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Canonical form transformation of second order differential equations with Lie symmetries

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Abstract. The most difficult part of solving an ordinary differential equation (ode) by Lie's symmetry theory consists of transforming it into a canonical form corresponding to its symmetry type. In this article, for all possible symmetry types of a quasilinear second order ode, theorems are obtained that reduce the transformation into canonical form to solving linear partial differential equations (pde's) or certain Riccati equations. They allow it to determine *algorithmically* the finite transformation functions to canonical form that are Liouvillian over the base field of the given ode. The knowledge of the infinitesimal symmetry generators is *not* required. Fundamental new concepts that are applied are the Janet base of a system of linear pde's and its decomposition into completely reducible components, i. e. the analogue to Loewy's decomposition of linear ode's.

1. Introduction

The most powerful methods for obtaining closed form solutions of nonlinear ordinary differential equations (ode's) are based on Lie's symmetry theory. Yet for more than fifty years after his death it has virtually never been applied for solving practical problems, only during the last decade some activity in this area has emerged. The collection of solved equations by Kamke [1] for example does not even mention his name, although almost all solutions given there are the consequence of a symmetry. This is essentially due to two reasons. On the one hand, for any nontrivial example the amount of calculations necessary for applying Lie's theory makes it impossible to be performed by pencil and paper. Secondly the theory as described by Lie does not allow it to design solution algorithms in a straightforward manner because various parts of it are not constructive.

After Lie had recognized that the symmetry of an ode is the fundamental new concept for finding its solutions in closed form, he has described essentially two versions of a solution procedure based on it. Originally he applied the symmetry

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group of an ode only for recognizing a certain canonical form of the given equation that may be integrated more easily, and it is indicated how the transformation to it may be achieved. Lie has outlined this proceeding already in 1883 [8] but never came back to it. In a second version of his integration theory the infinitesimal generators of the Lie algebra corresponding to the symmetry group have to be determined explicitly and a canonical form is constructed from it. This proceeding is described in Lie's book on the subject [2], see also the books by Olver [3] and Bluman and Kumei [4].

In this article the original approach of Lie is completed such that it may be applied for solving ode's. Based on the third part of Lie's series of articles [8], a complete and optimal answer for obtaining the transformation functions to canonical form is obtained *without* solving the determining system for the symmetries. For each of the eight possible symmetry types of a quasilinear second order ode a theorem is derived that describes the simplest possible system of equations the solutions of which are the transformation functions to the desired canonical form. These systems are *not* obtained by simplification according to some heuristics but by transforming them *algorithmically* into a Janet base. For any of these systems there are algorithms available allowing it to determine its solutions in well defined function fields, e. g. solutions that are Liouvillian over the base field. In this way Lie's approach may be formulated much more precisely. Thi! ! s is the subject of Section 3. I

n the subsequent Section 2 a few results on the symmetries of a second order ode are given without proof in order to make this article sufficiently self-contained. In Section 4 some extensions of the results presented in this article are discussed, in particular its relevance for designing solution algorithms based on Lie symmetries. A good survey on the computer algebra software that is available in this field may be found in Hereman's review [5]. For questions concerning complete reducibility and the unique decomposition into completely reducible factors the reader is referred to Loewy's article [7].

2. Symmetries of Second Order ODE's

A group of point transformations of the $x-y-$plane with r parameters is determined by equations of the form

$$\bar{x} = f(x, y, a_1, \ldots, a_r), \qquad \bar{y} = g(x, y, a_1, \ldots, a_r)$$

with suitable constraints for the functions f and g. If not stated otherwise, it is assumed that the parameter values $a_k = 0$ for $k = 1, \ldots, r$ correspond to the identity transformation $\bar{x} = x$, $\bar{y} = y$. They define uniquely a set of r infinitesimal generators

$$U_i = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y \qquad (1 \leq i \leq r)$$

by

$$\xi_i(x, y) = \frac{\partial f(x, y, a_1, \ldots, a_r)}{\partial a_i}|_{a_1=\ldots=a_r=0},$$

$$\eta_i(x, y) = \frac{\partial g(x, y, a_1, \ldots, a_r)}{\partial a_i}|_{a_1=\ldots=a_r=0}.$$

The notation $\partial_x = \partial/\partial x$ etc. is applied throughout this article. The second prolongation $U^{(2)}$ of U is by definition

$$U^{(2)} = \xi\partial_x + \eta\partial_y + \zeta^{(1)}\partial_{y'} + \zeta^{(2)}\partial_{y''}$$

with

$$\zeta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y {y'}^2,$$

$$\zeta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy}){y'}^2 - \xi_{yy}{y'}^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y''.$$

$U^{(2)}$ contains the information how derivatives up to order two are transformed under the action of the group.

Let an ode of order n be given as

$$\omega(x, y, y', \ldots, y^{(n)}) = 0. \tag{1}$$

If not stated explicitly otherwise it will be assumed that ω is polynomial in the derivatives with coefficients in some *base field* which is usually the field of rational functions in the independent and the dependent variable, i. e. $\omega \in \mathbf{Q}(x, y)[y', \ldots, y^{(n)}]$. The *Lie symmetries* of an equation (1) are those point transformations $x = \phi(u, v)$ and $y = \psi(u, v)$ with $v \equiv v(u)$ leaving the equation invariant. They may be determined as follows. The second prolongation of a generator U with a priori unspecified coefficients $\xi(x, y)$ and $\eta(x, y)$ is applied to ω. The resulting expression $U^{(2)}\omega$ must vanish $mod\,\omega$. This leads to a set of linear and homogeneous partial differential equations for the functions ξ and η, the so called *determining system* of the respective ode. Its representation in terms of a *Janet base* is particularly important [9], [10]. Its general solution depends on a finite set of constants, each of which corresponds to a one-parameter group of the equation $\omega = 0$. The totality of generators forms a Lie algebra corresponding to its symmetry group. A *symmetry type* comprises all groups that are pairwaise equivalent w.r.t. to point transformations.

The first fundamental result of Lie to be applied is a complete classification of all possible symmetry types of a second order quasilinear ode. He showed that for any such equation with a non-trivial symmetry there are either 1, 2, 3 or 8 generators. More precisely, there are eight symmetry types one of which contains a parameter, they are completely described in the subsequent Theorem. The enumeration of groups $\mathbf{g}_k$ is essentially the same as in Lie [8], part I.

Theorem 2.1. (Lie 1883) *Any symmetry generator of a second order quasilinear ode is similar to one in canonical variables u and v as given in the following listing. In addition the corresponding Janet base is given where $\alpha(u, v)$ and $\beta(u, v)$ are the coefficients of ∂_u and ∂_v respectively.*

One-parameter group

$\mathcal{S}^2_1 \equiv \mathbf{g}_{27}$: $\{\partial_v\}$. *Janet base* $\{\alpha, \beta_u, \beta_v\}$.

Two-parameter groups

$\mathcal{S}^2_{2,1} \equiv \mathbf{g}_{26}: \ \{\partial_u, \partial_v\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_u, \beta_v\}$.

$\mathcal{S}^2_{2,2} \equiv \mathbf{g}_{25}: \ \{\partial_v, u\partial_u + v\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u, \beta_v - \frac{1}{v}\beta, \alpha_{uu}\}$.

Three-parameter groups

$\mathcal{S}^2_{3,1} \equiv \mathbf{g}_{10}: \ \{\partial_u + \partial_v, u\partial_u + v\partial_v, u^2\partial_u + v^2\partial_v\}$. *Janet base*
$\{\alpha_v, \beta_u, \beta_v + \alpha_u + \frac{2}{u-v}(\beta - \alpha), \alpha_{uu} - \frac{2}{u-v}\alpha_u - \frac{2}{(u-v)^2}(\beta - \alpha)\}$.

$\mathcal{S}^2_{3,2} \equiv \mathbf{g}_{13}: \ \{\partial_u, 2u\partial_u + v\partial_v, u^2\partial_u + uv\partial_v\}$. *Janet base* $\{\alpha_u - \frac{2}{v}\beta, \alpha_v, \beta_v - \frac{1}{v}\beta, \beta_{uu}\}$.

$\mathcal{S}^2_{3,3} \equiv \mathbf{g}_{7}: \ \{\partial_u, \partial_v, u\partial_u + \gamma v\partial_v\}$, $\gamma \neq 1$. *Janet base* $\{\alpha_v, \beta_u, \beta_v - c\alpha_u, \alpha_{uu}\}$.

$\mathcal{S}^2_{3,4} \equiv \mathbf{g}_{20}$ *with* $r = 1$: $\{\partial_u, \partial_v, u\partial_u + (u+v)\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u - \alpha_u, \beta_v - \alpha_u, \alpha_{uu}\}$.

Eight-parameter group:

$\mathcal{S}^2_{8} \equiv \mathbf{g}_{3}: \ \{\partial_u, \partial_v, u\partial_v, v\partial_v, u\partial_u, v\partial_u, u^2\partial_u + uv\partial_v, uv\partial_u + v^2\partial_v\}$.
Janet base $\{\alpha_{vv}, \beta_{uu}, \beta_{uv}, \beta_{vv}, \alpha_{uuu}, \beta_{uuv}\}$.

This listing shows in particular that there does not exist any second order ode allowing a group of point symmetries with 4, 5, 6 or 7 parameters.

In general an ode is not given in canonical variables but in *actual* variables x and y. Let them be related to each other by $u = \sigma(x, y)$ and $v = \rho(x, y)$. In order to apply Lie's theory the symmetry type has to be identified for the equation in actual variables. This problem has been solved in terms of a set of criteria for the coefficients of the Janet bases for the respective determining systems [9]. In order to fix the notation, the types of Janet bases that actually do occur in this theorem are given below.

$$\mathcal{J}^{(2,2)}_{1,1}: \xi = 0, \ \eta_x + a\eta = 0, \ \eta_y + b\eta = 0.$$

$$\mathcal{J}^{(2,2)}_{1,2}: \eta + a\xi = 0, \ \xi_x + b\xi = 0, \ \xi_y + c\xi = 0.$$

$$\mathcal{J}^{(2,2)}_{2,3}: \begin{array}{c} \xi_x + a\eta + b\xi = 0, \ \xi_y + c\eta + d\xi = 0, \\ \eta_x + p\eta + q\xi = 0, \ \eta_y + r\eta + s\xi = 0. \end{array}$$

$$\mathcal{J}^{(2,2)}_{3,4}: \begin{array}{c} \xi_x + a\eta + b\xi = 0, \ \xi_y + c\eta + d\xi = 0, \\ \eta_y + p\xi_x + q\eta + r\xi = 0, \ \eta_{xx} + u\eta_x + v\eta + w\xi = 0. \end{array}$$

$$\mathcal{J}^{(2,2)}_{3,6}: \begin{array}{c} \xi_y + a\xi_x + b\eta + c\xi = 0, \ \eta_x + d\xi_x + e\eta + f\xi = 0, \\ \eta_y + p\xi_x + q\eta + r\xi = 0, \ \xi_{xx} + u\xi_x + v\eta + w\xi = 0. \end{array}$$

$$\mathcal{J}^{(2,2)}_{3,7}: \begin{array}{c} \xi_x + a\eta + b\xi = 0, \ \eta_x + d\xi_y + e\eta + f\xi = 0, \\ \eta_y + p\xi_y + q\eta + r\xi = 0, \ \xi_{yy} + u\xi_y + v\eta + w\xi = 0. \end{array}$$

A Janet base type is determined by its leading terms. The coefficients $a, b, \ldots, v, w$ depending on x and y are in the base field of the ode from which they originate.

Theorem 2.2. (Schwarz 1996) *The following criteria provide a decision procedure for the type of symmetry group of a second order ode if its Janet base for its determining system in a total-order lexicographic term ordering with* $\eta > \xi$, $y > x$ *is given.*

One-parameter group

$\mathcal{S}_1^2$: *Janet base of type* $\mathcal{J}_{1,1}^{(2,2)}$ *or* $\mathcal{J}_{1,2}^{(2,2)}$.

Two-parameter groups

$\mathcal{S}_{2,1}^2$: *Janet base of type* $\mathcal{J}_{2,3}^{(2,2)}$ *with* $a = d$ *and* $p = s$.

$\mathcal{S}_{2,2}^2$: *Janet base of type* $\mathcal{J}_{2,3}^{(2,2)}$ *with* $a \neq d$ *or* $p \neq s$.

Three-parameter groups

$\mathcal{S}_{3,1}^2$: *Janet base of type* $\mathcal{J}_{3,6}^{(2,2)}$ *with* $p = 1$ *and* $ad + 1 \neq 0$, *or of type* $\mathcal{J}_{3,7}^{(2,2)}$ *with* $p = 0$, $d \neq 0$.

$\mathcal{S}_{3,2}^2$: *Janet base of type* $\mathcal{J}_{3,4}^{(2,2)}$, *of type* $\mathcal{J}_{3,6}^{(2,2)}$ *with* $p = 1$ *and* $ad + 1 = 0$, *or of type* $\mathcal{J}_{3,7}^{(2,2)}$ *with* $p = d = 0$.

$\mathcal{S}_{3,3}^2$: *Janet base of type* $\mathcal{J}_{3,6}^{(2,2)}$ *with* $p \neq 1$ *and* $4ad + (p+1)^2 \neq 0$, *or of type* $\mathcal{J}_{3,7}^{(2,2)}$ *with* $p \neq 0$ *and* $p^2 \neq 4d$.

$\mathcal{S}_{3,4}^2$: *Janet base of type* $\mathcal{J}_{3,6}^{(2,2)}$ *with* $p \neq 1$ *and* $4ad + (p+1)^2 = 0$, *or of type* $\mathcal{J}_{3,7}^{(2,2)}$ *with* $p \neq 0$ *and* $p^2 = 4d$.

Eight-parameter group

$\mathcal{S}_8^2$: *Janet base of type* $\{\xi_{yy}, \eta_{xx}, \eta_{xy}, \eta_{yy}, \xi_{xxx}, \xi_{xxy}\}$.

To each symmetry type there corresponds a canonical form of a quasilinear second order ode allowing this symmetry. In general this canonical form is not unique. By definition, the totality of transformations leaving its structure unchanged is called its *structure invariance group.* For quasilinear equations of second order the canonical form and its structure invariance groups are described in the subsequent theorem. The structure invariance groups given there have an important meaning for the respective differential equation. In the first place they describe the degree of arbitrariness for the transformation functions to canonical form. This is an extremely important information on the solutions of the systems of pde's describing these transformations as it will be seen later on. Secondly they are a necessary prerequirement in order to obtain definite statements on the existence of exact solutions and for designing solution algorithms.

Theorem 2.3. *In canonical variables u and $v \equiv v(u)$ the second order quasi-linear equations with non-trivial symmetries have the following structure invariance groups of point transformations $u = \sigma(x,y)$ and $v = \rho(x,y)$. The unspecified functions $r(v')$ and $r(u,v')$ are assumed to be rational in its arguments.*
One-parameter group

$\mathcal{S}_1^2$: $v'' + r(u,v') = 0$ *allows the pseudogroup* $u = f(x)$, $v = g(x) + cy$, f *and* g *undetermined functions of* x, c *constant.*

Two-parameter groups

$\mathcal{S}_{2,1}^2$: $v'' + r(v') = 0$ *allows* $u = a_1x + a_2y + a_3, v = a_4x + a_5y + a_6$.

$\mathcal{S}_{2,2}^2$: $v''u + r(v') = 0$ *allows* $u = a_1x$, $v = a_2x + a_3y + a_4$.

Three-parameter groups

$\mathcal{S}_{3,1}^2$: $v''(u - v) + 2v'(v' + a\sqrt{v'} + 1) = 0$ *allows*

$$u = \frac{a_1(x + a_2)}{1 - a_1a_3(x + a_2)}, \quad v = \frac{a_1(y + a_2)}{1 - a_1a_3(y + a_2)}.$$

$\mathcal{S}_{3,2}^2$: $v''v^3 + a = 0$, $a \neq 0$ *constant, allows*

$$u = \frac{a_1(x + a_3)}{1 + a_4(x + a_3)}, \quad v = \frac{a_2y}{1 + a_4(x + a_3)}.$$

$\mathcal{S}_{3,3}^2$: $v'' + av'^{1-\frac{1}{\gamma - 1}} = 0$, a *constant*, $\gamma \neq 0, 1$, *allows* $u = a_1x + a_2$, $v = a_3y + a_4$.

$\mathcal{S}_{3,4}^2$: $v'' - ae^{-v'} = 0$, a *constant, allows* $u = a_1x + a_2$, $v = a_3x + a_1y + a_4$.

Eight-parameter group

$\mathcal{S}_8^2$: *Projective group of the plane.*

Proof. A general point transformation $u = \sigma(x,y)$ and $v = \rho(x,y)$ changes the first and the second derivative according to

$$v' \equiv \frac{dv}{du} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'}, \tag{2}$$

$$\begin{aligned} v'' = \frac{dv'}{du} = \frac{1}{(\sigma_x + \sigma_y y')^3}\{&(\sigma_x\rho_y - \sigma_y\rho_x)y'' + (\sigma_y\rho_{yy} - \sigma_{yy}\rho_y)y'^3 \\ &+[\sigma_x\rho_{yy} - \sigma_{yy}\rho_x + 2(\sigma_y\rho_{xy} - \sigma_{xy}\rho_y)]y'^2 \\ &+[\sigma_y\rho_{xx} - \sigma_{xx}\rho_y + 2(\sigma_x\rho_{xy} - \sigma_{xy}\rho_x)]y' + \sigma_x\rho_{xx} - \sigma_{xx}\rho_x\}. \end{aligned} \tag{3}$$

For $\mathcal{S}_1^2$, in order to avoid any dependence on y to be generated *via* $r(u, v')$, $\sigma_y = \rho_{xy} = \rho_{yy} = 0$ is required. This yields the above structure with

$$v' = \frac{1}{f'}(g' + cy'), \; v'' = \frac{1}{f'^3}(cf'y'' - cf''y' + f'g'' - f''g').$$

The expression for v'' shows that no further constraints are necessary. For the two parameter groups, any dependence on x and y in the transformed function r is avoided if

$$x = a_1 u + a_2 v + a_3, \quad y = a_4 u + a_5 v + a_6$$

where $a_1, \ldots, a_6$ are constant. Then the second derivative is transformed according to

$$y'' = \frac{a_1 a_5 - a_2 a_4}{(a_1 + a_2 v')^3} v''.$$

For $\mathcal{S}_{2,1}^2$ this assures the desired structure. For $\mathcal{S}_{2,2}^2$ the second derivative must be proportional to the independent variable, this requires in addition $a_2 = a_3 = 0$.

For $\mathcal{S}_{3,1}^2$ the transformed first derivative must be proportional to y', this requires $\sigma_y = \rho_x = 0$. The condition for the invariance of the second order invariant leads to a fairly complicated system of pde's for σ and ρ. A Janet base for the complete system is

$$\sigma_y = 0, \; \sigma_{xx} + \frac{2\sigma_x^2}{\rho - \sigma} = \frac{2\sigma_x}{x - y} = 0, \; \rho_x = 0, \; \rho_y \sigma_x - (\frac{\rho - \sigma}{x - y})^2 = 0.$$

Its general solution containing three constants are the transformation functions given above.

For $\mathcal{S}_{3,2}^2$ the transformed second derivative must be proportional to y'' and must be independent of x and y'. This is assured if

$$\sigma_y = 0, \; \rho_{yy} = 0, \; 2\sigma_x \rho_{xy} - \sigma_{xx}\rho_y = 0, \; \sigma_x \rho_{xx} - \sigma_{xx}\rho_x = 0$$

are valid. Then there holds $v'' = \rho_y/\sigma_x^2 \cdot y''$. In the transformed equation the coefficient of the second derivative must be independent of x and proportional to y^3. This requires in addition

$$(\frac{\rho_y \rho^3}{\sigma_x})_x = 0, \; (\frac{\rho_y \rho^3}{\sigma_x})_y \frac{\sigma_x^2}{\rho_y \rho^3} - \frac{3}{y} = 0.$$

The combined constraints may be transformed into the Janet base

$$\sigma_y = 0, \; y\rho_y - \rho = 0, \; \sigma_{xx}\rho - 2\rho_x \sigma_x = 0, \; \rho_{xx}\rho - 2\rho_x^2 = 0$$

with the general solution

$$\sigma = \frac{C_1 x + C_2}{C_3(C_3 x + C_4)}, \; \rho = \frac{y}{C_3 x + C_4}.$$

By a suitable change of the integration constants C_k the above expression for group is obtained.

For the group $\mathcal{S}_{3,3}^2$ the transformed first derivative must be proportional to y', this requires $\sigma_y = \rho_x = 0$. The transformed second derivative cannot contain a term proportional to a power of y', this requires $\sigma_{xx} = \rho_{yy} = 0$. The general solution of these equations are the transformation functions given in the Theorem.

For the group $\mathcal{S}^2_{3,4}$ the transformed first derivative must have the form $-y' + constant$, this requires

$$\sigma_y = 0,\ \rho_y - \sigma_x = 0,\ \sigma_x\rho_{xy} - \sigma_{xy}\rho_x = 0,\ \sigma_x\rho_{xx} - \sigma_{xx}\rho_x = 0$$

or the equivalent Janet base $\sigma_y = 0$, $\sigma_{xx} = 0$, $\rho_y - \sigma_x = 0$, $\rho_{xx} = 0$ with the solution given above.

Finally the structure invariance group of $v'' = 0$ is obviously identical to its symmetry group $\mathcal{S}^2_8$, i. e. the eight parameter projective group of the plane. ■

The structure invariance groups of the preceding Theorem have been identified by Lie [8], part III, in a different context.

3. Transformation to Canonical Form

This section is organized by the size of the symmetry groups, i. e. one-, two-, and three-parameter groups and the eight-parameter projective group are considered successively. The construction of the system of pde's for the transformation functions is based on the following general principle. On the one hand, the coefficients of the Janet base in actual variables may be expressed in terms of the transformation functions $\sigma(x,y)$ and $\rho(x,y)$. This representation is explicitly given in [9], page 183-184. On the other hand they are rationally known in terms of the coefficients of the given ode. Equating the two expressions yields a system of nonliner pde's for σ and ρ. If it is transformed into a Janet base in a proper term order, the equations are obtained in the desired form. Lie [8], part III, page 377-388 (see also Engels' comments on page 715-722 of the *Gesammelte Abhandlungen*, vol. 5) gives a thorough discussion of the structure of these equations base d on its group properties following from its invariance under the respective structure invariance group. For algorithmic purposes, i. e. if the goal is a set of equations for which solution algorithms for predetermined function fields are available, the method that is applied below seems to be more appropriate. In any case, Lie's and Engels' discussions of the subject are of fundamental importance for understanding the structure of these equations. In a second step they have to be solved. In order to obtain a well defined problem, the function field where the solutions are searched for has to be specified, and a solution algorithm for obtaining them must be available. In many instances there occur systems of first order partial differential equations that are quadratic in the unknown functions. For obvious reasons they have been baptized *partial Riccati like systems* in a recent publication by Z. Li and the author [6] where they are discussed in detail and solution algorithms are given. Th is proceeding will be described now for the various symmetry types one after another. As usual the notation $\Delta = \sigma_x\rho_y - \sigma_y\rho_x$ is applied.

One-Parameter Symmetry Group. This is the simplest type of invariance that may occur for any ode. The freedom involved in the canonical form transformation, i. e. two unspecified functions of the independent variable and a constant, correspond to the respective quantities generating the structure invariance group.

Theorem 3.1. *If a second order ode has the symmetry group $\mathcal{S}_1^2$, two types of Janet bases may occur.*

a.) If the Janet base is of type $\mathcal{J}_{1,1}^{(2,2)}$ the transformation functions σ and ρ are given by

$$\sigma = f(x), \quad \rho = \int e^{F(x,y)} dy + g(x). \tag{4}$$

F is determined by the path integral

$$F(x,y) = \int_{x_0}^{x} a(\bar{x}, y_0) d\bar{x} + \int_{y_0}^{y} b(x, \bar{y}) d\bar{y}$$

and $f(x)$, $g(x)$ are undetermined functions of x.

b.) If the Janet base is of type $\mathcal{J}_{1,2}^{(2,2)}$ two alternatives may occur. If $a \neq 0$, σ and ρ are determined by

$$\frac{dy}{dx} + a = 0, \text{ solution } \phi(x,y) = C, \ \sigma \equiv \sigma(\phi), \ \rho = -\int e^{G(x,y)} dx + \psi(y) \tag{5}$$

where $\psi(y)$ is an unspecified function of y and G is given by the path integral

$$G(x,y) = \int_{x_0}^{x} b(\bar{x}, y_0) d\bar{x} + \int_{y_0}^{y} c(x, \bar{y}) d\bar{y}.$$

If $a = 0$, the transformation functions σ and ρ are given by

$$\sigma = f(y), \quad \rho = \int e^{G(x,y)} dx + g(y) \tag{6}$$

G is defined as above, $f(y)$ and $g(y)$ are undetermined functions of y.

Proof. In case *a.)* the relations $(\log \rho_y)_x = a$ and $(\log \rho_y)_y = b$ follow from the Janet base for this group, they have the solution (4) given above. The first alternative (5) for case *b.)* is obtained as solution of

$$\frac{\sigma_x}{\sigma_y} = a, \quad (\log \frac{\Delta}{\sigma_y})_x = b \ \ (\log \frac{\Delta}{\sigma_y})_y = c$$

which follow again from the Janet base of the determining system. If $a = 0$ a few obvious simplifications lead to the second alternative (6). ■

Two-Parameter Symmetry Groups. The structure invariance groups corresponding to the symmetry types $\mathcal{S}_{2,1}^2$ and $\mathcal{S}_{2,2}^2$ have six or four parameters respectively. These numbers correspond to the degrees of freedom in the canonical form transformations as it is shown in the following theorem.

Theorem 3.2. *If a second order ode has a Janet base of type $\mathcal{J}_{2,3}^{(2,2)}$, its symmetry group has two parameters. There are two cases to be distinguished.*

i.) If $a = d$ and $p = s$, the symmetry group is $\mathcal{S}_{2,1}^2$. The transformation function σ is determined by

$$\sigma_{xx} - q\sigma_y - b\sigma_x = 0,\ \sigma_{xy} - p\sigma_y - a\sigma_x = 0,\ \sigma_{yy} - r\sigma_y - c\sigma_x = 0$$

and an identical set of equations for ρ. The general solution depends on six constants.

ii.) If $a \neq d$ or $p \neq s$, the symmetry group is $\mathcal{S}_{2,2}^2$. The transformation functions σ and ρ are determined by the system

$$\begin{gathered}\sigma_x + (s-p)\sigma = 0,\ \sigma_y - (d-a)\sigma = 0,\\ \rho_{xx} - q\rho_y + (s-b-p)\rho_x = 0,\ \rho_{xy} - p\rho_y - d\rho_x = 0,\\ \rho_{yy} + (a-d-r)\rho_y - c\rho_x = 0.\end{gathered}$$

The general solution depends on four constants.

Proof. For the group $\mathcal{S}_{2,1}^2$ the coefficients $a, b, \ldots, s$ in the Janet base may be expressed in terms of the transformation functions σ and ρ as

$$\begin{gathered}\sigma_y\rho_{xy} - \sigma_{xy}\rho_y = a\Delta,\ \sigma_y\rho_{xx} - \sigma_{xx}\rho_y = b\Delta,\\ \sigma_y\rho_{yy} - \sigma_{yy}\rho_y = c\Delta,\ \sigma_y\rho_{xy} - \sigma_{xy}\rho_y = d\Delta,\\ \sigma_x\rho_{xy} - \sigma_{xy}\rho_x = -p\Delta,\ \sigma_x\rho_{xx} - \sigma_{xx}\rho_x = -q\Delta,\\ \sigma_x\rho_{yy} - \sigma_{yy}\rho_x = -r\Delta,\ \sigma_x\rho_{xy} - \sigma_{xy}\rho_x = -s\Delta.\end{gathered} \tag{7}$$

As usual $\Delta = \sigma_x\rho_y - \sigma_y\rho_x$. This system may be transformed into a Janet base with total degree, then lexicographic term ordering with $\rho > \sigma > s > r > \ldots > b > a$, $y > x$ with the result

$$\begin{gathered}d - a = 0,\ \ s - p = 0,\\ b_y - a_x + qc - pa = 0,\ c_x - a_y + ra - pc + cb - a^2 = 0,\\ q_y - p_x + rq - qa - p^2 + pb = 0,\ r_x - p_y + qc - pa = 0,\\ \sigma_{xx} - q\sigma_y - b\sigma_x = 0,\ \sigma_{xy} - p\sigma_y - a\sigma_x = 0,\\ \sigma_{yy} - r\sigma_y - c\sigma_x = 0,\ \rho_{xx} - q\rho_y - b\rho_x = 0,\\ \rho_{xy} - p\rho_y - a\rho_x = 0,\ \rho_{yy} - r\rho_y - c\rho_x = 0.\end{gathered} \tag{8}$$

The lower equations not involving σ and ρ represent the integrability conditions for the Janet base coefficients. The upper half of this Janet base represents the two identical linear systems for σ and ρ respectively.

For the group $\mathcal{S}_{2,2}^2$ the system relating σ and ρ to the Janet base coefficients is

$$\begin{gathered}(\sigma_y\rho_{xy} - \sigma_{xy}\rho_y)\sigma + \sigma_y\Delta = -a\sigma\Delta,\\ (\sigma_y\rho_{xx} - \sigma_{xx}\rho_y)\sigma + \sigma_x\Delta = -b\sigma\Delta b,\\ \sigma_y\rho_{yy} - \sigma_{yy}\rho_y = -c\Delta,\ \ \sigma_y\rho_{xy} - \sigma_{xy}\rho_y = -d\Delta,\\ \sigma_x\rho_{xy} - \sigma_{xy}\rho_x = p\Delta,\ \ \sigma_x\rho_{xx} - \sigma_{xx}\rho_x = q\Delta,\\ (\sigma_x\rho_{yy} - \sigma_{yy}\rho_x)\sigma - \sigma_y\Delta = r\sigma\Delta,\\ (\sigma_x\rho_{xy} - \sigma_{xy}\rho_x)\sigma - \sigma_x\Delta = s\sigma\Delta.\end{gathered} \tag{9}$$

In the same term ordering as system (8) above the Janet base

$$
\begin{gathered}
b_y - a_x + qc - pd = 0,\ c_x - a_y + ra - pc - da + cb = 0,\\
d_x - a_x + sa - pd = 0,\ d_y - a_y + sc - rd + ra - pc = 0,\\
q_y - p_x - sp + rq - qa + pb = 0,\ r_x - p_y - sa + qc = 0,\\
s_x - p_x - sb + qd - qa + pb = 0,\ s_y - p_y - sa + pd = 0,\\
\sigma_x + (s-p)\sigma = 0,\ \sigma_y + (a-d)\sigma = 0\\
\rho_{xx} + (s-p-b)\rho_x = 0,\\
\rho_{xy} - p\rho_y - d\rho_x = 0,\\
\rho_{yy} + (a-r-d)\rho_y - c\rho_x = 0.
\end{gathered}
$$

is obtained. The three highest equations determine ρ, the two subsequent equations σ and the rest are the integrability conditions for the coefficients $a, b, \ldots, s$. This completes the proof.

■

Three-Parameter Symmetry Groups. The three parameters of the structure invariance group for the symmetry type $\mathcal{S}^2_{3,1}$ are obtained by combining the integration constant of a first order Riccati like system and a second order Janet base for σ or ρ respectively.

Theorem 3.3. *If a second order ode has the symmetry group $\mathcal{S}^2_{3,1}$ two types of Janet bases may occur.*

a.) If the Janet base is of type $\mathcal{J}^{(2,2)}_{3,6}$ three cases have to be distinguished. If $d \neq 0$, the transformation functions σ and ρ are determined by the system

$$
\begin{gathered}
R_x + R^2 - PR + Q = 0,\\
R_y + \tfrac{z+1}{d}R^2 - [\tfrac{z+1}{d}P + (\tfrac{z+1}{d})_x]R + \tfrac{z-1}{d}Q = 0,\\
\sigma_{xx} + (2R-P)\sigma_x = 0,\ \ \sigma_y - \tfrac{z+1}{d}\sigma_x = 0,\\
\rho - \tfrac{1}{R}\sigma_x - \sigma = 0.
\end{gathered}
\tag{10}
$$

If $d = 0$ and $\sigma_x \neq 0$, $\rho_x = 0$, the system for σ and ρ is

$$
\begin{gathered}
R_x + R^2 - rR = 0,\ R_y - \tfrac{a}{2}R^2 + (ar - c)R + P = 0,\\
\sigma_{xx} + (2R - r)\sigma_x = 0,\ \sigma_y + \tfrac{a}{2}\sigma_x = 0,\ \rho - \tfrac{1}{R}\sigma_x - \sigma = 0.
\end{gathered}
\tag{11}
$$

Finally if $d = 0$, $\sigma_x = 0$ and $\rho_x \neq 0$, σ and ρ are determined by

$$
\begin{gathered}
R_x + P = 0,\ R_y + R^2 - QR - \tfrac{a}{2}P = 0,\\
\sigma_{yy} + (2R - Q)\sigma_y = 0,\ \ \sigma_x = 0,\ \ \rho - \tfrac{1}{R}\sigma_y - \sigma = 0.
\end{gathered}
\tag{12}
$$

The functions $P(x,y)$ and $Q(x,y)$ are defined by

$$
\begin{gathered}
P(x,y) = \frac{1}{(ad+1)d}[(z+1)(ad+1)f - z(cd + r - e)d],\\
Q(x,y) = \frac{d}{4z^2}[z^2(c_x - aw - v) + c(cd + 2r - 3e) - a(r-e)^2 - acde]
\end{gathered}
\tag{13}
$$

and $z^2 = ad + 1$.

b.) If the Janet base is of type $\mathcal{J}_{3,7}^{(2,2)}$ the transformation function σ and ρ are determined by

$$\begin{aligned} &R_x + R^2 - (ac+b)R + B = 0,\\ &R_y - \tfrac{1}{c}R^2 - \frac{c_x - (ac+b)c}{c^2}R - \tfrac{1}{c}B = 0,\\ &S_x - S^2 + (ac-b)S - B = 0,\\ &S_y + \tfrac{1}{c}S^2 - \frac{c_x + (ac-b)c}{c^2}S + \tfrac{1}{c}B = 0. \end{aligned} \tag{14}$$

Here $c^2 + d = 0$ and

$$B = e^{2\int b dx + 2\int \frac{c_x + bc}{c^2} dy}. \tag{15}$$

The system for σ and ρ is

$$\begin{aligned} &\sigma_y - \tfrac{1}{c}\sigma_x = 0,\ \sigma_{xx} + (2R - ac - b)\sigma_x = 0,\\ &\rho_y + \tfrac{1}{c}\sigma_x = 0,\ \rho_{xx} - (2S - ac + b)\rho_x = 0. \end{aligned}$$

Proof. At first case *a.*) will be considered. If $d \neq 0$, the Janet base for σ and ρ in total degree ordering with $\rho > \sigma$, $y > x$ is

$$\begin{aligned} &\sigma_y - \tfrac{z+1}{d}\sigma_x = 0,\ \sigma_{xx}(\rho - \sigma) + 2\sigma_x^2 - P\sigma_x(\rho - \sigma) = 0,\\ &\rho_y + \tfrac{z-1}{d}\rho_x = 0,\ \rho_x\sigma_x - Q(\rho - \sigma)^2 = 0. \end{aligned} \tag{16}$$

The new function $R \equiv \sigma_x/(\rho - \sigma)$ is introduced with the result

$$\begin{aligned} &\sigma_x - R(\rho - \sigma) = 0,\ \ \sigma_y - \tfrac{z+1}{d}R(\rho - \sigma) = 0,\\ &\rho_x - \tfrac{Q}{P}(\rho - \sigma) = 0,\ \ \rho_y - \tfrac{z-1}{d}\tfrac{Q}{R}(\rho - \sigma) = 0. \end{aligned}$$

P, Q and z are defined as above. The integrability conditions of this system are the two first equations of 10. If a lexicographic term order with $\rho > \sigma > R$ is applied, the complete system (10) is obtained.

If $d = 0$, $\sigma_x \neq 0$ and $\rho_x = 0$, the Janet base for σ and ρ is

$$\begin{aligned} &\sigma_y + \tfrac{a}{2}\sigma_x = 0,\ \ \rho_x = 0,\\ &\rho_y\sigma_x - P(\rho - \sigma)^2 = 0, \sigma_{xx}(\rho - \sigma) + 2\sigma_x^2 - r\sigma_x(\rho - \sigma) = 0. \end{aligned}$$

Introducing again $R = \sigma_x/(\rho - \sigma)$ as a new function, the system

$$\begin{aligned} &\sigma_x - R(\rho - \sigma) = 0,\ \sigma_y + \tfrac{1}{2}aR(\rho - \sigma) = 0,\\ &\rho_x = 0,\ \rho_y R - P(\rho - \sigma) = 0 \end{aligned}$$

for σ and ρ is obtained. Its integrability conditions are the first two equations of (11), the remaining equations are obtained by applying the lexicographic term order $\rho > \sigma > R$ and $y > x$.

Finally if $d=0$, $\sigma_x=0$, $\rho_x\neq 0$, the Janet base for σ and ρ in total-degree, $\rho>\sigma$, $x>y$ term ordering is

$$\sigma_x=0,\ \ \sigma_{yy}(\rho-\sigma)+2\sigma_y^2-Q\sigma_y(\rho-\sigma)=0,$$
$$\rho_x\sigma_y-P(\rho-\sigma)^2=0,\ \ \rho_y\sigma_y+\tfrac{a}{2}P(\rho-\sigma)^2=0.$$

Now the new function $R=\sigma_y/(\rho-\sigma)$ is introduced. It yields the system

$$\sigma_{yy}+(2R-Q)\sigma_y=0,\ \sigma_x=0,$$
$$\rho_yR+\tfrac{a}{2}P(\rho-\sigma)=0,\ \rho_xR-P(\rho-\sigma)=0$$

for σ and ρ. Its integrability conditions are the first two equations of (12), the remaining equations are obtained by applying the lexicographic term order $\rho>\sigma>R$ and $x>y$.

In case $b.$) the type $\mathcal{J}_{3,7}^{(2,2)}$ Janet base for σ and ρ is

$$\begin{gathered}\sigma_y-\tfrac{1}{c}\sigma_x=0,\ \ \rho_y+\tfrac{1}{c}\rho_x=0,\\ \sigma_{xx}(\rho-\sigma)+2\sigma_x^2-(ac+b)\sigma_x(\rho-\sigma)=0,\\ \rho_{xx}(\rho-\sigma)-2\rho_x^2+(ac-b)\rho_x(\rho-\sigma)=0.\end{gathered}\tag{17}$$

Defining $R\equiv\sigma_x/(\rho-\sigma)$ and $S\equiv\rho_x/(\rho-\sigma)$ leads to the equations

$$\begin{gathered}R_x+R^2+RS-(ac+b)R=0,\\ R_y-\tfrac{1}{c}R^2-\tfrac{1}{c}RS-\frac{c_x-(ac+b)c}{c^2}R=0,\\ S_x-S^2-RS+(ac-b)S=0,\\ S_y+\tfrac{1}{c}S^2+\tfrac{1}{c}RS-\frac{c_x+(ac-b)c}{c^2}S=0.\end{gathered}$$

Combining the two equations determining the x-derivatives and the y-derivatives respectively leads to

$$(RS)_x-2bRS=0,\ \ (RS)_y-\frac{2(c_x+bc)}{c^2}RS=0$$

from which (15) follows. ■

The four parameters of the structure invariance group for the symmetry type $\mathcal{S}_{3,2}^2$ are obtained by combining the integration constant of a second order Riccati like system and a second order Janet base for σ or ρ respectively.

Theorem 3.4. *If a second order ode has the symmetry group $\mathcal{S}_{3,2}^2$, three types of Janet bases may occur.*

a.) If the Janet base is of type $\mathcal{J}_{3,4}^{(2,2)}$ the transformation functions σ and ρ are determined by

$$\begin{gathered}R_{xx}+R_x^2-bR_x-\tfrac{1}{4}(r_x+r^2-br-v)=0,\ R_y-\tfrac{1}{2}a=0,\\ \sigma_{xx}-(2R_x-b)\sigma_x=0,\ \sigma_y=0,\ \rho-e^{-R}=0.\end{gathered}\tag{18}$$

b.) If the Janet base is of type $\mathcal{J}^{(2,2)}_{3,7}$ *with* $d = p = 0$ *the transformation functions* σ *and* ρ *are determined by*

$$R_{yy} + R_y^2 - qR_y - \tfrac{1}{4}(a_y + a^2 - aq - w) = 0,\ R_x - \tfrac{1}{2}r = 0, \qquad (19)$$
$$\sigma_{yy} + (2R_y - q)\sigma_y = 0,\ \sigma_x = 0,\ \rho - e^{-R} = 0.$$

c.) If the Janet base is of type $\mathcal{J}^{(2,2)}_{3,6}$ *with* $p = 1$ *and* $ad + 1 = 0$ *the transformation functions* σ *and* ρ *are determined by*

$$R_{xx} + R_x^2 + afR_x - \tfrac{Q}{P} = 0,\ R_y + aR_x - \tfrac{1}{2}P = 0, \qquad (20)$$
$$\sigma_{xx} + (2R_x + af)\sigma_x = 0,\ \sigma_y + a\sigma_x = 0,\ \rho - e^{-R} = 0$$

where $P(x, y) = ar - ae - c \equiv -2\Delta/(\sigma_x\rho) \neq 0$ *and*

$$Q(x, y) = \tfrac{1}{4}[(dc + r - e)(c_x - v) + a(r - e)^3$$
$$-wP(x, y) + 2fP(x, y)^2 + c(er - cde - r^2)].$$

Proof. The proof will be given in detail for *case a.)* for Janet base type $\mathcal{J}^{(2,2)}_{3,4}$. If its coefficients are compared to the corresponding expressions in terms of the transformation functions (see also Theorem (2.2)), the following system of pde's is obtained.

$$(\log \rho^2)_y = a,\ (\log \tfrac{\sigma_x}{\rho^2})_x = b,\ (\log \tfrac{\rho_y}{\rho})_y = p,\ q = 0,$$
$$(\log \tfrac{\rho_y}{\rho})_x = r,\ (\log \tfrac{\rho_y^2}{\sigma_x}) = u,$$
$$(\log \rho_x)_y(\log \sigma_x)_x - (\log \rho_{xx})_y + 3(\log \rho)_x(\log \tfrac{\sigma_x}{\rho_x})_x = v,$$
$$\tfrac{\rho_x}{\rho_y}[(\log \tfrac{\rho_{xx}}{\sigma_{xx}})_x + 3(\log \tfrac{\sigma_x}{\rho_x})_x(\log \tfrac{\sigma_x}{\rho})_x] = w.$$

¿From this system a Janet base in total degree, then lexicographic term order $\rho > \sigma > w > v > \ldots > b > a$ is generated.

$$u - 2r - b = 0,\ a_x - ra = 0,\ a_y - qa = 0,\ b_y - ra = 0,$$
$$r_y - q_x = 0,\ w_y - v_x + wq - 2wa - vr + 2vb = 0,$$
$$q_{xx} - v_y - 2r_xa + 2q_xr - q_xb - 3r^2a + 2rba = 0,$$
$$r_{xx} - v_x + 2r_xr - 3r_xb - b_xr - 2wa + 2vb - 2r^2b + 2rb^2 = 0,$$
$$\sigma_y = 0,\ \sigma_{xx}\rho - 2\sigma_x\rho_x - \sigma_x\rho b = 0,$$
$$\rho_y + \tfrac{1}{2}\rho a = 0,\ \rho_{xx}\rho - 2\rho_x^2 - \rho_x\rho b + \tfrac{1}{4}\rho^2(r_x - v + r^2 - br) = 0.$$

Introducing the new function $R = -\log \rho$ into the last four equations yields system (18). For *case b.)* a similar calculation leads to the following system for σ and ρ.

$$\sigma_x = 0,\ \sigma_{yy}\rho - 2\sigma_y\rho_y - \sigma_y\rho q = 0,\ \rho_x + \tfrac{1}{2}\rho r = 0,$$
$$\rho_{yy}\rho - 2\rho_y^2 - \rho_y\rho q + \tfrac{1}{4}\rho^2(a_y - w - qa + a^2) = 0.$$

Introducting again $R = -\log\rho$ yields system (19). Finally in *case c.)* the four highest equations of the Janet base for the full system expressing σ and ρ in terms of the coefficients $a, b, \ldots, v, w$ are

$$\sigma_y + a\sigma_x = 0,\ = 0,\ \rho_y + a\rho_x + \tfrac{1}{2}P = 0,$$
$$\sigma_{xx}\rho - 2\rho_x\sigma_x + \sigma_x\rho f a = 0,\ \rho_{xx}\rho - 2\rho_x^2 + af\rho_x\rho + \rho^2\frac{Q}{P} = 0.$$

where P and Q are defined above. Substituting $R = -\log\rho$ yields (20). This completes the proof. ∎

For both the $\mathcal{S}^2_{3,3}$ and the $\mathcal{S}^2_{3,4}$ symmetry type the four parameter structure invariance correspondes originates from the second order Janet bases for the transformation functions σ and ρ as it is shown next.

Theorem 3.5. *If a second order ode has the symmetry type $\mathcal{S}^2_{3,3}$, two types of Janet bases may occur.*

a.) If $\sigma_x\rho_y - \gamma\sigma_y\rho_x \neq 0$ the Janet base is of type $\mathcal{J}^{(2,2)}_{3,6}$. There are two alternatives. If $d = 0$ the transformation functions σ and ρ are determined either by

$$\sigma_x = 0,\ \sigma_{yy} - (ar + q + \frac{c}{\gamma})\sigma_y = 0,\ \rho_y + \frac{\gamma a}{\gamma - 1}\rho_x = 0,\ \rho_{xx} + \gamma r\rho_x = 0$$

where $\gamma p + 1 = 0$ and $\gamma p \neq 0$, or by

$$\sigma_y - \frac{a}{\gamma - 1}\sigma_x = 0,\ \sigma_{xx} - u\sigma_x = 0,\ \rho_x = 0,\ \rho_{yy} - (ar + q + \gamma c)\rho_y = 0$$

where $\gamma + p = 0$. If $d \neq 0$ and $\gamma + 1 \neq 0$, the transformation functions σ and ρ are determined by

$$\sigma_y - \frac{p+\gamma}{(\gamma+1)d}\sigma_x,\ \ \sigma_{xx} + \frac{R(x,y,)}{T(x,y)}\sigma_x = 0,$$
$$\rho_y - \frac{\gamma p + 1}{(\gamma+1)d}\rho_x = 0,\ \ \rho_{xx} + \frac{S(x,y)}{T(x,y)}\rho_x = 0$$

where

$$R(x,y) = \gamma^2(cd^2 - adf - de + dr - f) + \gamma[(p+2)d(cd + r - e - af) - (2p+1)f]$$
$$+(p+1)(cd^2 - adf - de + dr - f),$$
$$S(x,y) = \gamma^3 d(cd + r - e) + \gamma^2[(p+2)d(cd + r - e) - (ad+1)pf]$$
$$+\gamma[(p+1)d(cd + r - e) - (adp + 1)f],$$
$$T(x,y) = d[p + ad + \gamma(\gamma+2)(ad+1)].$$

The constant γ is a solution of

$$\gamma^2 + \frac{2ad + p^2 + 1}{ad + p}\gamma + 1 = 0.$$

b.) If $\sigma_x\rho_y - \gamma\sigma_y\rho_x = 0$ *the Janet base is of type* $\mathcal{J}_{3,7}^{(2,2)}$*. The transformation functions* σ *and* ρ *are determined by*

$$\sigma_y - \frac{\gamma+1}{\gamma p}\sigma_x = 0, \ \ \sigma_{xx} + \frac{ap-(\gamma+1)b}{\gamma+1}\sigma_x = 0,$$
$$\rho_y - \frac{\gamma+1}{p}\rho_x = 0, \ \ \rho_{xx} + \frac{ap+(\gamma+1)b}{\gamma+1}\rho_x = 0.$$

The constant γ *is a solution of*

$$\gamma^2 - (\frac{p^2}{d} - 2)\gamma + 1 = 0.$$

Proof. The original expressions of the Janet base coefficients in terms of σ and ρ are too voluminous to be given here explicitly. If they are transformed into a Janet base in total degree, then lexicographic term order with $\rho > \sigma$ and $y > x$, the above linear system for σ and ρ is obtained. The relation for γ is an additional constraint for the coefficients following from the integrability conditions and the group structure. ■

Theorem 3.6. *If a second order ode has the symmetry type* $\mathcal{S}_{3,4}^2$ *two types of Janet bases may occur.*

a.) If $\sigma_x\rho_y - \sigma_y\rho_x \neq \sigma_x\sigma_y$ *the Janet base is of type* $\mathcal{J}_{3,6}^{(2,2)}$*. Three cases have to be distinguished. If* $d \neq 0$*,* $p+1 \neq 0$ *and* $T(x,y) \equiv 2ad + p + 1 \neq 0$*, the transformation functions* σ *and* ρ *are determined by*

$$\sigma_y + \frac{2a}{p+1}\sigma_x = 0, \quad \sigma_{xx} + \frac{R(x,y)}{T(x,y)}\sigma_x = 0,$$
$$\rho_y - \frac{p+1}{2d}\rho_x - \frac{p-1}{2d}\sigma_x = 0, \ \rho_{xx} + \frac{R(x,y)}{T(x,y)}\rho_x + \frac{S(x,y)}{dT(x,y)}\sigma_x = 0$$

where

$$R(x,y) = (p+1)(cd+r-4) - (p-1)af,$$
$$S(x,y) = (p+1)[(cd+r-e)d+f] - (p-3)adf.$$

If $d \neq 0$*,* $p+1 = T = 0$ *the system for* σ *and* ρ *is*

$$\sigma_y = 0, \ \sigma_{xx} - (r-e)\sigma_x = 0,$$
$$\rho_y + \frac{1}{d}\sigma_x = 0, \ \rho_{xx} - (r-e)\rho_x + (r - e + \frac{f}{d})\sigma_x = 0.$$

If $d = p+1 = T = 0$ *there is always* $a \neq 0$*, the system for* σ *and* ρ *is*

$$\sigma_x = 0, \ \sigma_{yy} - (ar+c-q)\sigma_y = 0,$$
$$\rho_x + \frac{1}{a}\sigma_y = 0, \ \rho_{yy} - (ar+c-q)\rho_y - (ar + c - \frac{b}{a})\sigma_y = 0.$$

b.) If $\sigma_x\rho_y - \sigma_y\rho_x = \sigma_x\sigma_y$ *the Janet base is of type* $\mathcal{J}_{3,7}^{(2,2)}$ *and there holds* $p \neq 0$*. The system for* σ *and* ρ *is*

$$\sigma_y - \frac{2}{p}\sigma_x = 0, \ \ \sigma_{xx} + (\frac{1}{2}ap - b)\sigma_x = 0,$$
$$\rho_y - \frac{2}{p}\rho_x - \frac{2}{p}\sigma_x = 0, \ \ \rho_{xx}(\frac{1}{2}ap - b)\rho_x + \frac{1}{2}ap\sigma_x = 0.$$

Proof. At first case $a.)$ is considered. In terms of σ and ρ the coefficients a, d and p are

$$a = -\frac{\sigma_y^2}{\sigma_x\sigma_y - \Delta},\ d = \frac{\sigma_x^2}{\sigma_x\sigma_y - \Delta}, p = \frac{\sigma_x\sigma_y - \Delta}{\sigma_x\sigma_y - \Delta}$$

where as usual $\Delta = \sigma_x\rho_y - \sigma_y\rho_x$. It follows that

$$p + 1 = \frac{2\sigma_x\sigma_y}{\sigma_x\sigma_y - \Delta},\ T(x,y) = -\frac{2\sigma_x\sigma_y\Delta}{\sigma_x\sigma_y - \Delta}.$$

The three alternatives correspond to $\sigma_x \neq 0$ and $\sigma_y \neq 0$, $\sigma_x \neq 0$, $\sigma_y = 0$ and $\sigma_x = 0$, $\sigma_y \neq 0$ respectively. In the latter case there follows $a \neq 0$. In case $b.)$ there holds $p = 2\sigma_x/\sigma_y$, consequently $p = 0$ entails $\sigma_x = 0$ which is not possible due to the constraint $\sigma_x\sigma_y = \Delta \neq 0$, i. e. $p \neq 0$ is assured. ■

The Projective Group as Symmetry Group. The largest group of point symmetries that any second order ode may allows has been shown to be the eigth-parameter projective group of the plane. Because there is obviously no degree of freedom in the canonical form corresponding to this symmetry group, the structure invariance group is identical to the symmetry group. Its eight parameters correspond to the integration constants of the Riccati like system (24) and the two third order Janet bases (23) for σ and ρ.

Theorem 3.7. *(Lie 1883) Any second order ode*

$$y'' + A(x,y)y'^3 + B(x,y)y'^2 + C(x,y)y' + D(x,y) = 0 \tag{21}$$

satisfying the constraints

$$\begin{aligned} &D_{yy} + BD_y - AD_x + (B_y - 2A_x)D + \tfrac{1}{3}B_{xx} \\ &\qquad -\tfrac{2}{3}C_{xy} + \tfrac{1}{3}C(B_x - 2C_y) = 0, \\ &2AD_y + A_yD + \tfrac{1}{3}C_{yy} - \tfrac{2}{3}B_{xy} + A_{xx} \\ &\qquad -\tfrac{1}{3}BC_y + \tfrac{2}{3}BB_x - A_xC - AC_x = 0. \end{aligned} \tag{22}$$

allows the projective group as symmetry group. It is similar to an equation $v''(u) = 0$. For $D \neq 0$ the transformation functions $u = \sigma(x,y)$ and $v = \rho(x,y)$ are solutions of

$$\begin{aligned} \sigma_{xx} - D\sigma_y + (C - 2C_2)\sigma_x &= 0, \\ \sigma_{xy} - C_2\sigma_y + B_2\sigma_x &= 0, \\ \sigma_{yy} - (B - 2B_2)\sigma_y + A\sigma_x &= 0 \end{aligned}\ , \tag{23}$$

and an identical system for ρ such that $\sigma_x\rho_y - \sigma_y\rho_x \neq 0$. The coefficients $B_2 = b$ and $C_2 = -a$ are determined by the Riccati system

$$\begin{aligned} a_x + a^2 + Ca - Db + D_y + BD &= 0, \\ a_y + ab - \tfrac{1}{3}A_x + \tfrac{2}{3}C_y + AD &= 0, \\ b_x + ab - \tfrac{2}{3}B_x + \tfrac{1}{3}C_y + AD &= 0, \\ b_y + b^2 - Bb + Aa - A_x + AC &= 0. \end{aligned} \tag{24}$$

Substituting them into (23) generates Janet bases (23) for σ and ρ.

Proof. A general point transformation $u = \sigma(x,y)$, $v = \rho(x,y)$ of $v''(u) = 0$ generates an equation of the form (21). Writing $B = B_1 + 2B_2$ and $C = C_1 + 2C_2$, the explicit form of the coefficients is

$$\frac{\sigma_y\rho_{yy} - \sigma_{yy}\rho_y}{\sigma_x\rho_y - \sigma_y\rho_x} = A, \quad \frac{\sigma_x\rho_{yy} - \sigma_{yy}\rho_x}{\sigma_x\rho_y - \sigma_y\rho_x} = B_1,$$

$$\frac{\sigma_y\rho_{xy} - \sigma_{xy}\rho_y}{\sigma_x\rho_y - \sigma_y\rho_x} = B_2 \quad \frac{\sigma_y\rho_{xx} - \sigma_{xx}\rho_y}{\sigma_x\rho_y - \sigma_y\rho_x} = C_1,$$

$$\frac{\sigma_x\rho_{xy} - \sigma_{xy}\rho_x}{\sigma_x\rho_y - \sigma_y\rho_x} = C_2, \quad \frac{\sigma_x\rho_{xx} - \sigma_{xx}\rho_x}{\sigma_x\rho_y - \sigma_y\rho_x} = D.$$

These equations may be considered as a system of pde's determining σ and ρ in terms of the coefficients of (21). B_1 and C_1 are expressed in terms of B_2 and C_2 respectively, and the resulting system is transformed into a Janet base with lexicographical term ordering $\rho > \sigma > B_2 > C_2 > A > \ldots > D$. The full Janet base comprises twelve equations the upper half of which is given by (23) and an identical system for ρ. The two lowest equations are the constraints (22) for $A, \ldots, D$ guaranteeing the projective symmetry. In between there is the system of four equations

$$\begin{aligned} C_{2,x} - C_2^2 + DB_2 + CC_2 - D_y - BD &= 0, \\ C_{2,y} + B_2C_2 + \tfrac{1}{3}B_x - \tfrac{2}{3}C_y - AD &= 0, \\ B_{2,x} - B_2C_2 - \tfrac{2}{3}B_x + \tfrac{1}{3}C_y + AD &= 0, \\ B_{2,y} + B_2^2 - BB_2 - AC_2 - A_x + AC &= 0. \end{aligned} \tag{25}$$

Combined with the two lowest equations they may be considered as integrability conditions for the linear homogeneous system (23). Equations (25) express the functions B_2 and C_2 in terms of the known coefficients A, B, C and D. Substituting $C_2 = -a$ and $B_2 = b$ yields the Riccati system (24) for a and b. This completes the proof. ■

These results will be illustrated now by a few examples. All results related to the decomposition of Janet bases into largest completely reducible components may be found in a forthcoming publication [12]. Square brackets mean always taking the least common multiple.

Example 1. The equation

$$y''y'x - y''y - y'^2 - 2y' - 1 = 0$$

generates the type $\mathcal{J}_{2,3}^{(2,2)}$ Janet base

$$\xi_x - \frac{1}{x}\xi = 0, \quad \xi_y = 0,$$

$$\eta_x - \frac{1}{x+y}\eta + \frac{y}{x(x+y)}\xi = 0, \quad \eta_y - \frac{1}{x+y}\eta - \frac{1}{x+y}\xi = 0$$

with $a = d = 0$ and $p = s = -1/(x+y)$, by Theorem 2.2 it follows that its symmetry group is $\mathcal{S}_{2,1}^2$. By Theorem 3.2, $i.$) the type $\mathcal{J}_{3,2}^{(1,2)}$ Janet base for the transformation function σ is

$$\sigma_{xx} - \frac{y}{x(x+y)}\sigma_y + \frac{1}{x}\sigma_x = 0, \ \sigma_{xy} + \frac{1}{x+y}\sigma_y = 0, \ \sigma_{yy} + \frac{1}{x+y}\sigma_y = 0$$

and an identical one for ρ. Its socle is $\{\sigma_x, \sigma_y\}$. Dividing it out yields a type $\mathcal{J}_{2,3}^{(2,2)}$ Janet base for the quotient in terms of $\sigma_1 = \sigma_x$ and $\sigma_2 = \sigma_y$. It is completely reducible and decomposes into type $\mathcal{J}_{1,2}^{(2,2)}$ Janet bases according to

$$\{\sigma_{1,x} - \frac{y}{x(x+y)}\sigma_2 + \frac{1}{x}\sigma_1, \sigma_{1,y} + \frac{1}{x+y}\sigma_2,\ \sigma_{2,x} + \frac{1}{x+y}\sigma_2, \sigma_{2,y} + \frac{1}{x+y}\sigma_2\} =$$
$$[\{\sigma_2 + \frac{(C-1)x}{Cy+x}\sigma_1,\ \sigma_{1,x} + (\frac{1}{x} + \frac{(C-1)y}{(Cy+x)(x+y)})\sigma_1, \sigma_{1,y} - \frac{(C-1)x}{(Cy+x)(x+y)}\sigma_1\}]$$

where the constant C parametrizes the components. The two special choices $C = 1$ and $C = 0$ lead to the representation

$$[\{\sigma_2, \sigma_{1,x} + \frac{1}{x}\sigma_1, \sigma_{1,y}\},\ \{\sigma_2 - \sigma_1, \sigma_{1,x} + \frac{1}{x+y}\sigma_1, \sigma_{1,y} + \frac{1}{x+y}\sigma_1\}] = 0$$

from which the solutions $\sigma_1 = 1/x$, $\sigma_2 = 0$ and $\sigma_1 = \sigma_2 = 1/(x+y)$ respectively and finally $\sigma = \log x$ and $\sigma = \log(x+y)$ are obtained. A possible choice of the transformation functions therefore is

$$\sigma = \log x,\ \ \rho = \log(x+y) \longrightarrow x = e^u,\ \ y = e^v - e^u.$$

It yields the canonical form

$$(v'' + v'^2 - v')(v' - 1) - v'^2 = 0.$$

The transformations of the structure invariance group leave a lot of freedom for the canonical form. For example, replacing u and v by $u + v$ and $u - v$ respectively yields

$$v'' - \frac{5}{4}v'^3 + \frac{3}{2}v' - \frac{1}{4v'}.$$

Example 2. Equation 6.133 of Kamke's collection has the Janet base

$$\begin{aligned}
\xi_y + \xi_x - \frac{1}{x+y}\eta - \frac{1}{x+y}\xi &= 0,\\
\eta_x - \xi_x + \frac{1}{x+y}\eta + \frac{1}{x+y}\xi &= 0,\\
\eta_y + \xi_x - \frac{2}{x+y}\eta - \frac{2}{x+y}\xi &= 0,\\
\xi_{xx} - \frac{3}{x+y}\xi_x + \frac{3}{(x+y)^2}\eta + \frac{3}{(x+y)^2}\xi &= 0
\end{aligned}$$

of type $\mathcal{J}_{3,6}^{(2,2)}$ for its point symmetries with $a = p = 1$, $d = -1$. By Theorem 2.2 its symmetry type is $\mathcal{S}_{3,2}^2$. Therefore *case c.)* of the Theorem 3.4 applies. The system (20) is

$$R_{xx} + R_x^2 + \frac{1}{x+y}R_x - \frac{\frac{1}{4}}{(x+y)^2} = 0,\ \ R_y + R_x + \frac{1}{x+y} = 0.$$

These equation yield the solution

$$R_x = \frac{1}{x-y+C_1} - \frac{\frac{1}{2}}{x+y},\ \ R_y = -\frac{1}{x-y+C_1} - \frac{\frac{1}{2}}{x+y}$$

for the first derivatives of R with the integration constant C_1. The integrability conditions of the Janet base for σ in (20) require $C_1 \to \infty$, i. e. the system for σ is $\{\sigma_y + \sigma_x, \sigma_{xx}\}$. It is completely reducible with the decomposition

$$[\{\sigma_x - \frac{1}{x-y+C_2}\sigma, \sigma_y + \frac{1}{x-y+C_2}\sigma\}] = [\{\sigma_x, \sigma_y\}, \{\sigma_x + \frac{1}{x-y}\sigma, \sigma_y - \frac{1}{x-y}\sigma\}.$$

The two components of the latter representation correspond to the values $C_2 \to \infty$ and $C_2 = 0$ respectively. A special solution is $\sigma = x - y$. To the chosen value of C_1 there corresponds the system $R_x = R_y = (1/2)/(x+y)$. A special solution is $R = \frac{1}{2}\log(x+y)$ and consequently $\rho = \sqrt{x+y}$. The inverse of the transformation

$$u = x - y,\ v = \sqrt{x+y}\ \textit{is}\ x = \frac{1}{2}(v^2 + u),\ y = \frac{1}{2}(v^2 - u)$$

yields the canonical form $v''v^3 + \frac{1}{4} = 0$ for equation 6.133 of Kamke's collection. The transformations of the structure invariance group may be applied to generate any value for the constant term. Choosing $a_1 = 2$, $a_2 = 1$ and $a_3 = a_4$ for example yields $v''v^3 + 1 = 0$.

Example 3. By the above criterion (22) equation 6.180

$$x^2(y-1)y'' - 2x^2y'^2 - 2x(y-1)y' - 2y(y-1)^2 = 0$$

of the collection by Kamke with $A = 0$, $B = -2/(y-1)$, $C = -2/x$ and $D = -2y(y-1)/x^2$ has the symmetry type $\mathcal{S}_8^2$. The system (24) for a and b is

$$a_x + a^2 - \frac{2}{x}a + \frac{2y(y-1)}{x^2}b + \frac{2}{x^2} = 0,$$
$$a_y + ab = 0,\ \ b_x + ab = 0,\ \ b_y + b^2 + \frac{2}{y-1}b = 0.$$

Its general solution is rational and may be written as

$$a = \frac{K_2 r_{1,x} + r_{2,x}}{K_1 + K_2 r_1 + r_2} + \frac{1}{x} = \frac{K_1 s_{1,x} + s_{2,x}}{K_2 + K_1 s_1 + s_2},$$
$$b = \frac{K_2 r_{1,y} + r_{2,y}}{K_1 + K_2 r_1 + r_2} = \frac{K_1 s_{1,y} + s_{2,y}}{K_2 + K_1 s_1 + s_2} - \frac{1}{y(y-1)}$$

where

$$r_1 = \frac{y}{x(y-1)},\ \ r_2 = x,\ \ s_1 = \frac{x(y-1)}{y},\ \ s_2 = \frac{x^2(y-1)}{y}$$

and K_1 and K_2 are the integration constants. Choosing the special values $K_1 = 0$ and $K_2 \to \infty$ in the second representation involving the functions s_1 and s_2, $a = 0$ and $b = -1/y(y-1)$ are obtained. They correspond to $C_2 = 0$, $B_2 = 1/y(y-1)$ and yield the Janet base

$$\sigma_{xx} + \frac{2y(y-1)}{x^2}\sigma_y - \frac{2}{x}\sigma_x = 0,$$
$$\sigma_{xy} - \frac{1}{y(y-1)}\sigma_x = 0,\ \sigma_{yy} + \frac{2}{y}\sigma_y = 0$$

for σ and an identical one for ρ. They are completely reducible. The components of the socle in the former case are

$$\sigma_x - \frac{(C_2+2x)(y-1)}{C_1 y + (C_2+x)x(y-1)}\sigma = 0,\ \sigma_y - \frac{(C_2+x)x}{C_1 y^2 + (C_2+x)xy(y-1)}\sigma = 0.$$

The three particular components

$$[\{\sigma_x,\ \sigma_y\}, \{\sigma_x - \frac{2}{x}\sigma,\ \sigma_y - \frac{1}{y^2-y}\sigma\}, \{\sigma_x - \frac{1}{x}\sigma,\ \sigma_y - \frac{1}{y^2-y}\sigma\}]$$

are obtained by choosing $C_1 \to \infty, C_2 = 0$, $C_1 = C_2 = 0$ and $C_1 = 0, C_2 \to \infty$ respectively. From this the fundamental system

$$1,\ \frac{x(y-1)}{y},\ \frac{x^2(y-1)}{y}$$

is obtained. Therefore a possible selection for the transformation functions is

$$u = \frac{x(y-1)}{y},\ v = \frac{x^2(y-1)}{y} \longrightarrow x = \frac{u}{v},\ y = \frac{u}{u-v^2}.$$

4. Summary

The results described in this article provide optimal constructive methods for all the necessary steps in order to transform any second order quasilinear ode with non-trivial Lie symmetries into a canonical form corresponding to its symmetry type. The reason is that the equations describing the transformations to canonical form are obtained *algorithmically* by transforming them into a Janet base and not by *ad hoc* manipulations. The answer is not changed in any significant way if other term orderings are applied. The function fields for the transformation functions are determined by the solution algorithms that are available for the system of equations that are given in Theorems 4 to 10, i. e. they are Liouvillian over the base field determined by the given ode. Furthermore the freedom for this canonical form is completely described in terms of its structure invariance group.

These results are important prerequisites for designing solution algorithms based on Lie's theory. The general solution scheme for an ode along these lines decomposes into the following steps.

- ▷ Determine the type of the symmetry group.
- ▷ Transform the equation to the canonical form corresponding to its symmetry type.
- ▷ Solve the canonical form and generate the solution of the given equation from it.

The first step is achieved best of all by generating a Janet base for the determining system as it has been described in Theorem 2.2, from its coefficients the symmetry type may be read off immediately. The second and most difficult step has been solved in this article. For the largest possible symmetry group, i. e. the 8-parameter projective group of the plane, the complete solution scheme has been worked out in detail in [11].

Lie's second approach for solving ode's [2] requires to know the infinitesimal generators of the symmetries explicitly. This means the determining system, a system of linear homogeneous pde's for two functions of the dependent and the independent variable, has to be solved. There does not seem to exists an algorithm for determining Liouvillian solutions of such systems. Furthermore, even if a solution has been obtained, there remains the problem of identifying the type of the Lie algebra and generating canonical generators for it. For symmetry groups with more than three parameters this is a highly non-trivial problem. Knowing all these difficulties, in his original approach Lie tried to set up a system of equations for the finite transformation functions after the type of the symmetry group is known, but he did not succeed. From the results described in this article the reason is obvious. It does not seem to be possible to set up *generically* a system of equat! ! ions for the desired transformat

ion functions in terms of the coefficients of the given ode. Rather starting from the Janet base of the determining system such a set of equations may be obtained for each symmetry type that in some cases is even considerably simpler than the system for the infinitesimal generators. Most important however is the fact that the Liouvillian solutions of these systems may be obtained algorithmically.

There are numerous extensions of the results described in this article. The most obvious ones are:

▷ Generate a similar scheme for third order equations. This is straightforward *in principle* because the same method applies, due to the large number of more than fourty possible symmetry types however it is rather extensive.

▷ It would be highly desirable to extend the type of equations admitted for the above analysis, e. g. abolishing the requirement of quasilinearity or admitting more general base fields.

▷ A classification of symmetry types and canonical forms may be given for partial differential equations in an analogous way. For the simplest partial pde's determining a single function $z(x, y)$ depening on x and y all the necessary prerequisites like e. g. a classification of the relevant groups is available in the literature.

Many details on Janet bases, its decomposition into irreducible components and solving Riccati like systems of partial differential equations may be found in [12].

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