Journal of Lie Theory
Volume **10** (2000) 331–343

Index of Lie algebras of seaweed type

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Communicated by E. B. Vinberg

Abstract. We present a solution to the problem of finding the index of parabolic subalgebras of $\mathrm{GL}(n)$ and expose their relationship to interesting combinatorial objects, namely, *Meanders*.

1. Introduction

In this paper we provide an explicit computation of index for a family of subalgebras of $\mathrm{GL}(n)$. This moves the problem of finding Frobenius parabolic subalgebras of simple Lie algebras one step ahead; a partial solution of this problem was provided by A. G. Elashvili in [1, 2]. Surprisingly the answer involves consideration of *Meanders*—an object rarely seen before in representation theory.

2. Index of Lie algebra, definitions

The definitions presented below are done, in part, to settle on notation used throughout the paper. For more extended discussion of them the reader might consult the sources mentioned in the references.

All vector spaces in the material below are assumed to be finite dimensional.

Definition 2.1. A Lie algebra $\mathfrak{g}$ is a vector space together with a bracket operation $[\cdot,\cdot]$ that satisfies the following conditions:

$$[X,Y] = -[Y,X] \quad \text{(skew-symmetry)} \tag{1}$$

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \quad \text{(Jacobi identity)} \tag{2}$$

The bracket operation on a Lie algebra $\mathfrak{g}$ gives rise to a skew-symmetric form. Let f be a linear function on $\mathfrak{g}$. Then

$$B_f(X,Y) := f([X,Y])$$

is skew-symmetric in X and Y. The dual space of $\mathfrak{g}$ we will denote by $\mathfrak{g}^*$. It possesses a natural action of the Lie algebra $\mathfrak{g}$ (called the coadjoint action):

$$\mathrm{ad}^*_X : f \to B_f(-X,\cdot)$$

ISSN 0949–5932 / \$2.50

Definition 2.2. The index of the skew-symmetric form B_f is defined as $\dim\ker B_f$.

The index varies with f, for example when $f=0$, the index of B_f is equal to $\dim\mathfrak{g}$.

Definition 2.3. The index of a Lie algebra $\mathfrak{g}$ is defined by the following formula:

$$\operatorname{ind}\mathfrak{g} = \min_f \operatorname{ind} B_f$$

Proposition 2.4. *The set of f such that $\operatorname{ind} B_f = \operatorname{ind}\mathfrak{g}$ is open in $\mathfrak{g}^*$ under both Zariski and Euclidian topologies.*

Proposition 2.5. *Consider $\Omega = B_f$ as a skew-symmetric form over $\mathfrak{g}$ with coefficients in polynomials over $\mathfrak{g}^*$ (i.e. we allow f to vary). Let r be the maximal number such that $\wedge^r\Omega$ is non-zero. Then*

$$2r + \operatorname{ind}\mathfrak{g} = \dim\mathfrak{g} \tag{3}$$

Example 2.6. Let us compute the index for the Lie algebra $\mathrm{gl}(n)$ of $n\times n$ matrices. The dual space of this algebra can also be realized as $n\times n$ matrices:

$$F(X) = tr(FX)$$

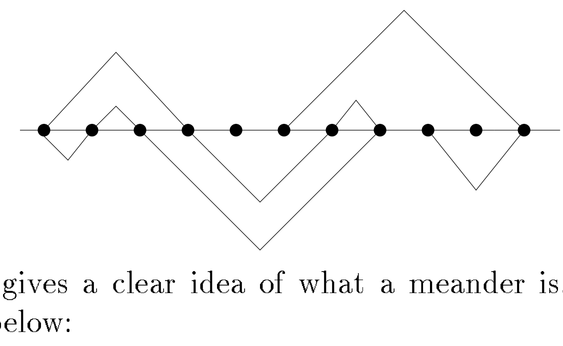
B_F can be computed explicitly:

$$B_F(X,Y) = tr(F[X,Y]) = tr(FXY - FYX) = tr(FXY - XFY) = tr([F,X]Y)$$

The last term shows that $X\in\ker B_F$ if and only if $[F,X]=0$. When all eigenvalues of F are different, the space of all X that satisfy this condition has dimension equal to n.

Since Ω has polynomial coefficients and because the set of all F with distinct eigenvalues is open, the number r should be equal to half the rank of B_F in a generic point F, which can be assumed to have distinct eigenvalues. Hence $\operatorname{ind}\mathrm{gl}(n) = n$.

3. Meanders



The picture above gives a clear idea of what a meander is. The mathematical definition is given below:

Definition 3.1. The *target space* T is $\mathbb{R}^2$ with an embedded line L. The *source space* M determines the type of the meander and is a disjoint union of circles, segments and points (one can also consider points as segments with 0 length).

The *meander representation* is defined as an embedding of M into the target space T that satisfies the following conditions:

- The endpoints of segments must map into L
- Single points must map into L
- The embedding is transversal to the line L.

The *meander* is defined as an equivalence class of meander representations under homeomorphisms of the target space.

It is useful to mark all intersection points of the embedding and the line L. The meander can be split into an upper half and a lower half, each of which consisting of a number of arcs and a number of points on the horizontal line L. Each arc defines a transposition of two marked points.

Definition 3.2. The product of transpositions over all arcs in the meander is the *permutation associated to this meander.*

Since the product can be taken in different orders, the associated permutation is not unique. However the following statement is true:

Theorem 3.3. *The number of cycles in a decomposition of the associated permutation is unique to the meander. It can be computed as twice the number of circles plus the number of segments plus the number of isolated points.*

Proof. Arcs that are disjoint define transpositions that commute between themselves. Thus the associated permutation can be represented as a product of commuting permutations that correspond to a single circle or segment.

Let us first concentrate on a single segment. By enumerating the points we can reduce this case to the following proposition:

Proposition 3.4. *The product of transpositions* $(1,2), ..., (n, n+1)$ *in any order is always a full cycle.*

Proof. We will proceed by induction. The cyclic change in the order the product is taken can be achieved by conjugation which does not change the number of cycles in a decomposition. Thus the cases $n = 1, 2$ are true.

Assume that case $n-1$ holds. By cyclic reordering we can put $(n, n+1)$ at the very end of the product. Now by assumption the product of the other transpositions is a n-cycle that does not include the number $n+1$. Multiplication by $(n, n+1)$ produces an $(n+1)$-cycle thus proving the proposition. ∎

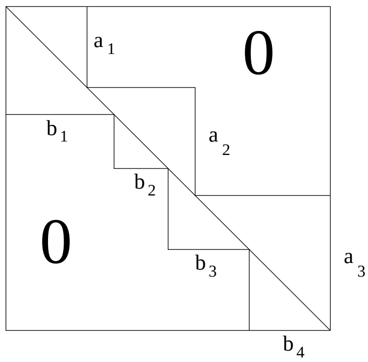
The circle case is solved by the following proposition:

Proposition 3.5. *The product of transpositions* $(1,2),...,(n,n+1)$ *and* $(n+1,1)$ *in any order is a product of two cycles.*

Proof. Indeed, by cyclic reordering we can always achieve that $(n+1,1)$ is at the very end of the product. By the proposition above, the product of the previous transpositions is a full $(n+1)$-cycle, the product of which with $(n+1,1)$ produces two disjoint cycles which lengths add up to $n+1$. ∎

This completes the proof of the theorem. ∎

4. Seaweed algebras



Seaweed algebras are subalgebras of the full matrix algebra. An example is shown in the picture above. The shape is fixed and the numbers $a_1,..,a_3$ and $b_1,...,b_4$ are invariants of this particular seaweed algebra. The precise mathematical definition is below.

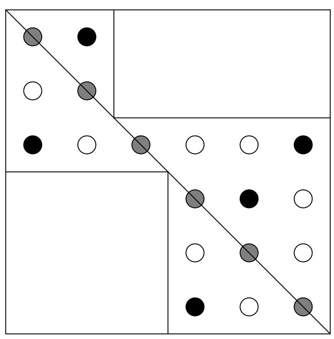
Definition 4.1. Let $\mathbf{k}$ be an arbitrary field. Fix two ordered partitions $\{a_i\}_{i=1}^{i=k}$ and $\{b_j\}_{j=1}^{j=l}$ of the number n. Let $\{e_i\}_{i=1}^{i=n}$ be the standard basis in $\mathbf{k}^n$. A subalgebra of $\mathrm{Mat}(n)$ that preserves the vector spaces $\{V_i = \mathrm{span}\,(e_1,...,e_{a_1+...+a_i})\}$ and $\{W_j = \mathrm{span}\,(e_{b_1+...+b_j+1},...,e_n)\}$ is called a subalgebra of seaweed type.

The dimension of the seaweed algebra is equal to

$$\sum_{i=1}^{k}\frac{a_i^2}{2}+\sum_{j=1}^{l}\frac{b_j^2}{2}$$

Seaweed algebras are associative. The standard bracket operation $[X,Y]=XY-YX$ gives the structure of a Lie subalgebra of $\mathrm{gl}(n)$. When one of the partitions is just $\{n\}$, we obtain a *parabolic* subalgebra.

It is convenient to distinguish special places in the matrix of an element of a given seaweed algebra.

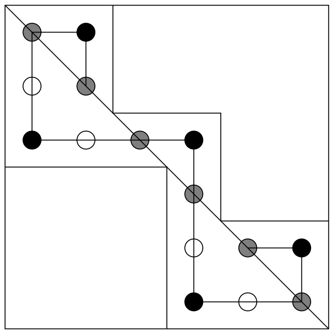


The diagonal entries are colored *gray*. The entries on the medians of small triangles are colored *black*. All other places are colored *white*. We assign color to matrix units according to the place of their non-zero entry.

Definition 4.2. The seaweed algebra $\mathfrak{g}'$ *dual* to given seaweed algebra $\mathfrak{g}$ is the algebra of matrices obtained by exchanging $\{a_i\}$ and $\{b_j\}$ in the definition 4.1.

For $X \in \mathfrak{g}'$ and $Y \in \mathfrak{g}$ we have $\langle X, Y\rangle := \operatorname{tr}(XY)$. Thus the dual space $\mathfrak{g}^*$ can be realized as the dual algebra $\mathfrak{g}'$.

Each seaweed algebra has a meander associated to it. It is produced by connecting each black entry with the gray entry in the same column or row on the picture of the matrix.



The role of the horizontal line L is taken by the main diagonal of the matrix.

5. Main theorem

Theorem 5.1. *The index of a seaweed algebra is equal to:*

- *twice the number of circles in the associated meander plus the number of segments plus the number of isolated points*
- *the number of cycles in the permutation corresponding to the associated meander.*

The proof splits naturally into four parts: first one has to establish the the index is at most the number of cycles in the associated permutation, then prove the theorem for seaweed algebras with connected meanders which makes it possible to prove that the estimate is precise for all algebras. We finish with the argument that the theorem is true for algebras over any field of sufficiently large (depending on the size of the matrices) or 0 characteristic. In the first three parts we will assume that our algebra is defined over $\mathbb{C}$.

6. Proof, part 1: estimate of the index

Recall that in section 2. we established that the maximal number r such that $\wedge^r\Omega$ is non zero satisfies

$$2r + \operatorname{ind}\mathfrak{g} = \dim\mathfrak{g}.$$

Here Ω is a skew-symmetric form with coefficients in polynomials over $\mathfrak{g}^*$. If we restrict these polynomials to a subset of $\mathfrak{g}^*$ the maximal number q such that q-th power of Ω is non-zero should not exceed r, thus satisfying

$$2q + \operatorname{ind}\mathfrak{g} \leq \dim\mathfrak{g}$$

This produces an estimate for $\operatorname{ind}\mathfrak{g}$:

$$\operatorname{ind}\mathfrak{g} \leq \dim\mathfrak{g} - 2q$$

The trick now is to specify a subset of $\mathfrak{g}^*$, such that the computation of q is both feasible and possesses the necessary precision.

Consider $\mathfrak{g}^*$ realized as the dual algebra $\mathfrak{g}'$. Let $\mathcal{F}$ denote the family of elements $F \in \mathfrak{g}^*$ which have all *white* and *gray* matrix entries set to zero.

For example (algebra (2,2,2):(3,3)):

$$F = \begin{pmatrix} 0 & 0 & f_{1,3} & 0 & 0 & 0 \\ f_{2,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{4,2} & 0 & 0 & f_{4,6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{6,5} & 0 \end{pmatrix}$$

We will denote the *black* entries of F by $f_{k,l}$.

Recall that if $E_{i,j}$ and $E_{k,l}$ are two matrix units (matrices with all components equal to 0 except (i,j) and (k,l) set to 1 correspondingly) their commutator is

$$[E_{i,j}, E_{k,l}] = \delta_{jk}E_{i,l} - \delta_{il}E_{k,j}$$

Recall also that $E_{i,j}$ is dual to $E_{j,i}$. Then

$$\Omega|_{\mathcal{F}} = \sum f_{l,k} \sum_{i=k}^{l} E_{i,k} \wedge E_{l,i} + \sum f_{l,k} \sum_{i \notin [k,l]} E_{i,k} \wedge E_{l,i}$$

For matters of exposition it is useful to split $\Omega|_{\mathcal{F}}$ into 3 forms:

$$\Omega|_{\mathcal{F}} = \omega_0 + \omega_1 + \omega_2$$

where

$$\begin{aligned} \omega_0 &:= \sum f_{l,k} \sum_{i \in]k,l[\cap \mathbb{Z}} E_{i,k} \wedge E_{l,i} \\ \omega_1 &:= \sum f_{l,k} \, (E_{k,k} - E_{l,l}) \wedge E_{l,k} \\ \omega_2 &:= \sum f_{l,k} \sum_{i \notin [k,l]} E_{i,k} \wedge E_{l,i} \end{aligned}$$

Since in ω_0 all monomials are wedges of different pairs of *white* units the rank of ω_0 is equal to the number of *white* units.

The rank of ω_1 is computed by considering each component of the meander graph separately.

Segments: let us change the numbering of the vertices so that the chain consists of edges $\{(1,2),..,(m-1,m)\}$. This corresponds to the following summand of ω_1

$$\gamma = \sum_{i=1}^{m-1} (E_{i,i} - E_{i+1,i+1}) \wedge t_i$$

Here t_i is the corresponding *black* unit times plus or minus the corresponding f_* variable.

The rank of the form γ is easy to compute: it is equal to $2m-2$. Since γ involves $2m-1$ elements, we are losing exactly 1 dimension.

Circles: similarly to the case of of a chain we will change the numbering so that the cycle consists of edges $\{(1,2),..,(m-1,m),(m,1)\}$

This corresponds to the following summand of ω_1:

$$\gamma = \sum_{i=1}^{m-1} (E_{i,i} - E_{i+1,i+1}) \wedge t_i + (E_{m,m} - E_{1,1}) \wedge t_m$$

Here t_i correspond to *black* entries and are equal to plus or minus *black* entry times the corresponding f_* variable.

The rank of the form γ is easy to compute: it is equal to $2m-2$. Since γ involves $2m$ elements exactly 2 dimensions are missing.

The forms ω_0 and ω_1 contain wedges of different matrix units. Thus the rank of $\omega_0 + \omega_1$ is the sum of the ranks of ω_0 and ω_1. Since ω_0 has rank equal to the number of *white* units and ω_1 has rank equal to the number of *black* and *gray* units minus the number of cycles in Ξ we conclude that the dimension of the kernel of $\omega_0 + \omega_1$ is equal to the number of cycles in the associated permutation (which we will denote by $\mathfrak{J}$).

Consider now forms $\mu_1 = (\omega_0 + \omega_1)^{\frac{\dim \mathfrak{g} - \mathfrak{J}}{2}}$, $\mu_2 = (\omega_0 + \omega_1 + \omega_2)^{\frac{\dim \mathfrak{g} - \mathfrak{J}}{2}}$ and $\mu_2' = (\omega_0 + \omega_2)^{\frac{\dim \mathfrak{g} - p}{2}} \omega_1^{\frac{p - \mathfrak{J}}{2}}$ (here p is the total number of *black* and *gray* matrix units).

The forms ω_0 and ω_2 do not have terms with *gray* entries, while each term of ω_1 contains exactly one *gray* entry. Thus if the expression μ_2' is non-zero, μ_2 should be non-zero as well. The form $\mu_2'' = \omega_0^{\frac{\dim \mathfrak{g} - p}{2}} \omega_1^{\frac{p - \mathfrak{J}}{2}}$ is non zero (the powers are equal to half the ranks of the corresponding forms) allowing us to concentrate on the $(\omega_0 + \omega_2)^{\frac{\dim \mathfrak{g} - p}{2}}$ component of μ_2'.

Since ω_0 has exactly $\frac{\dim \mathfrak{g} - p}{2}$ summands, we can represent $\omega_0^{\frac{\dim \mathfrak{g} - p}{2}}$ as a monomial in 1-forms times a monomial in f_*.

Our goal is to prove that the addition of ω_2 cannot introduce any terms to cancel out products with this monomial as a coefficient.

Indeed, each pair of wedge products in ω_2 consists of exactly one *white* matrix unit which lies in the same triangle as the *black* unit corresponding to this pair. Moreover, this unit must lie to the top or right of the *black* unit for the triangle in the upper half of the matrix (bottom or left for the triangle in the lower half). This implies that there are no pairs corresponding to the *black* units on the boundary of the matrix form of our seaweed algebra.

Thus if we try to construct a product of elements of $\omega_0+\omega_2$ with the same monomial coefficient in f_* the pairs corresponding to boundary units would have to be taken from ω_0 and to the same degree as in $\omega_0^{\frac{\dim \mathfrak{g}-p}{2}}$. But this implies that we cannot take any pairs from ω_2 corresponding to the next-to-boundary *black* units - since all *white* units to the top and right (in case of upper triangles, bottom and left in case of lower triangles) already participate in the product. Continuing we get that the only way to form a product with the same monomial in f_* is to use wedges only from ω_0. Thus μ_2' is non-zero and the rank of $\omega_0+\omega_1+\omega_2$ is at least $\dim \mathfrak{g} - \mathfrak{J}$.

This concludes the first part of the proof.

7. Proof, part 2: single component meanders

Proposition 7.1. *The statement of the theorem holds for all seaweed algebras with connected meander.*

Proof. We will consider three cases when the meander has only one component separately.

The case when meander consists of one marked point is trivial: it corresponds to the base field $\mathbb{C}$. The algebra is commutative and the index is equal to 1.

If the meander consists of one segment, than the index can not exceed 1. But every seaweed algebra has a center, thus the index is at least one. The invariant of the coadjoint action on $\mathfrak{g}^*$ is the trace.

The case when the meander consists of one circle is slightly less trivial. First of all observe that each marked point on the horizontal line must have upper and lower arcs connected to it. But arcs in the upper(and lower) half of the meander are disjoint. Thus the number of marked points is even.

Moreover none of the numbers $\{a_i\}$ or $\{b_j\}$ can be odd, since this would imply that the meander has a marked point which is connected to only one arc.

Hence, the dimension of our algebra is even. But since trace is an invariant and the rank of the form Ω is always even (Ω is skew-symmetric), we obtain that the corank of Ω is a positive even number, which can only be 2. ∎

8. Proof, part 3: the estimate is precise

Let $\mathfrak{g}$ denote a seaweed algebra. Let $\mathcal{M}$ denote it's meander graph. For each subset $S \subset \{1,..,n\}$ (n is the size of the matrices in $\mathfrak{g}$) we can construct a subalgebra $\mathfrak{g}_S$ of matrices with entries with one or both coordinates outside of S set to 0.

Each meander component defines a subset $\mathcal{M}_i \subset \{1,..,n\}$ of marked points on the diagonal. The meander associated to the seaweed algebra $\mathfrak{g}|_{\mathcal{M}_i}$ is exactly the component corresponding to $\mathcal{M}_i$.

The embedding $\mathfrak{g}|_{\mathcal{M}_i} \to \mathfrak{g}$ induces the projection $\mathfrak{g}^* \to \mathfrak{g}|^*_{\mathcal{M}_i}$ which has a natural section: $\mathfrak{g}|^*_{\mathcal{M}_i}$ can be embedded into $\mathfrak{g}^*$ as $\mathfrak{g}^*|_{\mathcal{M}_i}$.

Let us define the set $\mathfrak{H}$ as follows:

1. For each algebra $\mathfrak{g}|_{\mathcal{M}_i}$ pick a generic point $h_i \in \mathcal{F}|_{\mathcal{M}_i}$.
2. Let[1] $\mathfrak{H}_i$ be a manifold containing h_i of dimension ind $\mathfrak{g}|_{\mathcal{M}_i}$ that is transversal to coadjoint orbits in $\mathfrak{g}|^*_{\mathcal{M}_i}$. (This is possible because at h_i the form B_{h_i} has the smallest possible kernel - and thus the orbit that passes through h_i has the maximum possible dimension. Consequently in a small neighborhood of h_i the dimension of tangent spaces to the orbits does not vary and the tangent spaces themselves change smoothly, insuring the existence of $\mathfrak{H}_i$.)
3. Let $\mathfrak{H} = \mathfrak{H}_1 + ... + \mathfrak{H}_k$, where k is the number of components of the meander of $\mathfrak{g}$. The sum in taken in the sense of matrix representations of elements of $\mathfrak{H}_i$.

We will prove that the stabilizer of an element $H \in \mathfrak{H}$ is a direct sum of stabilizers of images H_i of element H under the projections on dual spaces of $\mathfrak{g}|_{\mathcal{M}_i}$.

The stabilizer of the element H is nothing else but the kernel of the form $\Omega|_{\mathcal{F}}$ evaluated in H, i.e. B_H. An element X belongs to the kernel of B_H if for every element $Y \in \mathfrak{g}$ we have

$$\langle H, [X, Y]\rangle = 0$$

Considering matrix representations of H, X and Y, we can restate this as following:

$$\langle H, [X, Y]\rangle = \operatorname{tr}(H[X, Y]) = \operatorname{tr}([H, X], Y)$$

That is, $[H, X]$ should be orthogonal to $\mathfrak{g}$.

Now if $H \in \mathfrak{g}^*$ is the sum of components H_i corresponding to different components of the meander graph, the image of H under the projection on $\mathfrak{g}|^*_{\mathcal{M}_i}$ would be exactly H_i (in view of the matrix representation of $\mathfrak{g}|_{\mathcal{M}_i}$).

Since for seaweed algebras of which the meander has one component the theorem is true, we deduce that the stabilizer of H_i has the needed dimension (1 or 2 depending on whether the component is a marked point, segment or circle). This stabilizer consists of matrices X_i such that $[H_i, X_i]$ is orthogonal to $\mathfrak{g}|_{\mathcal{M}_i}$. But, since $\mathfrak{g}|_{\mathcal{M}_i}$ is obtained by selecting elements with both indices in the set $\mathcal{M}_i$, $[H_i, X_i]$ is orthogonal to the whole algebra $\mathfrak{g}$. Now the observation that X_i commutes with H_j with $j \neq i$ implies that $X_i \in \operatorname{stab}_H$.

Thus we have proved that for elements of $\mathfrak{H}$ the stabilizer has at least dimension $\dim \mathfrak{g} - \mathcal{J}$. However, we know from the first part of the proof that for the point $h = h_1 + .. + h_k$ the dimension of the stabilizer is at most $\dim \mathfrak{g} - \mathcal{J}$. Thus (possibly reducing $\mathfrak{H}$ by intersection with open neighborhood of h), the dimension of the stabilizer for points of (reduced) $\mathfrak{H}$ is exactly $\dim \mathfrak{g} - \mathcal{J}$.

[1] It was pointed out to us by E.B.Vinberg that one can simplify the proof by choosing $\mathfrak{H}_i = \mathfrak{g}|^*_{\mathcal{M}_i}$.

To complete the proof, it is sufficient to show that $\mathfrak{H}$ is transversal to orbits of generic elements of $\mathfrak{H}$.

We will make use of the following lemma:

Lemma 8.1. *Let $\mathfrak{H}$ be a submanifold in the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. Let $H \in \mathfrak{H}$. Let ad^* denote the coadjoint representation of $\mathfrak{g}$. Then $T_H\mathfrak{H} + \mathrm{ad}^*[\mathfrak{g}](H) = \mathfrak{g}^*$ if and only if the image of $T_H\mathfrak{H}$ under the map $\mathfrak{g}^* \to \mathrm{stab}^*_H$ is stab^*_H*

Proof. Indeed, by definition, stab_H is the largest subspace of $\mathfrak{g}$ such that the projection $\mathfrak{g}^* \to \mathrm{stab}^*_H$ maps $\mathrm{ad}^*[\mathfrak{g}](H)$ to 0. ■

Now consider projections $p_i : \mathfrak{g}^* \to \mathfrak{g}|^*_{\mathcal{M}_i}$. Note that the image of $\mathfrak{H}$ under the projection p_i produces $\mathfrak{H}$ corresponding to the algebra $\mathfrak{g}|_{\mathcal{M}_i}$. These projections commute between themselves (again we consider the algebras $\mathfrak{g}|_{\mathcal{M}_i}$ embedded into $\mathrm{gl}(n)$). Since we know that the theorem is true in the case of algebras with connected meander graph, this implies that the image of $\mathfrak{H}$ under the projection onto the direct sum S of stabilizers of H_i in $\mathfrak{g}|^*_{\mathcal{M}_i}$ is S. But we know that the stabilizer of H in $\mathfrak{g}$ is equal to this direct sum. This implies that $T_H\mathfrak{H} + \mathrm{ad}^*[\mathfrak{g}](H) = \mathfrak{g}^*$. And because of the fact that $\dim\mathfrak{H} + \mathrm{rank}\ \Omega|_{\mathcal{F}} = \dim\mathfrak{g}$ for generic H, we have that $\mathfrak{H}$ is transversal to the coadjoint orbit of H.

Thus for an open subset of $\mathfrak{g}^*$ all orbits passing through points in this subset have dimension exactly equal to $\dim\mathfrak{g} - \mathcal{J}$. This is equivalent to the statement that $\mathrm{rank}\ \Omega|_{\mathcal{F}} = \dim\mathfrak{g} - \mathcal{J}$ for H in this subset. However $\Omega|_{\mathcal{F}}$ has polynomial coefficients. This implies that $\mathrm{rank}\ \Omega|_{\mathcal{F}} \le \dim\mathfrak{g} - \mathcal{J}$ for all $F \in \mathfrak{g}^*$ - which is precisely the statement of the theorem.

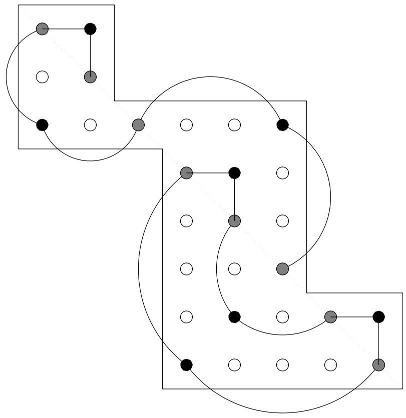
9. Proof, part 4: fields of arbitrary characteristic

Now that we know that the theorem holds over $\mathbb{C}$, let us examine what happens when we consider a different field. Let us choose a basis in $\mathfrak{g}$ of matrix units. Then Ω is composed of polynomials with integral coefficients. Wedge products of Ω thus also have integral coefficients. This means that if the characteristic of our field k is sufficiently large (a rough estimate is $(2\dim\mathfrak{g})^{\frac{\dim\mathfrak{g}-\mathrm{ind}\,\mathfrak{g}}{2}}$) or 0, the number r such that $\wedge^r\Omega$ is non-zero is independent of k.

10. Examples and discussion

Theorem 5.1 makes it possible to compute the index of a seaweed algebra in time linear in the size of the matrix. This is much easier than trying to compute the rank of Ω which is at least n^4. Here are two examples of application of the theorem 5.1:

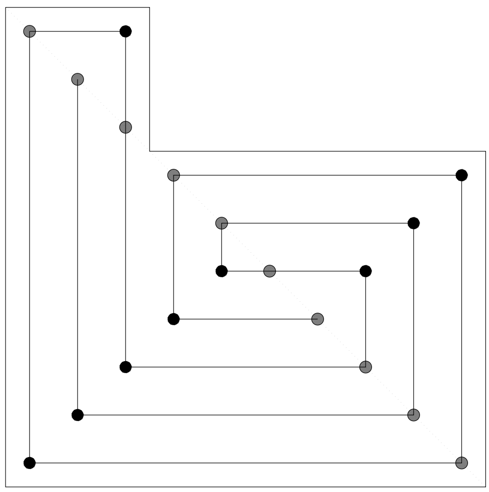
Example : $(2,4,2):(3,5)$



$$\begin{aligned}&[(12)]\,[(36)(45)]\,[(78)]\cdot[(13)]\,[(48)(57)]\\&=(1632)(47)(58)\qquad\text{(right multiplication)}\end{aligned}$$

The index is 3.

Example: $(3,7):(10)$

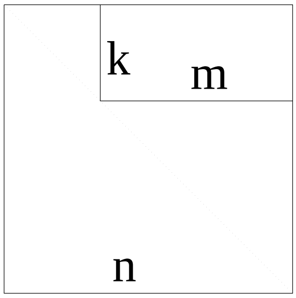


$$\begin{aligned}&[(1,3)]\,[(4,10)(5,9)(6,8)]\cdot\\&\quad[(1,10)(2,9)(3,8)(4,7)(5,6)]=\\&\qquad=(1,4,7,10,3,6,9,2,5,8)\end{aligned}$$

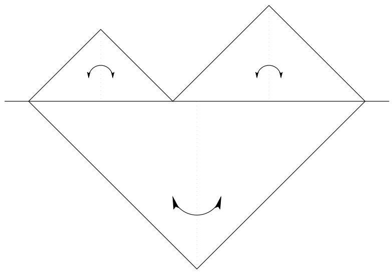
The index is 1.

We do not know any formula for the index in elementary functions. In the special case of a maximal parabolic subalgebra of $\mathrm{gl}(n)$, it is possible to come up with a simple expression. (To our knowledge, this problem was first solved by A.G.Elashvili [2].)

Maximal parabolic: $(k, m) : (n)$



Exercise 10.1. Prove that index of the maximal parabolic algebra $(k, m) : (n)$ is equal to the greatest common divisor of k and m using both the direct method and by computing the number of cycles in the associated permutation:



As was pointed to us by A. G. Elashvili, the proof of the Main theorem implies that in the dual space of any seaweed algebra there exists a Zariski open set of functionals $f \in \mathfrak{g}^*$ such that the stabilizer subgroups are conjugate to each other.

11. Acknowledgments

We express our gratitude to A. G. Elashvili for many interesting comments. Also, we would like to thank A. Borodine and J. Stasheff for suggestions on improving the presentation of material in this paper. Finally, the Communicating Editor's contributions are greatly appreciated.

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Received April 5, 1999
and in final form November 2, 1999