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On Representations of $\mathbf{SL}_n$ with Algebras of Invariants being Complete Intersections

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Abstract. We obtain the complete list of representations of SL_n such that the algebra of invariants is a hypersurface. We also give a list containing all the representations of SL_n such that the algebra of invariants is a complete intersection.

Introduction.

Let A be a $\mathbf{N}$-graded finitely-generated commutative algebra over an algebraically closed field $\mathbf{k}$ of characteristic 0 such that $A_0 = \mathbf{k}$. Then a choice of a minimal system $f_1, \cdots, f_n$ of homogeneous generators of A yields an exact sequence

$$0 \to I \to \mathbf{k}[T_1, \cdots, T_n] \to A \to 0,$$

where I is the ideal of syzygies of $f_1, \cdots, f_n$. Denote by tr.deg.A the transcendence degree of A and set hd$A = n -$ tr.deg.A. If A is a Cohen-Macaulay algebra, hd$A$ equals the *homological dimension* of A, i.e., the length of a minimal free resolution of $\mathbf{k}[T_1, \cdots, T_n]$-module A ([20]). The algebra A is called a hypersurface if I is a principal ideal, i.e., hd$A = 1$. Moreover, A is called a complete intersection if I is generated by hd$A$ elements or, equivalently, a minimal system of homogeneous generators of I is a regular sequence in $\mathbf{k}[T_1, \cdots, T_n]$. In this case the affine variety $X = \mathrm{Spec}(A)$ naturally embedded in $\mathbf{A}^n$ is the intersection of $n - \dim(X)$ hypersurfaces. In the sequel, c.i. will be shorthand of "complete intersection" and h-s of "hypersurface".

Let G be a reductive algebraic group over $\mathbf{k}$, $\rho : G \to GL(V)$ a finite-dimensional representation. The algebra $\mathbf{k}[V]^G$ of G-invariant regular functions on V is finitely-generated by Hilbert's theorem. Recall that ρ is called coregular

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if $\mathbf{k}[V]^G$ is a polynomial algebra. We will say that ρ is a c.i. (h-s) representation if $\mathbf{k}[V]^G$ is a c.i. (h-s).

In this paper we classify c.i. representations of SL_n. The starting point of this classification is the Classical Invariant Theory of the 19^{th} century. For example, classical texts consider all the c.i. representations of SL_2 (see, e.g. [7]). Coregular representations of connected simple groups are classified in [9], [1], [18]. Noncoregular irreducible c.i. representations of connected simple groups are classified in [14] and [22, 5.1]. All of them are h-s. Also, [14] contains the classification of c.i. representations of SL_2. In [22], [23] we classified the representations of connected simple groups admitting a finite coregular extension in $GL(V)$. All of them are h-s. Further, [15] provides some important examples of c.i. representations of simple groups. It should be noted that noncoregular c.i. representations listed above are for the most part representations of SL_n. The results of this paper are the following

Theorem 0.1. *A representation $\rho : SL_n \to GL(V)$ such that $V^{SL_n} = \{0\}$ is h-s if and only if either ρ or its dual is contained in Tables 1-8.*

Theorem 0.2. *For any c.i. representation $\rho : SL_n \to GL(V)$ such that $V^{SL_n} = \{0\}$, $\mathrm{hd}\mathbf{k}[V]^{SL_n} \geq 2$, either ρ or its dual is contained either in Table 9 or in List 10. All the representations in Table 9 are c.i.*

The paper is organized as follows. In § 1. we present Tables 1-9 and List 10. In § 2. we apply Classical Invariant Theory to calculate generators of the algebra of invariants for several series of representations, some of them being c.i. In § 3. we explain how we got our lists. In § 4. we prove Theorem 0.1. In § 5. we prove Theorem 0.2 and also discuss the indeterminate cases from List 10. In § 6. we discuss the situation for the other simple groups.

1. Tables.

For the group SL_n we denote by $\varphi_i, i = 1, \cdots, n-1$ the fundamental weights. A dominant weight is denoted by a monomial in φ_i-s. This monomial denotes also the irreducible representation with the corresponding highest weight. A reducible representation is denoted by a polynomial in φ_i-s like $\varphi_1\varphi_2 + \varphi_1^2$. An asterisk denotes the dual representation, for example $\varphi_i^* = \varphi_{n-i}$. Note that $\varphi_i, \varphi_1^i, \varphi_1\varphi_{n-1}$ is nothing but $\wedge^i\mathbf{k}^n$, $S^i\mathbf{k}^n$, and the adjoint representation, respectively.

For each $\rho : SL_n \to GL(V)$ from Tables 1-8 below we describe a minimal system of polyhomogeneous generators of $\mathbf{k}[V]^{SL_n}$ and the generator of the ideal of syzygies. Here "polyhomogeneous" means homogeneous with respect to each irreducible factor of a fixed decomposition of (SL_n, V). For some generators and syzygies we give only their polydegrees, for example $f(3,0,1)$. However when it is possible and reasonable, we describe them explicitly.

From now on, we fix tensor notation for coordinates on irreducible factors as follows: v^i for φ_1, α_i for φ_1^*, A^{ij} for φ_1^2, C_{ij} for φ_{n-1}^2, B^{ij} for φ_2, D_{ij} for φ_2^*, Q^{ijk} for φ_3, L^i_j for $\varphi_1\varphi_{n-1}$. Tensors corresponding to different isomorphic factors differ from each other by an index *in parentheses* as follows: $A^{ij}_{(2)}$, $D^{(q)}_{kl}$.

For describing invariants we use well-known notation of linear algebra: det$A$ for the determinant and tr$L$ for the trace of a matrix, pf$B$ for the pfaffian of a skew matrix, $[\alpha_{(1)} \cdots \alpha_{(n)}]$ for the determinant of a system of vectors or linear forms, $\alpha(v)$, $B(\alpha_i, \alpha_j)$, $A(\alpha) = A(\alpha, \alpha)$ for the corresponding contractions of covariant

Table 1. The serial h-s representations of SL_n.

$n \geq 4$ in entries 6-9,13,14, $n \geq 3$ in the other entries.

	ρ	Generators	Syzygy
1	$(n+1)\varphi_1+$ $+\varphi_1^*$	$d_i = [v_{(1)}, \cdots, \widehat{v_{(i)}}, \cdots, v_{(n+1)}]$, $l_i = \alpha(v_{(i)})$	$\sum_{i=1}^{n+1} (-1)^i d_i l_i = 0$
2	$n\varphi_1 + n\varphi_1^*$	$d = [v_{(1)}, \cdots, v_{(n)}], l_{ij} = \alpha_{(i)}(v_{(j)})$, $e = [\alpha_{(1)} \cdots \alpha_{(n)}]$,	$de = \det(l_{ij})_{i,j=1}^n$
3	$\varphi_1^2 + n\varphi_1^*$	$d = \det A, e = [\alpha_{(1)} \cdots \alpha_{(n)}]$, $f_{ij} = A(\alpha_{(i)}, \alpha_{(j)}), i \leq j$	$de^2 = \det(f_{ij})_{i,j=1}^n$
4	φ_1^2+ $(n-1)\varphi_1^*+$ $+\varphi_1$	$d = \det A, f_{ij} = A(\alpha_{(i)}, \alpha_{(j)}), i \leq j$, $f_{nn} = [A_1^1 \cdots A_{n-1}^1 v][A_1^2 \cdots A_{n-1}^2 v]$, $f_{ni} = \alpha_{(i)}(v), f_L = [A_1^1 \cdots A_{n-1}^1 v]*$ $*(A_1^2, \alpha_{(1)}) \cdots (A_{n-1}^2, \alpha_{(n-1)})$	$f_L^2 = \det(f_{ij})_{i,j=1}^n$ for $f_{in} = df_{ni}$
5	$\varphi_1^2 + r\varphi_1^*+$ $+2\varphi_1$, $r \leq n-3$	$d = \det A, A(\alpha_{(i)}, \alpha_{(j)}), i \leq j$, $\alpha_{(i)}(v_{(k)}), e_{kl} = [A_1^1 \cdots A_{n-1}^1 v_{(k)}]*$ $*[A_1^2 \cdots A_{n-1}^2 v_{(l)}], k \leq l$, $g = [A_1^1 \cdots A_{n-2}^1 v_{(1)} v_{(2)}]*$ $*[A_1^2 \cdots A_{n-2}^2 v_{(1)} v_{(2)}]$	$e_{12}^2 = e_{11}e_{22} + dg$
6	$\varphi_2 + n\varphi_1^*$ n even	$d = \mathrm{pf}B, e = [\alpha_{(1)} \cdots \alpha_{(n)}]$, $f_{ij} = B(\alpha_{(i)}, \alpha_{(j)}), i < j$	$de = \mathrm{pf}(f_{ij})_{i,j=1}^n$
7	$\varphi_2 + n\varphi_1^*+$ $+\varphi_1$	n even : all of 6 and $\alpha_{(i)}(v)$ n odd : e, f_{ij} of 6, $l_i = \alpha_{(i)}(v)$, $d = [B \cdots Bv]$	$de = \mathrm{pf}(f_{ij})_{i,j=1}^n$ $de = F(f_{ij}, l_k)$
8	φ_2+ $(n-1)\varphi_1^*+$ $+3\varphi_1$	$B(\alpha_{(k)}, \alpha_{(l)}), k < l, \alpha_{(k)}(v_{(i)})$, and n even : $\mathrm{pf}B, [B \cdots Bv_{(i)}v_{(j)}], i < j$ odd : $[B \cdots Bv_{(i)}], [B \cdots Bv_{(1)}v_{(2)}v_{(3)}]$	$(n-2, 1, \cdots, 1)$
9	φ_2+ $r\varphi_1^* + 4\varphi_1$ $r \leq n-3$	$B(\alpha_{(k)}, \alpha_{(l)}), k < l, \alpha_{(k)}(v_{(i)})$, and n even : $h_{ij} = [B \cdots Bv_{(i)}v_{(j)}], i < j$, $d = \mathrm{pf}B, p = [B \cdots Bv_{(1)}v_{(2)}v_{(3)}v_{(4)}]$ n odd : $d_i = [B \cdots Bv_{(i)}]$, $e_i = [B \cdots Bv_{(1)} \cdots \widehat{v_{(i)}} \cdots v_{(4)}]$	$dp = h_{12}h_{34}+$ $h_{13}h_{24} + h_{14}h_{23}$ $\sum_{i=1}^{4} d_i e_i = 0$

and contravariant tensors. For bilinear tensors X^{ij} and Y_{kl} we denote by XY the operator $XY_l^i = X^{ij}Y_{jl}$.

We also describe invariants of SL_n as contractions of copies of the above tensors with copies of SL_n-invariant covariant tensor $\det_{i_1 \cdots i_n}$ and its contravariant analog $\det^{-1}$. A contraction containing p copies of a tensor X defines an invariant of degree p with respect to the irreducible factor corresponding to X. Copies of a tensor corresponding to the same factor differ from each other either by an index *without* parentheses (Q_1^{ijk}, Q_2^{ijk} etc.) or as follows: $A_{(2)}^{i_1 j_1}, A_{(2)}^{i_2 j_2}$ etc.

We present contractions in a special short form as follows. If an upper index of a tensor is contracted with a lower index of another one, both tensors being different from det, we place the symbols of the tensors with the indicated indices in parentheses; for example, $(A^2_{n-1}, \alpha_{(n-1)})$ means that the second index of

Table 1. Continuation.

10	$2\varphi_1^2+\varphi_1$	$f_k : \det(A_{(1)} - \lambda A_{(2)}) = f_0\lambda^n + \cdots +$ $+f_n,\quad [A^{i_1}_{(1)} \cdots A^{i_l}_{(1)} A^{i_{l+1}}_{(2)} \cdots A^{i_{n-1}}_{(2)} v]*$ $*[A^{j_1}_{(1)} \cdots A^{j_l}_{(1)} A^{j_{l+1}}_{(2)} \cdots A^{j_{n-1}}_{(2)} v]$, $l = 0, \cdots, n-1$, $f_L(n(n-1)/2, n(n-1)/2, n)$	$f_L^2 = F(\cdots)$
11	$2\varphi_1^2+\varphi_1^*$	f_k of 10, $A_{(l)}(\alpha)$, $[A^{i_2}_{(1)} \cdots A^{i_m}_{(1)} A^{p_1}_{(2)} \cdots A^{p_{n+1-m}}_{(2)}]*$ $*[A^{j_1}_{(1)} \cdots A^{j_m}_{(1)} A^{q_2}_{(2)} \cdots A^{q_{n+1-m}}_{(2)}]*$ $*(A^{i_1}_{(1)}, \alpha)(A^{q_1}_{(2)}, \alpha), m = 1, \cdots, n,$ $f_L(n(n-1)/2, n(n-1)/2, n)$	$f_L^2 = F(\cdots)$
12	$\varphi_1^2 + \varphi_{n-1}^2+$ $+\varphi_1$	d, f_{nn} of 4, $\det C$, $\mathrm{tr}(AC)^i$, $C(v, (AC)^{i-1}v), 1 \le i \le n-1$, $f_L(n(n-1)/2, n(n-1)/2, n)$	$f_L^2 = F(\cdots)$
13	$\varphi_1^2 + \varphi_2^* + \varphi_1$ $n = 2m+1$	d, f_{nn} of 4, $f_i = \mathrm{tr}(AD)^{2i}$, $g_i = D(v, (AD)^{2i-1}v), i = 1, \cdots, m$, $f_L = [P \cdots Pv], P^{jk} = A^{jl}D_{lm}A^{mk}$	$f_L^2 =$ $eF(f_i) + dg_m$ $+dR(f_i, g_j)$
14	$\varphi_2 + \varphi_2^* + 3\varphi_1$ $n = 2m+1$	d_i, e_4 of 9, $\mathrm{tr}(BD)^i$, $D(v_{(k)}, (BD)^{i-1}v_l), i = 1, \cdots, m, k < l$	$(2m-1, m, 1)$
15	$\varphi_1\varphi_{n-1}+$ $+\varphi_1 + \varphi_1^*$	$\sigma_k = \mathrm{tr}L^k, 2 \le k \le n, f_l = \alpha(L^l v)$, $0 \le l \le n-1, d = [v\ Lv \cdots L^{n-1}v]$, $e = [\alpha\alpha L \cdots \alpha L^{n-1}]$	$de = \det M$ $M_{ij} =$ $= \alpha(L^{i+j}v)^2$

A_{n-1} is contracted with the unique index of $\alpha_{(n-1)}$. Further, we write brackets $[\cdots]$ for each copy of det and $\det^{-1}$ and place into the brackets the tensors with their indices contracted with this copy (the absence of indices means all the indices).

If our representation contains several isomorphic factors, say $p\varphi_1$, and a contraction involves $v_{(i)}, \cdots, v_{(j)}$, then it is assumed that each of $i, \cdots, j$ runs from 1 to p; if necessary, we give restrictions on $i, \cdots, j$

We illustrate the above conventions by an example. For $\rho = 3\varphi_2 + \varphi_1 + \varphi_3$ of SL_4 we wrote: $[B_{(k)}B^1_{(l)}v](B^2_{(l)}, \alpha), k < l$. This means:

$$[B_{(k)}B^1_{(l)}v](B^2_{(l)}, \alpha) = B^{i_1j_1}_{(k)} B^{i_2j_2}_{(l)} v^t \alpha_{j_2} \det_{i_1j_1i_2t}.$$

Since ρ contains 3 copies of φ_2, $k < l$ means $1 \le k < l \le 3$ and this contraction defines 3 different generators.

References like " f_{ij} of 6" mean that one should append f_{ij} from entry 6 of the same Table to the list of generators.

The notation f_L refers to a polynomial generating the ideal of functions on V vanishing on a codimension-1 Luna stratum of the quotient for a finite extension

[2] By the Hamilton-Cayley theorem, for any $p \in \mathbf{N}$ $\alpha(L^p v)$ is a polynomial in f_l, σ_k.

$H \subseteq GL(V)$ of $\rho(SL_n)$ (see [23]). Note that in these cases such a group H and such a stratum are unique. Furthermore, $f_L^2 = F(\cdots)$ means that F depends on all the generators but f_L.

We drop sometimes coefficients of monomials in the syzygy. However by multiplying the generators by some scalars, one can get such a form of syzygy.

Table 2. The h-s representations of SL_2.

This classification is contained in [16, Theorem 10.2] For SL_2 we write φ instead of φ_1. In entries 3,7,9 the action of SL_2 on φ^2 and φ^4 can be thought of as that of SO_3 on vectors A of $\mathbf{k}^3$ endowed with an invariant form $\langle *, * \rangle$ and on selfadjoint operators R on $\mathbf{k}^3$, respectively. The explicit form of generators in entries 5,6,10,11 can be found in [7] (or [21] for 5,6).

	ρ	Generators	Syzygy
1	4φ	$l_{ij} = [v_{(i)}v_{(j)}]$	$l_{12}l_{34} - l_{13}l_{24} + l_{14}l_{23} = 0$
2	$\varphi^2 + 2\varphi$	$d = \det A, f_{ij} = [A^1 v_{(i)}][A^2 v_{(j)}]$, $i \leq j, e = [v_{(1)}v_{(2)}]$	$de^2 = f_{11}f_{22} - f_{12}^2$
3	$3\varphi^2$	$f_{ij} = \langle A_{(i)}, A_{(j)} \rangle, i \leq j$, $d = [A_{(1)}A_{(2)}A_{(3)}]$	$d^2 = \det(f_{ij})_{i,j=1}^3$
4	$2\varphi^2 + \varphi$	all of 3 for $A_{(3)} = v^2$, but $f_{33} = 0$	
5	$\varphi^3 + \varphi$	$D(4,0), H(2,2), G(3,3), f(1,3)$	$G^2 + H^3 = Df^2$
6	$\varphi^3 + \varphi^2$	$D(4,0), \det A, (2,1), (2,3), f_L(4,3)$	$f_L^2 = F(\cdots)$
7	$\varphi^4 + \varphi^2$	$\mathrm{tr}R^2, \mathrm{tr}R^3, \langle A, A \rangle, \langle RA, A \rangle, \langle R^2 A, A \rangle$, $f_L = [A, RA, R^2 A]$	$f_L^2 = F(\cdots)$
8	$\varphi^4 + \varphi$	all of 7 for $A = v^2$, but $\langle A, A \rangle = 0$	
9	$2\varphi^4$	$\mathrm{tr}(R_{(1)}^k R_{(2)}^l), k + l = 2, 3 \text{ or } (k,l) = (2,2)$	$(6,6)$
10	φ^5	$(4), (8), (12), f_L(18)$	$f_L^2 = F(\cdots)$
11	φ^6	$(2), (4), (6), (10), f_L(15)$	$f_L^2 = F(\cdots)$

Table 3. The h-s representations of SL_3.

In entries 5,6 $X = A_{(1)}, Y = A_{(2)}, Z = A_{(3)}$.

	ρ	Generators	Syzygy
1	$2\varphi_1\varphi_2$	$\mathrm{tr}(L_{(1)}^k L_{(2)}^l), k + l = 2, 3 \, or \, (k,l) = (2,2)$, $f_L = \mathrm{tr}(L_{(1)}L_{(2)}L_{(1)}^2 L_{(2)}^2 - L_{(2)}^2 L_{(1)}^2 L_{(2)} L_{(1)})$	$f_L^2 = F(\cdots)$
2	$\varphi_1\varphi_2 + \varphi_1^2$	$(2,0), (3,0), (0,3), (2,3), (3,3), (4,3), (6,3)$	$(12,9)$
3	$\varphi_1^3 + \varphi_1$	$(4,0), (6,0), (3,3), (5,3), (4,6), f_L(12,9)$	$f_L^2 = F(\cdots)$
4	$\varphi_1^3 + \varphi_2$	$(4,0), (6,0), (1,3), (3,3), (8,6), f_L(12,9)$	$f_L^2 = F(\cdots)$
5	$3\varphi_1^2$	coefficients of $\det(X + \mu Y + \lambda Z)$, $[X_1^1 Y_1^1 Z_1^1][X_2^1 Y_2^1 Z_2^1][X_1^2 X_2^2 Z_1^2][Y_1^2 Y_2^2 Z_2^2]$	$(6,6,6)$
6	$2\varphi_1^2 + \varphi_2^2$	coefficients of $\det(X + \lambda Y), \det C, \mathrm{tr}(XC)$, $\mathrm{tr}(YC), \mathrm{tr}(XC)^2, \mathrm{tr}(YC)^2, \mathrm{tr}(XCYC)$, $[X_1^1 X_2^1 Y_2^2][Y_1^1 Y_2^1 X_2^2](X_1^2, C_1)(Y_1^2, C_2)$	$(6,6,6)$

Table 4. The h-s representations of SL_4.

In entries 3-5: $h_{klm;pq} = [B_{(k)}B^1_{(m)}v_{(p)}][B_{(l)}B^2_{(m)}v_{(q)}]$.

	ρ	Generators	Syzygy
1	φ_1^3	$(8),(16),(24),(32),(40),f_L(100)$	$f_L^2 = F(\cdots)$
2	$6\varphi_2$	$[B_{(i)}B_{(j)}], i \le j, f_L(1,\cdots,1)$	$f_L^2 = F(\cdots)$
3	$4\varphi_2+\varphi_1$	$[B_{(i)}B_{(j)}], i \le j, h_{klm;11}, k<l<m$	$(2,2,2,2,4)$
4	$3\varphi_2+2\varphi_1$	$[B_{(i)}B_{(j)}], [B_{(i)}v_{(1)}v_{(2)}], i \le j,$ $h_{123;pq}, p \le q$	$(2,2,2,2,2)$
5	$3\varphi_2+\varphi_1+\varphi_3$	$[B_{(i)}B_{(j)}], i \le j, \alpha(v), h_{123;11},$ $[B_{(k)}B^1_{(l)}v](B^2_{(l)},\alpha), k<l,$ $[B_{(1)}B^1_{(2)}B^1_{(3)}](B^2_{(2)},\alpha)(B^2_{(3)},\alpha)$	$(2,2,2,2,2)$
6	$\varphi_2^2+\varphi_1$	$(2,0),(3,0),(4,0),(5,0),(6,0)$ $(3,4),(5,4),(6,4),(7,4),(9,4)$	$(30,16)$
7	$\varphi_2^2+\varphi_2$	$(2,0),(3,0),(4,0),(5,0),(6,0)$ $(0,2),(1,2),(2,2),(3,2),(4,2),(5,2),$ $f_L(15,6)$	$f_L^2 = F(\cdots)$

Table 5. The h-s representations of SL_5.

	ρ	Generators	Syzygy
1	$2\varphi_2+3\varphi_4$	$[B_{(1)}B_{(2)}B^1_{(i)}](\alpha_{(j)},B^2_{(i)}),$ $B_{(i)}(\alpha_{(j)},\alpha_{(k)}), i=1,2, j<k$	$(2,2,1,1,1)$
2	$3\varphi_2+\varphi_1$	$[B_{(i)}B_{(j)}B^1_{(i)}][B_{(i)}B_{(k)}B^2_{(i)}],$ $[B_{(j)}B_{(k)}B^1_{(k)}][B_{(j)}B_{(i)}B^2_{(k)}], j<k,$ $i \ne j, i \ne k, [B_{(l)}B_{(m)}v], l \le m$	$(4,4,4,1)$
3	$2\varphi_2+\varphi_3+\varphi_1$	$\mathrm{tr}(B_{(1)}D)^p(B_{(2)}D)^q, p+q=1,2,$ $[B_{(i)}B_{(i)}B^1_{(j)}](B^2_{(j)},D_1)(v,D_2), i \ne j,$ $[B_{(l)}B_{(m)}v], l \le m,$ $[B_{(1)}B_{(1)}B^1_{(2)}][B_{(2)}B_{(2)}B^1_{(1)}](B^2_{(1)},D_1)*$ $*(B^2_{(2)},D_2), v^r v^s B^{kl}_{(1)}B^{mn}_{(2)}D_{kr}D_{ms}D_{ln}$	$(4,4,4,2)$

Table 6. The h-s representations of SL_6.

$d = [QQ_1^{12}Q_2^3][QQ_2^{12}Q_1^3], e = \mathrm{pf}B, f = [QBQ^3][QBB^1][QQ^{12}B^2],$
$g_i = [QBB^1](B^2,\alpha_{(i)}),\ h_i = [QQ^{12}B_1^1][QQ^3B_1^2B_2^1](B_2^2,\alpha_{(i)}),$
$p_{ij} = [QQ^{12}B^1](B^2,\alpha_{(i)})(Q^3,\alpha_{(j)})$.

	ρ	Generators	Syzygy
1	$\varphi_3+\varphi_2+2\varphi_5$	$d,e,f,B(\alpha_{(1)},\alpha_{(2)}),g_i,h_i,p_{ij},$ $[QBQ_1^1][QQ_1^2Q_2^{12}](Q_1^3,\alpha_{(1)})(Q_2^3,\alpha_{(2)})$	$(8,5,2,2)$
2	$\varphi_3+\varphi_2+\varphi_1+\varphi_5$	$d,e,f,g_1,h_1,p_{11},\alpha(v),[QBv],$ $[QQ^{12}v][QBQ^3],[QB_1^1B_2^1v][QB_1^2B_2^2v],$ $[QQ^{12}v](Q^3,\alpha),(2,3,1,1),(4,3,1,1)$	$(8,6,2,2)$

Table 7. The h-s representations of SL_7.

$d = [QQ_1^{12}Q_2^3Q_4^1][QQ_2^{12}Q_3^3Q_4^2][QQ_3^{12}Q_1^3Q_4^3]$,
$p = [QQ_1^{12}Q_2^3v_{(1)}][QQ_2^{12}Q_3^3v_{(2)}][QQ_3^{12}Q_1^3v_{(3)}]$, $r_{klm} = Q(\alpha_{(k)}, \alpha_{(l)}, \alpha_{(m)})$,
$f_{ij} = [QQ_1^{12}Q_2^3v_{(i)}][QQ_2^{12}Q_1^3v_{(j)}]$, $g_{ij} = [QQ_1^{12}Q_2^{12}](Q_1^3, \alpha_{(i)})(Q_2^3, \alpha_{(j)})$,
$h_{i;ts} = [QQ_1^{12}Q_2^{12}][QQ_3^{12}Q_1^3v_{(i)}](Q_2^3, \alpha_{(t)})(Q_3^3, \alpha_{(s)})$.

	ρ	Generators	Syzygy
1	$\varphi_3 + 3\varphi_1$	$d, p, f_{ij}, i \le j, e(5,2,2,2)$	$p^2 = de + F(f_{ij})$
2	$\varphi_3 + 2\varphi_1$ $+2\varphi_6$	$d, f_{ij}, g_{ij}, i \le j, \alpha_{(k)}(v_{(i)}), h_{i;12}$, $[QQ^{12}v_{(1)}v_{(2)}](Q^3, \alpha_{(i)}), (7,1,1,1,1)$	$(14,2,2,2,2)$
3	$\varphi_3 + \varphi_1$ $+3\varphi_6$	$d, f_{11}, g_{ij}, i \le j, \alpha_{(i)}(v), h_{1;ts}, t < s, r_{123}$, $[Q_1^{12}Q_2^{12}Q_3^{12}v](Q_1^3, \alpha_{(1)})(Q_2^3, \alpha_{(2)})(Q_3^3, \alpha_{(3)})$	$(13,2,2,2,2)$
4	$\varphi_3 + 4\varphi_6$	$d, g_{ij}, i \le j, r_{klm}, k < l < m, (6,1,1,1,1)$	$(12,2,2,2,2)$
5	$\varphi_3 + \varphi_2$	$d, (1,2), (8,2), (2,4), (9,4)$, $(3,6), (10,6), (10,6), f_L(15,9)$	$f_L^2 = F(\cdots)$
6	$\varphi_3 + \varphi_5$	$d, (6,2), (6,2), (5,4), (5,4)$, $(4,6), (4,6), (11,6), f_L(13,9)$	$f_L^2 = F(\cdots)$

Table 8. The h-s representations of SL_8.

	ρ	Generators	Syzygy
1	$\varphi_3 + 2\varphi_1$	$(16,0,0), (10,i,2-i), i = 0,1,2$, $(7,j,3-j), j = 0,1,2,3, (4,2,2), f_L(22,3,3)$	$f_L^2 = F(\cdots)$
2	$\varphi_3 + \varphi_1 +$ $+\varphi_7$	$(16,0,0), (10,2,0), (0,1,1), (6,0,2), (7,3,0)$, $(13,2,1), (3,1,2), (9,0,3), (16,2,2), f_L(24,3,3)$	$f_L^2 = F(\cdots)$
3	$\varphi_3 + 2\varphi_7$	$(16,0,0), (6,i,2-i), i = 0,1,2$, $(9,j,3-j), j = 0,1,2,3, (12,2,2), f_L(10,3,3)$	$df_L^2 = F(\cdots)$

Table 9. The known c.i. representations of SL_n with hd ≥ 2.
Set $m = \frac{n}{2}$ for n even, $m = \frac{n-1}{2}$ for n odd.

	n	ρ	hd
1	≥ 3	$(n+1)\varphi_1 + r\varphi_1^*$ $2 \leq r < n$	r
2	≥ 3	$\varphi_1^2 + 2\varphi_1 + (n-2)\varphi_1^*$	2
3	≥ 4	$\varphi_2 + 2\varphi_1 + n\varphi_1^*$	2
4	≥ 4	$\varphi_2 + 4\varphi_1 + (n-2)\varphi_1^*$	2
5	≥ 4	$2\varphi_2 + k\varphi_1 + l\varphi_1^*$ $k + l = 4, k \geq 2$ if n is odd	$3,$ if $n = l = 4$ $m,$ if $n = 2m < 2l + 2$ $n - l - 1,$ else
6	≥ 5	$\varphi_2 + \varphi_2^* + k\varphi_1 + l\varphi_1^*$ $k + l = 4, k \geq l$ $l \geq 1$ if n is odd	$m,$ if $(k,l) = (2,2)$ $m,$ if $(k,l) = (3,1), n = 2m$ $m + 1,$ else
7	≥ 4	$\varphi_1^2 + \varphi_2 + k\varphi_1 + l\varphi_1^*$ $k + l = 2, k \geq 1$ if n is odd	$2,$ if $n = 4, (k,l) = (0,2)$ $n - l - 1,$ else
8	≥ 5	$\varphi_1^2 + \varphi_2^* + k\varphi_1 + l\varphi_1^*$ $k + l = 2, l \geq 1$ if n is odd	$m,$ if $(k,l) = (0,2)$ $m,$ if $(k,l) = (1,1), n = 2m$ $m + 1,$ else
9	2	$2\varphi^3$	2
10	4	$\varphi_1^2 + 3\varphi_2$	2
11	4	$\varphi_1\varphi_3 + 2\varphi_2$	3
12	6	$\varphi_3 + l\varphi_1 + m\varphi_5$ $(l,m) = (3,3), (4,1), (4,2)$	$2,$ if $(l,m) = (3,3), (4,1),$ $4,$ if $(l,m) = (4,2)$
13	6	$\varphi_3 + \varphi_2 + 2\varphi_1$	2

List 10. Representations of SL_n that may be c.i.

$SL_n, n \geq 5$ odd: $\varphi_1^2 + \varphi_2 + 2\varphi_1^*$, $2\varphi_2 + \varphi_1 + 3\varphi_1^*$, $2\varphi_2 + 4\varphi_1^*$,
SL_5: $3\varphi_2 + \varphi_4$, $2\varphi_2 + \varphi_3 + \varphi_4$, SL_7: $\varphi_3 + 3\varphi_1 + \varphi_6$.

2. Classical methods.

Recall some classical results on invariants of SL_n (see, e.g. [17, §9]). In this section, vectors are denoted by characters x, y, z, etc., covectors by Greek letters.

Theorem 2.1. *A minimal system of generators of* $\mathbf{k}[l\mathbf{k}^n + m\mathbf{k}^{n*}]^{SL_n}$ *is:*

$$\alpha(x), [\alpha, \cdots, \beta], [x, \cdots, y].$$

The ideal of syzygies have generators of the form:

$$[x, y, \cdots, z]l(u) = [u, y, \cdots, z]l(x) + \cdots + [x, y, \cdots, u]l(z), \tag{1}$$

where either $l(u) = \alpha(u)$ *or* $l(u) = [u, v, x, \cdots, w]$*, similar relations for* α*-s, and*

$$[\cdots, v_i, \cdots][\cdots, \alpha_j, \cdots] = \det(\alpha_i(v_j)).$$

Consider now a representation of SL_n in the space $V = V_1 + \cdots + V_n$, where coordinates on each V_i is a tensor T_i over $\mathbf{k}^n$.

Theorem 2.2. *The linear space* $\mathbf{k}[V]^{SL_n}$ *is generated over* $\mathbf{k}$ *by the complete contractions of copies of the tensors* $T_1, \cdots, T_n$ *with copies of the covariant tensor* det *and the contravariant tensor* $\det^{-1}$ *such that either* det *or* $\det^{-1}$ *is not involved in the contraction.*

Furthermore, in some cases one can reduce the set of contractions that give generators. In [1] this was done for several series of representations. Unfortunately, this paper is not available in English and also the text is not completely clear even for a Russian reader. Therefore we present these results below.

Lemma 2.3. *A minimal system of generators of* $\mathbf{k}[\wedge^2\mathbf{k}^n + l\mathbf{k}^n + m\mathbf{k}^{n*}]^{SL_n}$ *is:*

$$\alpha(x), [\alpha \cdots \beta], B(\alpha, \beta), \text{and}$$

$$\text{for n even}: [B \cdots B], [B \cdots Bx \cdots y], \cdots, [x \cdots y],$$

$$\text{for n odd}: [B \cdots Bx], [B \cdots Bxyz], \cdots, [x \cdots y].$$

Proof. Let f be a polyhomogeneous generator of the algebra of invariants. By 2.2, f can be represented by a contraction of copies of tensors $B^{ij}, x, \cdots, y, \alpha, \cdots \cdots, \beta, \det, \det^{-1}$. We say that a copy B_p connects two copies of det, if the indices of B_p are contracted with these two copies. To prove that the above invariants generate the algebra of invariants, it is sufficient to present f by a contraction such that a det has no connections with other det-s.

The classical idea consists of two steps. First we polarize f, that is, we substitute in the contraction q "different" tensors B instead of q copies of B, where q is the degree of f by B. This new point of view makes f a multilinear invariant. Secondly we note that, as a multilinear invariant, f is completely determined by its action on "decomposable" tensors $B_p = b^{i_p} \otimes b^{j_p} - b^{j_p} \otimes b^{i_p}$. Substituting such $B_1, \cdots, B_q$ in the contraction, we obtain a linear combination of contractions of vectors $b^{i_1}, b^{j_1}, \cdots, b^{i_q}, b^{j_q}, x, \cdots, y$, covectors $\alpha, \cdots, \beta$, and det-s, skew by b^{i_p}, b^{j_p} for any p. Assume that $q > 0$ and some b is contracted with a det. Then f can be written as follows:

$$f = [b^{i_2}, b^{j_2}, \cdots, b^{i_p}, b^{j_p}, b^{i_1}, \cdots] l(b^{j_1}) - [b^{i_2}, b^{j_2}, \cdots, b^{i_p}, b^{j_p}, b^{j_1}, \cdots] l(b^{i_1}).$$

We apply the formula (1) and obtain:

$$f = [b^{j_1}, b^{j_2}, \cdots, b^{i_p}, b^{j_p}, b^{i_1}, \cdots] l(b^{i_2}) + [b^{i_2}, b^{j_1}, \cdots, b^{i_p}, b^{j_p}, b^{i_1}, \cdots] l(b^{j_2}) + \cdots.$$

Now the sum of the first and the second summands equals $-f$, if we forget the polarization. The same is true for the sum of the third and the fourth summands etc. Therefore we obtain: $f = -(2p+1)f + g$, where g is a contraction such that $b^{i_1}, b^{j_1}, \cdots, b^{i_p}, b^{j_p}$ are contracted with the same det. Thus, $f = \frac{g}{2p+2}$, and we can "cut off" all the connections. That this system of generators is minimal, follows from a consideration of their polydegrees. ■

Lemma 2.4. *A minimal system of generators of* $\mathbf{k}[S^2\mathbf{k}^n + l\mathbf{k}^n + m\mathbf{k}^{n*}]^{SL_n}$ *is:*

$$\alpha(x), [\alpha \cdots \beta], [x \cdots y], A(\alpha, \beta),$$
$$[A_1^1 \cdots A_p^1 x \cdots y][A_1^2 \cdots A_p^2 z \cdots w], p \geq 1,$$
$$[A_1^1 \cdots A_l^1 x \cdots y](A_1^2, \alpha) \cdots (A_l^2, \beta).$$

Proof. Note that if both indices of a copy of A are contracted with the same det, then the contraction equals zero. Hence, it is sufficient to prove, that if in a contraction giving a generator there is at least 2 det-s, then it equals $[A_1^1 \cdots A_p^1 x \cdots y][A_1^2 \cdots A_p^2 z \cdots w]$. An application of the formula (1) as in the proof of 2.3 completes the proof. ■

Lemma 2.5. *A system of generators of* $\mathbf{k}[2 \wedge^2 \mathbf{k}^n + l\mathbf{k}^n + m\mathbf{k}^{n*}]^{SL_n}$ *is (we set* $A = B_{(1)}, B = B_{(2)}$*):*

$$\alpha(x), [\alpha \cdots \beta], A(\alpha, \beta), B(\alpha, \beta),$$
$$[x \cdots yA \cdots AB \cdots BB_1^1 \cdots B_p^1](B_1^2, \alpha) \cdots (B_p^2, \beta). \tag{2}$$

Proof. Take any contraction with A-s contracted with a det. As in the proof of 2.3, we can transfer to these brackets all the a^{j_t}-s such that a^{i_t}-s are already there. Then we transfer all the b^{i_s}-s from these brackets to their b^{j_s}-s that are in other brackets. Clearly, the A-s and B-s that were completely in the first det before the latter procedure, remain there after it and we can repeat the procedure for A-s, then for B-s. Finally, we obtain one of the above generators. ■

Remark. All the above generators except some invariants from (2) with $p > 0$ can be included in a minimal system of generators. On the other hand, 2.3 implies $[xB \cdots BB^1](B^2, \alpha) = c\alpha(x)[B \cdots B], c \in \mathbf{k}$. Since the set of polydegrees for a minimal system of generators is symmetric with respect to the transposition of two φ_2 factors, an invariant from (2) can be included in a minimal system of generators only if it involves $\geq p$ copies of A. Sometimes this is also a sufficient condition (see § 5.).

Lemma 2.6. *For* $V = \wedge^2\mathbf{k}^n + \wedge^2\mathbf{k}^{n*} + l\mathbf{k}^n + m\mathbf{k}^{n*}$ *a minimal system of generators of* $\mathbf{k}[V]^{SL_n}$ *is:*

$$[\alpha \cdots \beta D \cdots D], [B \cdots Bx \cdots y], \operatorname{tr}(BD)^s, 2 \leq 2s \leq n-1,$$
$$\alpha((BD)^t x), 0 \leq 2t \leq n-2, D((BD)^p x, y), B(\alpha(BD)^p, \beta), 0 \leq 2p \leq n-3.$$

Proof. Consider a contraction giving a generator f. If a det is involved in the contraction, then, arguing as in the proof of 2.3, we may change the contraction to have that a det have no connections with other det-s. Then this f is nothing but $[B \cdots Bx \cdots y]$. The same is true for $\det^{-1}$. Furthermore, in any contraction without det and $\det^{-1}$ we can find a maximal chain like $ADA \cdots D$ such that any two neighbors are contracted with each other. The two extreme indices are contracted with each other (this gives $\mathrm{tr}(BD)^s$), or with x, y ($D((BD)^p x, y)$), or with α, β ($B(\alpha(BD)^p, \beta)$), or with α, x ($\alpha((BD)^t x)$). Which s, t, p give members of a minimal system of generators follows from a consideration in [18] of coregular subrepresentations with $(l, m) = (2,0), (1,1)$. ■

Lemma 2.7. *A system of generators of* $\mathbf{k}[S^2\mathbf{k}^n + \wedge^2\mathbf{k}^n + l\mathbf{k}^n + m\mathbf{k}^{n*}]^{SL_n}$ *is:*

$$\begin{gathered}\alpha(x), A(\alpha), A(\alpha,\beta), B(\alpha,\beta),\\ [\alpha,\cdots,\beta], [B\cdots Bx\cdots yA_1^1\cdots A_p^1](A_1^2,\alpha)\cdots(A_p^2,\beta),\\ [A_1^1\cdots A_l^1 B\cdots Bx\cdots yB_1^1\cdots B_q^1](B_1^2,\alpha)\cdots(B_q^2,\beta)*\\ *[A_1^2\cdots A_l^2 B\cdots Bx\cdots zB_{s+1}^1\cdots B_r^1](B_{s+1}^2,\alpha)\cdots(B_r^2,\gamma), l>0.\end{gathered} \tag{3}$$

Proof. Let f be a generator of the algebra of invariants. If f is a contraction without det-s or having only one det, then as in the proof of 2.3, we prove that f is one of the above invariants. Assume now that f is a contraction involving A and at least two det-s. Then as in the proof of 2.4, we can transform $f = [A_1^1\cdots A_s^1\cdots][A_1^2\cdots A_s^2\cdots]\cdots$ such that $A_1,\cdots,A_s$ are all the A-s contracted with both det-s. Further, as in the proof of 2.3, we transfer all the b^{i_d}-s from both det-s to their b^{j_d}-s that are outside these two brackets. We repeat the first procedure, then the second one and so on. In the end we will have that both our det-s have no connection with other det-s; since f is a generator, there are only two det-s. Moreover, for any b^{i_d} contracted with one of two det-s, b^{j_d} is either in the same det, or is contracted with α, or is in the other det. In the first and the second cases we are done. In the third case we apply the formula (1) to our contraction to bring b^{j_d} in the first det and proceed as in the proof of 2.3 (note that terms containing $[\cdots A_c^1\cdots A_c^2\cdots]$ are zero). ■

Lemma 2.8. *A system of generators of* $\mathbf{k}[S^2\mathbf{k}^n + \wedge^2\mathbf{k}^{n*} + l\mathbf{k}^n + m\mathbf{k}^{n*}]^{SL_n}$ *is:*

$$[D\cdots D\alpha\cdots\beta], [A_1^1\cdots A_k^1 x\cdots y][A_1^2\cdots A_k^2 x\cdots z],$$

$$\mathrm{tr}(AD)^{2s}, 2\le 2s\le n-1, \tag{4}$$

$$D(x,(AD)^{2t+1}x), 0\le 2t\le n-3, A(\alpha,\alpha(AD)^{2u}), 0\le 2u\le n-2, \tag{5}$$

$$D(x,(AD)^p y), A(\alpha,\beta(AD)^q), \alpha((AD)^r x), \tag{6}$$

and, if $l>0$, some contractions involving exactly one det.

Proof. Arguing as in the proofs of 2.3 and 2.4, we have that, if a contraction of a generator f involves $\det^{-1}$-s or at least two det-s, then f is one of the above invariants. If f involves exactly one det and $f = [A_1^1\cdots A_n^1]\cdots$, then f vanishes on the zero level of $\det A$; since the latter is irreducible, f is not a generator. If the contraction does not contain det and $\det^{-1}$, then as in the proof of 2.6, we see that f is one of the invariants from (4),(5),(6). The restrictions on s in (4) and on t, u in (5) follow from a description in [18], [22] of the algebra of invariants for the subrepresentations with $(l, m) = (1,0), (0,1)$. ■

3. Necessary Conditions.

Let $G \subseteq GL(V)$ be a reductive algebraic group. The algebra $\mathbf{k}[V]^G$ is finitely generated and the affine variety $V/\!\!/G = \operatorname{Spec} \mathbf{k}[V]^G$ is called the *quotient* whereas the natural morphism $\pi_V : V \to V/\!\!/G$ is called the *quotient map.* For any point v of a closed orbit $Gv \in V$ the isotropy group G_v is reductive by the Matsushima theorem. Choose a G_v-invariant complementary vector subspace N_v to $\mathrm{T}_v(Gv)$ in $\mathrm{T}_v(V)$. The representation (G_v, N_v) is called the *slice representation* of v. By Luna's slice theorem [12], the natural morphism $(v + N_v)/\!\!/G_v \to V/\!\!/G$ is étale at $\pi_{v+N_v}(v)$. This implies:

Proposition 3.1.

1. ([16, Theorem 1.2]) $\mathrm{hd}\mathbf{k}[N_v]^{G_v} \leq \mathrm{hd}\mathbf{k}[V]^G$

2. ([2]) *If $V/\!\!/G$ is a c.i., then $N_v/\!\!/G_v$ is a c.i. as well.*

Let V_1, V_2 be G-modules. Choose a minimal system of polyhomogeneous generators of $\mathbf{k}[V_1 + V_2]^G$. Clearly, the elements of this system of degree 0 with respect to V_2 constitute a minimal system of homogeneous generators for $\mathbf{k}[V_1]^G$. The same is true for the syzygies. Then from the definition of c.i. we obtain:

Proposition 3.2. *If $(G, V_1 + V_2)$ is a c.i., then (G, V_1) is a c.i. as well.*

Using 3.1 and 3.2, we reduce in most cases the question about c.i. property of a representation of SL_n to that for a finite group, a 1-dimensional torus, or SL_2. For finite groups we have:

Lemma 3.3. ([6], [10]) *Suppose that G is finite and $V/\!\!/G$ is a c.i. Then G is generated by elements r such that* $\mathrm{rk}(r - \mathrm{Id}) \leq 2$.

For 1-dimensional torus we use the following easy observation.

Proposition 3.4. *The representations of $\mathbf{k}^*$ defined by the following systems of weights are not c.i.:*

$$(t,t,t,t^{-1},t^{-1},t^{-1}), (t,t,t^2,t^{-1},t^{-1},t^{-2}), (t,t^2,t^2,t^{-1},t^{-2},t^{-2}).$$

Now we explain how we got our lists. Any irreducible factor of a c.i. representation is c.i. by 3.2; by [14], [22] such a factor is either coregular (see [9]) or $(SL_2, \varphi^k), k = 5, 6$, (SL_4, φ_1^3). The remaining candidates for reducible c.i. representations were found by applying 3.1, 3.2 to sums of the above irreducible factors. The absence of the c.i. property for slice representations follows from 3.3, 3.4, and the classification of c.i. representations of SL_2 from [14]. However there are some special cases such that 3.1 is not sufficient. Then we apply results of § 2. together with arguments using Poincaré series.

For $\alpha = (\alpha^1, \cdots, \alpha^r) \in \mathbf{N}^r$ denote by t^α the monomial $t_1^{\alpha^1} \cdots t_r^{\alpha^r}$. For a $\mathbf{N}^r$-graded finitely generated algebra A consider the Poincaré series of A: $\bigoplus_{\alpha \in \mathbf{N}^r} \dim A_\alpha t^\alpha$. Recall that this is the Taylor series at 0 of a rational function P_A. Denote by $q(A)$ the difference of degrees of the denominator and the numerator of P_A.

Lemma 3.5. *Let A be a $\mathbf{N}^r$-graded algebra generated by homogeneous elements of degrees $\alpha_1, \cdots, \alpha_n$ such that the ideal of syzygies is generated by elements of degrees $\beta_1, \cdots, \beta_m$. If A is a c.i., then the Poincaré series of A is of the form:*

$$P_A = \frac{(1-t^{\beta_1})\cdots(1-t^{\beta_m})}{(1-t^{\alpha_1})\cdots(1-t^{\alpha_n})}, q(A) = \alpha_1 + \cdots + \alpha_n - \beta_1 - \cdots - \beta_m.$$

Proof. For a $\mathbf{N}^r$-graded algebra B and a homogeneous element $b \in B$ of degree β such that b is not zero divisor, we have for the Poincaré series: $P_{B/(b)} = P_B - P_{(b)} = P_B(1-t^\beta)$. Since the ideal of syzygies of A is generated by a regular sequence, we get the claim by applying the above equality m times. ■

Now we consider the special cases.

$\rho = (n+2)\varphi_1$. By 2.1, $\mathbf{k}[V]^{SL_n}$ is generated by $\binom{n+2}{2}$ elements and the ideal of syzygies is generated by $\binom{n+2}{4}$ elements. We have tr.deg.$\mathbf{k}[V]^{SL_n} = 2n+1$; hence, ρ is not a c.i.

$\rho = \varphi_1^2 + 3\varphi_1$. By 2.4, $\mathbf{k}[V]^{SL_n}$ is generated by 14 generators of degrees: $(n,0,0,0)$, $(n-1,i,j,k), i+j+k=2$, $(n-2,k,l,m), k+l+m=4, k,l,m \leq 2$, and either $(n-3,2,2,2)$ for $n \geq 4$ or $(0,1,1,1)$ for $n=3$. Moreover, we have tr.deg.$\mathbf{k}[V]^{SL_n} = 7$. From the syzygy of entry 5, Table 1 we get here 6 syzygies of degrees $(2n-2,k,l,m), k+l+m=4, k,l,m \leq 2$. Assume that ρ is a c.i. Then its Poincaré series is like in 3.5 and we have: $q(\mathbf{k}[V]^{SL_n}) = (2n-9,6,6,6) - \gamma$ for $n \geq 4$, and $q(\mathbf{k}[V]^{SL_3}) = (-3,5,5,5) - \gamma$, where γ is the degree of the seventh generator of the ideal of syzygies. By [11], $q(\mathbf{k}[V]^{SL_n}) \in \mathbf{N}^4$ and $q(\mathbf{k}[V]^{SL_n}) = (4n-6,4,4,4)$ for $n \geq 5$. A contradiction.

$\rho = \varphi_2 + 5\varphi_1$. By 2.3, $\mathbf{k}[V]^{SL_n}$ is generated by 16 polynomials. We have tr.deg.$\mathbf{k}[V]^{SL_n} = 11$. From the syzygy of entry 9, Table 1 we get here 5 syzygies of degrees $(n-2,1,1,1,1,0), \cdots, (n-2,0,1,1,1,1)$. As above, we assume that ρ is a c.i., calculate $q(\mathbf{k}[V]^{SL_n})$, and obtain a contradiction with [11].

Therefore all the non-coregular representations of SL_n but those appearing in Tables 1-9 and in List 10 above are not c.i. The estimate hd$\mathbf{k}[V]^{SL_n} \geq 2$ for the representations from List 10 can be checked either by applying 3.1.1 or (for SL_5 and SL_7) by constructing sufficiently many generators. On the other hand, one can prove:

Proposition 3.6. *For any representation ρ from List 10 and any closed orbit $SL_n v \subseteq V, v \neq 0$ the slice representation $((SL_n)_v, N_v)$ is a c.i.* □

4. Proof of Theorem 0.1.

Cases considered in other papers.

From now on, "case $n.m$" will refer to entry m from Table n.

Table 1. Cases 1.1-1.9 follow from 2.3, 2.4. Cases 1.10-1.13, 1.15 are considered in [22]. Case 1.14 is a subrepresentation of one of representations

from the Table in [15, 4.3]; hence, it is a c.i. Moreover, Panyushev's deformation arguments prove that it is a h-s[3] and the generators are easy to find.

Table 2. Representations of SL_2 were studied by Classical Invariant Theory of the 19-th century and all the representations in Table 2 were considered, e.g. in [7] (cf. also [21]).

Tables 3-8. Cases 3.1, 3.3, 3.4, 7.1 see in [22]. Cases 7.5, 7.6, 8.1, 8.2 see in [23]. Case 4.1 see in [2]. Since $(SL_4, \varphi_2) = (SO_6, \mathbf{k}^6)$, case 4.2 is classical and case 4.7 is contained in [22], Table 1.9. Case 5.1 see in [18, Table 1a'].

Further, a representation from the Tables can sometimes be obtained as a slice-representation of another one, up to trivial factors. Then by 3.1, we reduce to prove the h-s property for the second one. We present such relations in the following diagrams with arrows directed from a representation to a slice-representation:

$$7.2 \to 6.2 \to 5.3, \qquad 7.3 \to 6.1 \to 4.5, \qquad 7.4 \to 4.4.$$

Case 3.2. Case 3.2 can be considered as follows. For the group Sp_6 let φ_i denote the highest weight irreducible factor of $\wedge^i \mathbf{k}^6$. Let (G, V) be the finite extension of the group $(Sp_6, \varphi_3 + \varphi_2)$ by the operator acting by multiplication by $\sqrt{-1}$ on the first factor and trivially on the second one. Then by [22], (G, V) is coregular. Moreover, using the method of [23], one can prove that $V/\!\!/G$ has the unique codimension-1 Luna stratum corresponding to an isotropy group L of order 2. It turns out that $(N_{Sp_6}(L), V^L)$ is isomorphic to $(SL_3, \varphi_1\varphi_2 + \varphi_1^2)$. Using the technique of [23], one can prove that the closure of this stratum is normal; hence, by [13], $\mathbf{k}[V^L]^{N_{Sp_6}(L)}$ is isomorphic to the quotient of $\mathbf{k}[V]^G$ by the principal ideal vanishing on the stratum. By [23, 1.17], this ideal is generated by f_L^2, where f_L is the unique generator of $\mathbf{k}[V]^{G^0}$ that is not G-invariant. This f_L is nothing but h in the notation of entry 24 of Table 1 in [22]. This completes the proof.

Method for other cases. Thus we reduced to prove cases 3.5, 3.6, 4.3, 4.6, 5.2, 7.2, 7.3, 7.4, 8.3. In all these cases but 4.6 we apply a criterion from [23] for the algebra of invariants to be a hypersurface. For convenience of a reader we formulate it below.

Let G be an algebraic group with a semisimple identity component G^0. Suppose that V is a coregular G-module such that $\mathbf{k}[V]^G = \mathbf{k}[V]^{G^0} = \mathbf{k}[f_1, \cdots, f_s]$ and the nilcone $\mathcal{N} \subseteq V$ contains a dense orbit $G^0 z$. Let H be a principal isotropy group of (G, V); this means that generic *closed* G-orbits in V are isomorphic to G/H. Moreover, assume that generic G-orbits in V are closed, in other words, H is a generic isotropy group. Consider a G-module U having r irreducible factors so that $\mathbf{k}[U]$ carries a $\mathbf{N}^r$-grading preserved by the action of G. Define a natural partial order in $\mathbf{N}^r$ as follows: $\alpha \preceq \beta$ if and only if $\alpha^i \leq \beta^i$ for $i = 1, \cdots, r$, $\alpha \prec \beta$ means $\alpha \preceq \beta$ and $\alpha \neq \beta$. For any $f \in \mathbf{k}[V + U]$ denote by $\overline{f}$ the restriction of f to the subvariety $z + U \subseteq V + U$.

[3]Moreover, the representations of cases 1.2 with $n = 4$, 1.8 with $n = 4$, 1.9 with $r \leq 2$, 1.13, 1.14 are lacking in the Table of [15, 4.3]

Lemma 4.1. ([23, 4.5])

Assume that:

(a) $\mathbf{k}[U]^H$ *is a h-s generated by elements of degrees* $\beta_1, \cdots, \beta_t \in \mathbf{N}^r$ *with the unique syzygy of degree* $\beta \in \mathbf{N}^r$

(b) there exist $g_1, \cdots, g_t \in \mathbf{k}[V+U]^G$ *of degrees* $(\alpha_1, \beta_1), \cdots, (\alpha_t, \beta_t)$ *such that* $\mathbf{k}[\overline{g_1}, \cdots, \overline{g_t}]$ *is a h-s with the unique syzygy of degree* $\gamma \in \mathbf{N}^r$, $\gamma \not\prec \beta$.

Then $\mathbf{k}[V+U]^G = \mathbf{k}[f_1, \cdots, f_s, g_1, \cdots, g_t]$ *is a h-s.*

Example of application of Lemma 4.1. Now we show how one can apply Lemma 4.1. Consider case 7.2, i.e., the group

$$(G, V+U) = (SL_7, \varphi_3 + (2\varphi_1 + 2\varphi_6)).$$

For $(G,V) = (SL_7, \varphi_3)$, all the assumptions of Lemma 4.1 are fulfilled: $\mathbf{k}[V]^G = \mathbf{k}[d]$ and the orbit of $z = e_1\wedge e_2\wedge e_5 + e_3\wedge e_4\wedge e_6 + e_1\wedge e_3\wedge e_7 + e_2\wedge e_4\wedge e_7$ is dense in the zero level of d ([25]). By [5], H is a simple group of type G_2; therefore $(H,U) = (G_2, 4\mathbf{k}^7)$. By [19], $\mathbf{k}[U]^H$ is a h-s generated by invariants of the following degrees: (i,j,k,l), where either $i+j+k+l=2$ or $i,j,k,l = 0,1, i+j+k+l=3$, and $(1,1,1,1)$. By [11], $q(\mathbf{k}[U]^H) = (7,7,7,7)$. Hence, by 3.5, the degree of the unique syzygy is $(2,2,2,2)$. Therefore the hypothesis (a) of Lemma 4.1 is fulfilled. Moreover, the degrees of the invariants given in Table 7, entry 2 satisfy the hypothesis (b).

Denote by $x_1, \cdots, x_7$ the dual basis for $e_1, \cdots, e_7$. For $a, b_i, c_j, d_k \in \mathbf{k}$ set:

$$Z = (z, ae_7, b_1e_1 + \cdots + b_7e_7, x_1 + c_3x_3 + c_7x_7, d_1x_1 + d_2x_2 + d_3x_3 + x_4 + d_7x_7).$$

Using the explicit description of G_z, it can be shown that G_zZ is dense in $z+U$. Hence, in the hypothesis (b) of Lemma 4.1 we can replace the restrictions to $z+U$ by that to Z.

Let $h_{(p,q,r,s)}$ denote the generator from Table 7, entry 2 of degree $(*,p,q,r,s)$ with $0 < p+q+r+s < 4$. One can easily compute the restrictions of $h_{(p,q,r,s)}$ to Z as 14 polynomials in 14 variables a, b_i, c_j, d_k. To check that these are algebraically independent, we calculated their Jacobian (by an appropriate choice of polynomials and variables the Jacobian matrix is "almost triangular") and found it to be nonzero.

Therefore to check the hypothesis (b) of Lemma 4.1, we only need to prove the existence of $f \in \mathbf{k}[V+U]^G$ of degree $(*,1,1,1,1)$ such that the restriction of f to Z and the above 14 polynomials have no syzygies of degree $\prec (2,2,2,2)$.

We believe that it would be possible to find an explicit form for f. However we find it interesting to give a theoretical proof of the existence of f as follows. Let $\tilde{d}$ be the restriction of d to V^H. By [18, 3.13], the inclusion $V^H \subseteq V$ induces an isomorphism $\mathbf{k}[V+U]^G_d \cong \mathbf{k}[V^H+U]^{N_G(H)}_{\tilde{d}}$. Consider a generator $\tilde{f}$ of degree $(1,1,1,1)$ of $\mathbf{k}[U]^H$ as an element of $\mathbf{k}[V^H+U]$. Since $N_G(H) = HZ(G)$, $\tilde{f}$ is $N_G(H)$-invariant. Thanks to the above isomorphism, there exists $f \in \mathbf{k}[V+U]^G$ of degree $(7l,1,1,1,1), l \geq 1$ such that the restriction of f to V^H+U is $\tilde{d}^l\tilde{f}$. Moreover, assume that l is the minimal possible.

Besides, since $\mathbf{k}[U]^H$ has a unique generator of degree $(1,1,1,1)$, we can choose $\tilde{f}$ and f to be semi-invariant with respect to the group $S_2 \times S_2$ permuting the isomorphic G-factors of U.

Note that for any $p_1, \cdots, p_m \in \mathbf{k}[V+U]^G, R \in \mathbf{k}[T_1, \cdots, T_m]$ a relation $R(\overline{p_1}, \cdots, \overline{p_m}) = 0$ is equivalent to d dividing $R(p_1, \cdots, p_m)$ in $\mathbf{k}[V+U]$. In particular, since l is the minimal possible, $\overline{f}$ does not vanish.

Let R be a polynomial in 15 variables such that $F = R(h_{(p,q,r,s)}, f)$ is polyhomogeneous and $F \in (d)$. Assume also that $\deg F$ is the minimal possible; then $\deg F = (*, \gamma)$ for γ from Lemma 4.1. We see that $\gamma = (1,1,1,1)$ contradicts either $\tilde{f}$ being a generator of $\mathbf{k}[U]^H$ or the minimality of l. Since the set $\{h_{(p,q,r,s)}\}$ is $S_2 \times S_2$-stable and f is $S_2 \times S_2$-semi-invariant, F is $S_2 \times S_2$-semi-invariant as well. In particular, $\gamma = (m,m,n,n)$. Assuming $\gamma \prec (2,2,2,2)$, we have γ is either $(2,2,1,1)$ or $(1,1,2,2)$.

Assume $\gamma = (1,1,2,2)$. We write down the monomials in $f, h_{(p,q,r,s)}$ of degree $(*,1,1,2,2)$, and easily show that $l = 1$ and F must be of the form:

$$pfh_{(0,0,1,1)} + qh_{(1,1,0,0)}h^2_{(0,0,1,1)} + rh_{(1,1,0,0)}h_{(0,0,2,0)}h_{(0,0,0,2)} + sh_{(1,0,1,1)}h_{(0,1,1,1)},$$

where $p,q,r,s \in \mathbf{k}$, and either $r \neq 0$ or $s \neq 0$. Hence, the relation $\overline{F} = 0$ implies that the restriction of $h_{(0,0,1,1)}$ to Z divides that of $rh_{(1,1,0,0)}h_{(0,0,2,0)}h_{(0,0,0,2)} + sh_{(1,0,1,1)}h_{(0,1,1,1)}$ for a nontrivial pair (r,s). Using the explicit form of our invariants restricted to Z, we check this is wrong.

Similarly, we prove that $\gamma = (2,2,1,1)$ yields a contradiction. Therefore the hypothesis (b) holds and $\mathbf{k}[V]^G$ is a hypersurface. To find l, we consider the Poincaré series of $\mathbf{k}[V]^G$. By [11], $q(\mathbf{k}[V]^G) = (35,7,7,7,7)$. Hence, the degree of the syzygy is $(7l+7,2,2,2,2)$. We assume $l > 1$ and consider the monomials of this degree in the generators. It turns out that each monomial contains either f or d. This is impossible, because d and f are irreducible in $\mathbf{k}[V+U]$. Thus $l = 1$ and we are done.

The proofs for cases 3.5, 3.6, 4.3, 5.2, 7.3, 7.4 are similar and easier. We emphasize that such proofs are possible whenever one can calculate almost all generators restricted to a subvariety in $z + U$ intersecting generic G_z-orbits. For this reason, this method can not be applied directly to case 8.3.

Case 8.3. Case 8.3 resembles 8.1 and 8.2. For each of 8.1,8.2 the h-s property is a corollary of the fact that a finite extension of the group is coregular ([23]). The group 8.3 does not admit a coregular extension. Nevertheless, consider a nice finite extension as follows.

Let $(G^0, V) = (SL_8, \varphi_3 + 2\varphi_7)$ be the group 8.3. Let γ act by multiplication by λ^3 on the first factor of V and by λ^7 on the second and the third one, where λ is a primitive root of unity of degree 16. Note that γ^2 belongs to G^0 and consider a finite extension $G = G^0\langle\gamma\rangle$. Applying the technique of [23] to G, we prove that $V/\!\!/G$ has a unique Luna stratum C of codimension 1 such that the corresponding isotropy group L is of order 2. Further, the ideal I_L in $\mathbf{k}[V]^G$ vanishing on C is generated by f_L^2, where f_L can be defined as the unique generator of $\mathbf{k}[V]^{G^0}$ such that $\gamma(f_L) = -f_L$, the other generators of $\mathbf{k}[V]^{G^0}$ being G-invariant ([23, 1.17]). From this we deduce as in [23, case 5.3]: $\deg f_L = (10+16k, 3, 3), k \in \mathbf{N}$. Using the normality criterion from [23], we prove that the closure $\overline{C}$ is normal. Therefore by [13], the restriction of functions to V^L yields an isomorphism:

$$\mathbf{k}[\overline{C}] = \mathbf{k}[V]^G/I_L \cong \mathbf{k}[V^L]^{N_G(L)}.$$

Suppose that $\mathbf{k}[V^L]^{N_G(L)}$ is a h-s with generators of the degrees from Table 8, entry 3 (but $(10,3,3)$). Then the above isomorphism implies $\text{hd}\mathbf{k}[V]^G \leq 1$ and actually $V/\!\!/G$ is a h-s, because G is not coregular. Moreover, the generators of $\mathbf{k}[V]^G$ are f_L^2 and some invariants of the above degrees. Replacing f_L^2 by f_L, we get the generators of $\mathbf{k}[V]^{G^0}$. Note that the equality $p = 3(q+r)$ holds for the degree (p,q,r) of any generator but d, f_L. With the help of this observation, one can prove the form $df_L^2 = F(\cdots)$ of the syzygy. Applying [11] as above, we get $\deg f_L = (10,3,3)$ and we are done.

Thus we need to study the action

$$(N_G(L), V^L) = (SL_5 \times SL_3 \times \mathbf{k}^*, \theta \otimes t^{-15} + \varphi_2 \otimes \varphi_1 \otimes t + 2\varphi_4 \otimes \theta \otimes t^{-3}),$$

where θ stands for a trivial representation, t is a basic character of $\mathbf{k}^*$. We can not apply Lemma 4.1 here, because G is not semisimple. We adjust 4.1 to this case as follows. Let $\widetilde{V}, \widetilde{U} \subseteq V^L$ be the subspaces of the subrepresentations $\theta \otimes t^{-15} + \varphi_2 \otimes \varphi_1 \otimes t$ and $2\varphi_4 \otimes \theta \otimes t^{-3}$, respectively. Let $\widetilde{\mathcal{N}} \subseteq \widetilde{V}$ be the nilcone. Let $\widetilde{H}$ be a principal isotropy group of $(N_G(L), \widetilde{V})$; clearly, $\widetilde{H}$ is also a generic isotropy group. However $\widetilde{\mathcal{N}}$ is reducible and does not contain a dense $N_G(L)$-orbit. Denote by $\overline{f}$ the restriction of $f \in \mathbf{k}[V^L]$ to $\widetilde{\mathcal{N}} \times \widetilde{U}$. Then the hypotheses $(a), (b)$ of Lemma 4.1 make sense and one can check as in the proof of [23, 4.5] that these are sufficient for $\mathbf{k}[V^L]^{N_G(L)}$ being a h-s. We check the hypotheses (a) and (b) and prove that $\mathbf{k}[V^L]^{N_G(L)}$ is a h-s with generators of the degrees from Table 8, entry 3 (but $(10,3,3)$). This concludes the proof.

Case 4.6. Set $(G, U+W) = (SL_4, \varphi_2^2 + \varphi_1)$. Choose a basis of $\mathbf{k}^4$. Let V_4 be the group of Klein acting on $\mathbf{k}^4$ by permutations of the basis vectors. Denote by $H \subseteq G$ the finite subgroup generated by V_4 and the operators having a diagonal matrix in this basis with eigenvalues $\pm\sqrt{-1}$. One can easily check that H is a principal isotropy group of (G,U).

Furthermore, the algebra $\mathbf{k}[W]^H$ is a h-s. Namely, if x,y,z,w is the dual basis, then $\mathbf{k}[W]^H$ is generated by:

$$x^2y^2 + z^2w^2, x^2z^2 + y^2w^2, x^2w^2 + y^2z^2, x^4 + y^4 + z^4 + w^4, xyzw, \tag{7}$$

with a unique syzygy R of degree 16.

We applied the technique of [4] to obtain the Poincaré series for modules of covariants. Let M be the isotypic component of G-module $\mathbf{k}[U]$ of highest weight φ_1^4. Our calculations show that the Poincaré series of M has the form:

$$P_M = \frac{t^3 + t^5 + t^6 + t^7 + t^9}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)}.$$

From this one easily deduces that there exist elements of degrees $(3,4), (5,4)$, $(6,4)$, $(7,4), (9,4)$ of $\mathbf{k}[U+W]^G$ such that for generic point $u \in U$ the restriction of these to $u+W$ are the generators of $\mathbf{k}[u+W]^{G_u} \cong \mathbf{k}[W]^H$. Let $A \subseteq \mathbf{k}[U+W]^G$ be the subalgebra generated by these invariants and $\mathbf{k}[U]^G$. Then the above property implies that A is a h-s with the ideal of syzygies generated by a syzygy of degree $(a,16), a \in \mathbf{N}$.

Moreover, we calculated the Poincaré series P of the $\mathbf{N}^2$-graded algebra $\mathbf{k}[U+W]^G$; P is a fraction with numerator $1 - t^{30}s^{16}$ and denominator

$$(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)*$$

$$*(1-t^3s^4)(1-t^5s^4)(1-t^6s^4)(1-t^7s^4)(1-t^9s^4).$$

Assuming $a > 30$, we obtain $\dim A_{(30,16)} > \dim(\mathbf{k}[U+W]^G)_{(30,16)}$, a contradiction. Hence, $a \leq 30$, and if we prove the inequality $a \geq 30$, then the Poincaré series of $\mathbf{N}^2$-graded algebra A is equal to P. Hence, $A = \mathbf{k}[U+W]^G$ and we are done.

So we need to prove $a \geq 30$. Restricting the generators of A to the subspace $U^H + W$, we obtain some $N_G(H)$-invariant functions of the same degrees. Moreover, since generic orbit intersects $U^H + W$, the syzygies of these restrictions are the same as ones of the generators. Clearly, the restrictions of the generators of degree $(*, 4)$ are linear combinations of the polynomials from (7) with homogeneous coefficients in $\mathbf{k}[U^H]$. Replacing the polynomials from (7) by the variables $T_1, \cdots, T_5$, and substituting the above linear combinations in the syzygy of A, we obtain a polynomial of degree 4 in $T_1, \cdots, T_5$, with coefficients in $\mathbf{k}[U^H]$. Moreover, this polynomial has the form $Q(u)R(T_1, \cdots, T_5)$, where $Q \in \mathbf{k}[U^H]^N$ is a homogeneous polynomial of degree a, R is the syzygy of the polynomials from (7).

One can easily prove, $N_G(H)/H \cong S_6$ and the action $S_6 : U^H$ can be thought of as that of S_6 on the hyperplane $u_1 + \cdots + u_6 = 0$ in $\mathbf{k}^6$, where S_6 acts by permutations of $u_1, \cdots, u_6$. Let $D \in \mathbf{k}[U^H]^{S_6}$ be the discriminant:

$$D = \prod_{1 \leq i \neq j \leq 6} (u_i - u_j).$$

Then for $u \in U^H$, $D \neq 0$ if and only if $G_u = H$. In other words, the equation $D = 0$ defines a unique codimension 1 Luna stratum $\mathcal{D}$ of $U^H/N \cong U/\!\!/G$. We claim that the above polynomial Q vanishes on $\mathcal{D}$. This implies $a \geq 30$, since $\deg D = 30$ and Q is N-invariant.

Let us prove the claim. Take a generic point $u \in \mathcal{D}$, set $L = (N_G(H))_u \supsetneq H$. The action of L on the linear span T of the polynomials from (7) is not trivial. Indeed, otherwise we would have $\mathbf{k}[W]^L = \mathbf{k}[W]^H$ and this is impossible, since $L \supseteq H, L \neq H$. Substituting u in the generators of A, we obtain some elements of T^L. Assume that a basis of T is chosen such that T^L is generated by the first $m < 5$ elements. Then, substituting u in the relation, we obtain a polynomial in $T_1, \cdots, T_m$ with zero coefficients, since the first m basis elements of T are algebraically independent. But these coefficients are equal $Q(u)$. Thus $Q(u) = 0$. This completes the proof.

5. Proof of Theorem 0.2.

We prove that the representations in Table 9 are c.i. We also describe a minimal system of generators of $\mathbf{k}[V]^{SL_n}$.

For case 9.1, Theorem 2.1 yields generators and syzygies.

For cases 9.2, 9.3, 9.4, the generators are described in 2.3, 2.4 and we have $\mathrm{hd}\mathbf{k}[V]^{SL_n} = 2$. For the classical case 9.9 generators are described in [7] and $\mathrm{hd} = 2$ in this case, too. Then we apply:

Lemma 5.1. ([16], Remark to Proposition 1.5) *Let $G \subseteq GL(V)$ be a connected semisimple group such that $\mathrm{hd}\mathbf{k}[V]^G = 2$. Then $\mathbf{k}[V]^G$ is a c.i.*

For cases 9.5-9.8, the c.i. property and the upper bound $\mathrm{hd}\mathbf{k}[V]^{SL_n} \le n-1$ are proved in [15]. Applying 3.1, we find a lower bound $\mathrm{hd}\mathbf{k}[V]^{SL_n} \ge m$ for these cases. Minimal systems of generators are as follows.

For case 9.6, a minimal system of generators is described in 2.6.

For case 9.5, a minimal system of generators consists of all the invariants from 2.5 with the following restriction: in (2) we take the contractions involving $\ge p$ copies of A. This restriction is necessary in general (see Remark after 2.5). If n is odd, then all such invariants are members of a minimal system of generators for a subrepresentation. If n is even, we had to calculate these invariants in special points to prove that they can not be obtained from the invariants of smaller degrees.

In cases 9.7,9.8 we apply 2.7, 2.8 together with the results of [15]. For a reductive group G and two affine G-varieties X and Y, Panyushev introduced a special filtration of $\mathbf{k}[X\times Y]^G$ such that

$$\mathrm{gr}\mathbf{k}[X\times Y]^G \cong (\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T, \tag{8}$$

where U is a maximal unipotent subgroup of G, T is a maximal torus normalizing U, U^{op} is the opposite maximal unipotent group (this means $U\cap U^{op}=\{e\}$). Moreover, assume that $\mathbf{k}[X]$ and $\mathbf{k}[Y]$ carry G-stable $\mathbf{N}^l$- and $\mathbf{N}^m$-gradings, respectively. Then the above filtration respects the $\mathbf{N}^{l+m}$-grading of $\mathbf{k}[X\times Y]^G$. Therefore $\mathrm{gr}\mathbf{k}[X\times Y]^G$ inherits the $\mathbf{N}^{l+m}$-grading and (8) is an isomorphism of $\mathbf{N}^{l+m}$-graded algebras. Denote by $\mathcal{G}$ the set of polydegrees for a minimal system of generators of $(\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T$. Then $\mathbf{k}[X\times Y]^G$ has a (minimal or not minimal) system of generators with the same set $\mathcal{G}$ of polydegrees. Moreover, if the set of polydegrees for a minimal system of generators of $\mathbf{k}[X\times Y]^G$ does not contain $\gamma\in\mathcal{G}$, then $(\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T$ has a syzygy of polydegree $\preceq\gamma$, where $\preceq$ refers to the partial order from Lemma 4.1. For any group of cases 9.7,9.8 one can present $V=X\times Y$, where $\mathbf{k}[X]^U, \mathbf{k}[Y]^{U^{op}}$ are polynomial by [3], and check that $(\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T$ is a c.i. with $\mathrm{hd}=n-1$.

For case 9.8, the generators from 2.8 with polydegrees from $\mathcal{G}$ are the generators of algebras of invariants for proper subrepresentations (see [18] and cases 1.13, 1.5) and:

$$D(x,(AD)^p y), 0\le p\le n-3, A(\alpha,\beta(AD)^q), \alpha((AD)^r x), 0\le q,r\le n-2,$$

$$[P\cdots Pxy], [P\cdots PxA^1](A^2,\alpha), \text{where } P^{ij}=A^{ib}D_{bc}A^{cj}.$$

Thus we obtain either $\mathrm{tr.deg.}\mathbf{k}[V]^{SL_n}+m$ or $\mathrm{tr.deg.}\mathbf{k}[V]^{SL_n}+m+1$ generators. Taking into account the above estimate $\mathrm{hd}\mathbf{k}[V]^{SL_n}\ge m$, it is not difficult to prove that the above system of generators is minimal.

For case 9.7, the invariants from 2.7 with polydegrees from $\mathcal{G}$ are the generators of algebras of invariants for proper subrepresentations (see [18]), their polarizations and:

$$[B\cdots BxA^1](A^2,\alpha), [B\cdots BA^1_1A^1_2](A^2_1,\alpha)(A^2_2,\beta),$$

$$[A^1_1\cdots A^1_pB\cdots Bxy][A^2_1\cdots A^2_pB\cdots B], [A^1_1\cdots A^1_pB\cdots Bxy][A^2_1\cdots A^2_pB\cdots Bxy],$$

$$[A^1_1\cdots A^1_qB\cdots Bx][A^2_1\cdots A^2_qB\cdots BB^1](B^2,\alpha),$$

$$[A_1^1\cdots A_q^1B\cdots BxB^1][A_1^2\cdots A_q^2B\cdots B](B^2,\alpha),$$
$$[A_1^1\cdots A_q^1B\cdots BxB_1^1][A_1^2\cdots A_q^2B\cdots BxB_2^1](B_1^2,\alpha)(B_2^2,\alpha), \tag{9}$$
$$[A_1^1\cdots A_r^1B\cdots BB_1^1B_2^1][A_1^2\cdots A_r^2B\cdots B](B_1^2,\alpha)(B_2^2,\beta),$$
$$[A_1^1\cdots A_s^1B\cdots BB_1^1B_2^1][A_1^2\cdots A_s^2B\cdots BB_3^1B_4^1](B_1^2,\alpha)(B_2^2,\beta)(B_3^2,\alpha)(B_4^2,\beta),$$
$$p\geq 1, q\geq 2, r\geq 4, s\geq 4.$$

Let g be one of the above invariants. If $\mathrm{gr}\mathbf{k}[V]^{SL_n}$ has no syzygies of degree γ, $\gamma\preceq \deg g$, then, as we noted above, g is a member of a minimal system of generators. This is a case for all the above invariants but that from (9) with $q=2$ and that from the last series with $s=4$ (in both cases $n=2m$). Here we had to calculate some invariants in special points to prove that these are not generators.

For case 9.11, we have $(SL_4,\varphi_1\varphi_3+2\varphi_1)=(SO_6,\wedge^2\mathbf{k}^6+2\mathbf{k}^6)$. This is a slice representation of $(SL_6,\varphi_1^2+\varphi_4+2\varphi_5)$, up to a trivial factor. Hence, the c.i. property and the upper bound $\mathrm{hd}\mathbf{k}[V]^{SL_n}\leq 3$ follow from case 9.8 and 3.1. On the other hand, applying 3.1, we obtain $\mathrm{hd}\mathbf{k}[V]^{SL_n}\geq 3$, hence, $\mathrm{hd}\mathbf{k}[V]^{SL_n}=3$. Substituting $A^{ij}=\delta^{ij}$ in the generators of the algebra of invariants of SL_6 (see above), we obtain the generators for this case.

For case 9.13, (8) and [3] imply the c.i. property and the upper bound $\mathrm{hd}\leq 3$. Arguing as above, we obtain $\mathrm{hd}\mathbf{k}[V]^{SL_n}=2$. Besides the generators of the algebra of invariants for proper subrepresentations, we obtain generators of the following degrees: (2,2,1,1), (2,2,1,1), (4,2,1,1), (4,1,2,2).

For case 9.10, we present $V=X\times Y$ for $(G,X)=(SL_4,S^2\mathbf{k}^4)$, $(G,Y)=(SL_4,3\wedge^2\mathbf{k}^4)=(SO_6,3\mathbf{k}^6)$. By [3], $\mathbf{k}[X]^U$ is polynomial. On the other hand, by [24], $\mathbf{k}[Y]^U$ is generated by $\mathbf{k}[Y]^G$ and the multilinear antisymmetric invariants having the following degrees and weights with respect to the standard maximal torus of SL_4: $(1,\varphi_2),(2,\varphi_1\varphi_3),(3,\varphi_1^2),(3,\varphi_3^2)$. Applying (8), we obtain that, besides the generators from the subrepresentations of ρ (see [18]), $\mathbf{k}[V]^G$ has generators of degrees (1,1,1,1) and (3,1,1,1). Therefore $\mathrm{hd}\mathbf{k}[V]^G=2$ and by 5.1, $\mathbf{k}[V]^G$ is a c.i.

For case 9.12, we present $V=X\times Y$ for $(G,X,Y)=(SL_6,\wedge^3\mathbf{k}^6,l\mathbf{k}^6+m\mathbf{k}^{6*})$. By [3], $\mathbf{k}[X]^U$ is polynomial. On the other hand, by [24], $\mathbf{k}[Y]^U$ is generated by $\mathbf{k}[Y]^G,\mathbf{k}[l\mathbf{k}^6]^U$, and $\mathbf{k}[m\mathbf{k}^{6*}]^U$. Moreover, $\mathbf{k}[l\mathbf{k}^6]^U$ is generated by the multilinear antisymmetric invariants having the following degrees and weights with respect to the standard maximal torus of SL_6: $(1,\varphi_1),(2,\varphi_2),\cdots,(l,\varphi_l)$. A similar assertion holds for $\mathbf{k}[m\mathbf{k}^{6*}]^U$.

Applying (8), we obtain that all the generators of $\mathbf{k}[V]^G$ arise from the coregular subrepresentations of ρ (see [18]). Moreover, for $(l,m)=(3,3),(4,1)$ we have $\mathrm{hd}\mathbf{k}[V]^G=2$. Then by 5.1, $\mathbf{k}[V]^G$ is a c.i.

Furthermore, using [11], one can show that the ideal of syzygies for the case $(SL_6,\varphi_3+4\varphi_1+\varphi_5)$ is generated by elements of degree (3,4,1) and (5,4,1) (here we give the total degrees by the isotypic components). Now consider the group $(SL_6,\varphi_3+4\varphi_1+2\varphi_5)$. We have: $\mathrm{hd}\mathbf{k}[V]^G=4$. Let $x\in X$ be a generic point, $H=G_x$. Clearly, $(H,Y)\cong(SL_3\times SL_3',4(\varphi_1+\varphi_1')+2(\varphi_2+\varphi_2'))$. By 2.1, $\mathrm{hd}\mathbf{k}[Y]^H=4$ and the ideal of syzygies of $\mathbf{k}[Y]^H$ is generated by 4 elements of degree (4,1). The restrictions of the generators of $\mathbf{k}[V]^G$ to $x \times Y$ turn out to be the generators of $\mathbf{k}[x \times Y]^H$. It can be deduced that the ideal of syzygies of $\mathbf{k}[V]^G$ is generated by two pairs of relations of degrees (3,4,1) and (5,4,1) arising from two subrepresentations of the form $\varphi_3 + 4\varphi_1 + \varphi_5$. Hence, $\mathbf{k}[V]^G$ is a c.i.

Thus we proved Theorem 0.2. For the groups from List 10 the situation is as follows. For the serial cases,

$$G = SL_n, n = 2m+1, \rho = \varphi_1^2 + \varphi_2 + 2\varphi_{n-1}, 2\varphi_2 + \varphi_1 + 3\varphi_{n-1}, 2\varphi_2 + 4\varphi_{n-1},$$

one can deduce a minimal system of generators from 2.5 and 2.7. For instance, we obtain for $(SL_5, 2\varphi_2 + \varphi_1 + 3\varphi_4)$: $\mathrm{hd}\mathbf{k}[V]^{SL_5} = 2$. By 5.1, this is a c.i. representation.

However, for these serial cases and general n, $\mathrm{hd}\mathbf{k}[V]^{SL_n} > 2$. In such a situation above we applied the deformation arguments. Here V is of the form $V = X \times Y$, where $(G, Y) = (SL_{2m+1}, \varphi_2 + 2\varphi_{n-1})$ and either $X = Y$, or $\mathbf{k}[X]^U$ is polynomial by [3]. Applying an idea of [3], one can prove that $\mathbf{k}[Y]^{U^{op}}$ is a hypersurface, find generators of $\mathbf{k}[Y]^{U^{op}}$ and their unique syzygy. Unfortunately, $(\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T$ is not a c.i. Further, the Poincaré series of $\mathbf{k}[V]^G$ is equal to that of $(\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T$ and one can try to calculate the latter. Let GL_n act on the φ_i factor of ρ as on $\wedge^i \mathbf{k}^n$. So we have an action $GL_n : V$. The action of the center of GL_n gives rise to a $\mathbf{N}$-grading of both $\mathbf{k}[V]^G$ and $(\mathbf{k}[X]^U \otimes \mathbf{k}[Y]^{U^{op}})^T$. We calculated the Poincaré series of the latter algebra for the second and the third representations. This Poincaré series satisfies the condition 3.5 for Poincaré series of c.i. Thus all the necessary conditions of c.i. are fulfilled for these cases. But we have no idea how one can prove that these three representations are c.i.

For the groups

$$(SL_5, 2\varphi_2 + \varphi_3 + \varphi_4), (SL_7, \varphi_3 + 3\varphi_1 + \varphi_6),$$

one can find $\mathrm{tr.deg.}\mathbf{k}[\mathrm{V}]^{\mathrm{G}} + 2$ generators of $\mathbf{k}[V]^G$. By 5.1, the c.i. property would follow from the fact that these are all the generators.

To prove this, one can try to apply an analog of 4.1, as follows. We present our group (G, V) in the form

$$(G, V) = (G, W + U) = (SL_5, (\varphi_2 + \varphi_3) + (\varphi_2 + \varphi_4)), (SL_7, (\varphi_3) + (3\varphi_1 + \varphi_6)).$$

We have: W is a coregular G-module such that $\mathbf{k}[W]^G = \mathbf{k}[f_1, \cdots, f_s]$ and the nilcone $\mathcal{N} \subseteq W$ contains a dense orbit Gz. For a principal isotropy group H of (G, W), $\mathbf{k}[U]^H$ is a h-s. Hence, we obtain easily the Poincaré series of $\mathbf{k}[U]^H$ with respect to the grading of $\mathbf{k}[U]$ considered in 4.1. Let $g_1, \cdots, g_t$ be the above $\mathrm{tr.deg.}\mathbf{k}[\mathrm{V}]^{\mathrm{G}} + 2 - \mathrm{s}$ generators of $\mathbf{k}[V]^G$, different from $f_1, \cdots, f_s$. Let $\overline{g_1}, \cdots, \overline{g_t}$ be their restrictions to $z + U \subseteq W + U$. Then to prove $\mathbf{k}[V]^G = \mathbf{k}[f_1, \cdots, f_s, g_1, \cdots, g_t]$, it is sufficient to check that the Poincaré series of $\mathbf{k}[\overline{g_1}, \cdots, \overline{g_t}]$ is the same as that of $\mathbf{k}[U]^H$. One can calculate $\overline{g_1}, \cdots, \overline{g_t}$ explicitly. To calculate the Poincaré series of a subalgebra generated by a given system of homogeneous polynomials is a priori possible with the help of (powerful) computers. But we find the proofs of such kind too different from what we are doing in this paper.

For the group $(SL_5, 3\varphi_2 + \varphi_4)$ we have no idea how to prove the c.i. property.

6. C.i. representations of simple groups.

Examples of c.i. representations of connected simple groups different from SL_n can be found in [22], [15]. Further, applying 3.1 to the groups of entries 5, 7 in Table 9 and entry 1 in Table 6, we get new examples:

$$(SO_n, \wedge^2\mathbf{k}^n + 2\mathbf{k}^n), (Sp_n, \wedge^2\mathbf{k}^n + 4\mathbf{k}^n), (Sp_n, S^2\mathbf{k}^n + 2\mathbf{k}^n), (Sp_6, \wedge^3\mathbf{k}^6 + 2\mathbf{k}^6).$$

As a matter of fact, almost all non-coregular c.i. representations of SO_n, Sp_n are contained among the above examples. However for $Spin_n$ there is a number of representations that seem to be c.i. and need to be considered.

By [19], [22], the representations $(G_2, 4\mathbf{k}^7)$ and $(G_2, \wedge^2\mathbf{k}^7)$ are h-s. It is easy to prove that these are all the non-coregular c.i. representations of G_2.

Further, for F_4, E_6, E_7, the remaining candidates for c.i. representations are:

$$(F_4, 3\mathbf{k}^{26}), \quad (E_6, p\mathbf{k}^{27} + q(\mathbf{k}^{27})^*), p+q=4, \quad (E_7, 3\mathbf{k}^{56}).$$

By [8], $(F_4, 3\mathbf{k}^{26})$ is a h-s.

For E_8 the unique c.i. representation is the adjoint one.

This will be the object of another paper.

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