mathfrak{so}_m(\mathbb{R},1);\, Y \mapsto I\cdot Y \cdot I^{-1},$$

i.e. $\mathfrak{a}$ is a maximal abelian subspace of the Cartan complement $\mathfrak{p}$ of the Cartan decomposition defined by ι, i.e.

$$\begin{aligned} \mathfrak{so}_m(\mathbb{R},1) = \mathfrak{k}\oplus\mathfrak{p} \quad &\text{with} \quad \mathfrak{k} = \left\{ \begin{pmatrix} A & \\ & 0 \end{pmatrix} \,\middle|\, A \in \mathfrak{so}_{m-1}\mathbb{R} \right\} \\ &\text{and} \quad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ x^{\mathrm{tr}} & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R}^{m-1} \right\}. \end{aligned}$$

The restricted roots[8] with respect to X are 0, 1 and -1. We write $\mathfrak{g}_\lambda$ for the restricted root space belonging to the restricted root λ and obtain

$$\mathfrak{g}_1 = \{Y \in \mathfrak{g} \,|\, (\operatorname{ad} X)(Y) = Y\} = \left\{ \begin{pmatrix} 0 & -x & x \\ x^{\mathrm{tr}} & 0 & 0 \\ x^{\mathrm{tr}} & 0 & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R}^{m-2} \right\}.$$

In fact, $\mathfrak{g}_1$ is an abelian subalgebra of $\mathfrak{g}$ since $[\mathfrak{g}_1,\mathfrak{g}_1] \subseteq \mathfrak{g}_2 = \{0\}$. Notice that $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{g}_1$ and $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{g}_{-1}$ are the Iwasawa decompositions of $\mathfrak{g}$ defined by $\mathfrak{a}$.

In what follows we need the notion of parabolic subgroups of an almost simple Lie group. We will introduce this concept for the groups considered here only, for the general case we refer to [12, Chap.1.2].

We start with the Iwasawa decomposition $\Delta = K\cdot C\cdot N_1$, where $K = \exp\mathfrak{k}$, $N_1 = \exp\mathfrak{g}_1$ and where C denotes the real Cartan subgroup $C = \exp\mathfrak{a}$ of Δ. Let M and $M^\star$ be the centralizer and the normalizer, respectively, of C in K. Then $\Pi = M\cdot C\cdot N_1$ is a closed subgroup of Δ, which we call the *standard minimal parabolic subgroup* of Δ. We emphasize the fact that Π is uniquely determined by the generator X of $\mathfrak{a}$. If we replace X by an arbitrary element Y generating some other real Cartan subalgebra of $\mathfrak{g}$, then we obtain another closed subgroup Π_Y of Δ by the same construction. These subgroups are called the *minimal parabolic subgroups* of Δ. Since every two real Cartan subalgebras of $\mathfrak{g}$ are conjugate, we see from the construction that every minimal parabolic subgroup is conjugate to a minimal parabolic subgroup defined by an element X' of $\mathfrak{a}$. Obviously, $\Pi_{(rY)} = \Pi_Y$ holds for every positive real number r and thus X' can be taken as X or $-X$. In order to see that $\Pi = \Pi_X$ and Π_{-X} are conjugate, too, we use the *Weyl group*

[8] A restricted root is a real weight of the adjoint representation; the corresponding real weight spaces are called restricted root spaces.

$W(\Delta, C) = M^\star/M$ of Δ. In our situation, this Weyl group is a group of order 2: For every $w \in M^\star$, the automorphism $\mathrm{Ad}\, w$ leaves invariant the one–dimensional algebra $\mathfrak{a}$ and therefore $(\mathrm{Ad}\, w)|_{\mathfrak{a}} = \pm \mathrm{id}_{\mathfrak{a}}$. Moreover, w induces the identity on $\mathfrak{a}$ if, and only if, w is an element of the centralizer M of C in K. Consequently, $M^\star \cdot M^\star \subseteq M$ and thus $W(\Delta, C) = M^\star/M$ is a group of order 2 (since the Weyl group of a noncompact, almost simple group never is trivial, see the considerations in Warner [12, Chap. 1.2]). Given an element $w \in M^\star \setminus M$, we conclude from $(\mathrm{Ad}\, w)(X) = -X$ that $w \Pi_X w^{-1} = \Pi_{-X}$.

We should add another interesting conclusion: The set of orbits of Δ on the set of all parabolic subgroups (i.e. subgroups containing a minimal parabolic subgroup) of Δ is in one–to–one correspondence with the elements of the Weyl group, see [12, Thm.1.2.1.1]. In our situation, $|W(\Delta, C)| = 2$ implies that every proper subgroup of Δ containing a minimal parabolic subgroup is a minimal parabolic subgroup itself.

Let us return to geometry. According to (5.3) the real Cartan subgroup C of Δ fixes exactly the weight lines

$$W = \bigoplus_{\lambda \in \Lambda, \lambda < 0} P_\lambda \qquad \text{and} \qquad S = \bigoplus_{\lambda \in \Lambda, \lambda > 0} P_\lambda .$$

Here, Λ is the set of real weights of $\varphi_\star$ with respect to X (cp. (3.3)) and P_λ is the real weight space belonging to $\lambda \in \Lambda$. We claim

Lemma 7.2. *The stabilizer Δ_S of S equals the minimal parabolic subgroup $\Pi = M \cdot C \cdot N_1$ and fixes no line except S in $\mathcal{L}_0$.*

Moreover, if w represents the non–trivial element of the Weyl group $W(\Delta, C)$, then w interchanges W and S.

Proof. Let λ' be the maximal element of Λ. For $Y \in \mathfrak{g}_1$ and $v \in P_{\lambda'}$, the equation

$$\begin{aligned} \varphi_\star(Y)(v) &= [\varphi_\star(X); \varphi_\star(Y)](v) \\ &= \varphi_\star(X)\varphi_\star(Y)(v) - \varphi_\star(Y)\varphi_\star(X)(v) \\ &= \varphi_\star(X)(\varphi_\star(Y)(v)) - \lambda' \cdot \varphi_\star(Y)(v) \end{aligned}$$

shows that $\varphi_\star(X)(\varphi_\star(Y)(v)) = (1 + \lambda') \cdot \varphi_\star(Y)(v)$. Consequently, $\varphi_\star(Y)(v) = 0$ since λ' is a maximal real weight. Therefore, the group $N_1 = \exp \mathfrak{g}_1$ operates trivially on the subspace $P_{\lambda'}$ of S and hence leaves S invariant.

Thus, N_1 is a subgroup of Δ_S. Moreover, M centralizes $C \leq \Delta_S$ and thus fixes S, too. This shows that Δ_S contains the minimal parabolic subgroup Π and thus is a parabolic subgroup of Δ. The Weyl group $W(\Delta, C)$ has order 2 and thus either $\Delta_S = \Pi$ or $\Delta_S = \Delta$. The latter case cannot occur, otherwise Δ leaves S invariant and realizes on S a representation having only positive weights contradicting (3.4). Therefore, $\Delta_S = \Pi$ and, analogously, $\Delta_W = \Pi_{-X}$. Since $\Pi \neq \Pi_{-X}$, the line W is not invariant under Π. Moreover, $C \leq \Pi$ fixes no lines except W and S. From these considerations we infer that Π cannot fix a line in $\mathcal{L}_0 \setminus \{S\}$.

Now consider an element $w \in M^\star$ representing the non–trivial element of the Weyl group $W(\Delta, C)$. Then $\mathrm{Ad}\, w$ induces the map $Y \mapsto -Y$ on $\mathfrak{a}$. Consequently, w interchanges the real weight spaces P_λ and $P_{-\lambda}$ for every real weight λ. This proves the remaining assertion. ■

Corollary 7.3. *The stabilizer of a weight line $L \in \mathcal{S}$ is a minimal parabolic subgroup of Δ. Conversely, a minimal parabolic subgroup of Δ fixes exactly one line $L \in \mathcal{L}_0$ and, moreover, L is a weight line.*

Proof. Let C' be a real Cartan subgroup of Δ which fixes L. Then C' is conjugate to C by an element $\gamma \in \Delta$, see (3.1), and C fixes $\gamma(L)$. Without loss of generality we assume $\gamma(L) = S$. Consequently, $\Delta_L = \gamma^{-1}\Delta_S\gamma = \gamma^{-1}\Pi\gamma$ is conjugate to Π and thus is a minimal parabolic subgroup of Δ.

Conversely, a minimal parabolic subgroup is conjugate to $\Pi = \Delta_S$ by an element, say, $\delta \in \Delta$ and thus equals the stabilizer of $\delta(S) \in \mathcal{S}$. ■

Since any two minimal parabolic subgroups of Δ are conjugate to each other, we immediately obtain:

Corollary 7.4. *Δ operates transitively on its weight sphere $\mathcal{S}$.*

Thus, the weight sphere $\mathcal{S}$ is a homogeneous space Δ/Π, where $\Pi = \Delta_S$ is a minimal parabolic subgroup of Δ. Moreover, the central element $-e \in \Delta$ operates trivially on $\mathcal{L}_0$, i.e. $\mathcal{S}$ can be considered as a homogeneous space of $\Delta/\pm e \cong \mathrm{PSO}_m(\mathbb{R},1)$ modulo the minimal parabolic subgroup $\Pi' = \Pi/\pm e$. In fact, this homogeneous space may also be derived from the natural action of $\mathrm{PSO}_m(\mathbb{R},1)$ on the corresponding projective quadric, which is an $(m-2)$–sphere. Hence $\mathbb{S}_{m-2} \approx \mathrm{PSO}_m(\mathbb{R},1)/\Pi' \approx \mathcal{S}$ and the action of Δ on $\mathcal{S}$ is equivalent to the natural action of $\mathrm{PSO}_m(\mathbb{R},1)$ on $\mathbb{S}_{m-2}$, i.e. the assertions stated in part (1) and (2) of Theorem B are proved. Part (3) of Theorem B is a consequence of Theorem A.

It remains to show that no orbit except $\mathcal{S}$ is closed in $\mathcal{L}_0$. Consider a line $L \in \mathcal{L}_0 \setminus \mathcal{S}$. Fix a real Cartan subgroup $C \leq \Delta$. Then C fixes two lines $W, S \in \mathcal{S}$. Since $C \leq \mathbb{G}_{W,S}$ is a compression subgroup, we can apply (1.2) and find a parametrization $\rho : \mathbb{R} \to C$ such that $\lim_{t\to\infty} \rho(t)(L) = S$. The closure of the orbit $\Delta(L)$ thus contains $S \in \mathcal{S} = \Delta(S)$. We conclude that $\Delta(L)$ is not closed in $\mathcal{L}_0$.

References

[1] Betten, D., *4–dimensionale Translationsebenen mit irreduzibler Kollineationsgruppe*, Arch. Math. **24** (1973), 552–560.

[2] Buchanan, T., and H. Hähl, *On the kernel and the nuclei of 8–dimensional locally compact quasifields*, Arch. Math. **29** (1977), 472–480.

[3] Hähl, H., *Achtdimensionale lokalkompakte Translationsebenen mit zu* $\mathrm{SL}_2\mathbb{C}$ *isomorphen Kollineationsgruppen*, J. Reine Angew. Math. **330** (1982), 76–92.

[4] —, *Eine Kennzeichnung der Oktavenebene*, Indag. Math. **49** (1987), 29–39.

[5] Jacobson, N., "Lie Algebras," Dover Publications, 1962.

[6] Knapp, A. W., "Lie Groups Beyond an Introduction," Birkhäuser, 1996.

[7] Mostow, G. D., *Some New Decomposition Theorems for Semi–Simple Lie Groups*, Memoirs Amer. Math. Soc. **14** (1955), 44–55.

[8] Onishchik, A. L., and E. B. Vinberg, "Lie Groups and Algebraic Groups," Springer, 1990.

[9] —, "Lie Groups and Lie Algebras III," Encyclopaedia of Mathematical Sciences **41**, Springer, 1994.

[10] Salzmann, H., D. Betten, Th. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel,, "Compact Projective Planes," Walter de Gruyter, 1995.

[11] Tits, J., „Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen", Lecture Notes in Math. **40**, Springer, 1967.

[12] Warner, G., "Harmonic Analysis on Semi–Simple Lie Groups I," Springer, 1972.

Harald Löwe
Technische Universität Braunschweig
Institut für Analysis
Abteilung für Topologie und Grundlagen der Analysis
Pockelsstr. 14
38106 Braunschweig, Germany
harald@geometrie.tu-bs.de

Received October 10, 1998
and in final form Mai 26, 1999