Journal of Lie Theory
Volume **11** (2001) 545–557

Jacobi Forms on Symmetric Domains and Torus Bundles over Abelian Schemes

Min Ho Lee

Communicated by M. Moskowitz

Abstract. We introduce Jacobi forms on Hermitian symmetric domains using automorphy factors associated to torus bundles over ableian schemes. We discuss families of modular forms determined by such Jacobi forms and prove that these Jacobi forms reduce to the usual Jacobi forms of several variables when the Hermitian symmetric domain is a Siegel upper half space.

1. Introduction

Jacobi forms on the Poincaré upper half plane, which were developed systematically by Eichler and Zagier [5], occur naturally in various contexts in number theory, and they have been studied extensively in recent years (see e.g. [3], [6]). Jacobi forms of several variables have been considered mainly on Siegel upper half spaces (cf. [15], [16]) and a few other special types of Hermitian symmetric domains (see [4], [7]). The goal of this paper is to introduce Jacobi forms on Hermitian symmetric domains which allow equivariant holomorphic maps into Siegel upper half spaces.

Torus bundles over abelian schemes occur naturally in the study of compactifications of locally symmetric spaces. Let $\mathcal{D}$ be a Hermitian symmetric domain, and let $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ be a discrete subgroup of arithmetic type. Then the locally symmetric space $\Gamma\backslash\mathcal{D}$ can be compactified by the method of Baily and Borel (cf. [2]). Thus $\Gamma\backslash\mathcal{D}$ can be canonically embedded as a Zariski open subset in a projective variety $\overline{\Gamma\backslash\mathcal{D}}$. Unfortunately, the Baily-Borel compactification $\overline{\Gamma\backslash\mathcal{D}}$ may have complicated singularities along the boundary, which prevents us from using it to obtain information on automorphic forms on $\mathcal{D}$, for example. On the other hand the method of toroidal compactification provides smooth compactifications of locally symmetric spaces (see [1], [11], [12]). Toroidal compactifications are defined using torus embeddings, and there are in general different toroidal compactifications associated to different systems of cone decompositions involved. The toroidal compactification of $\Gamma\backslash\mathcal{D}$ dominates the Baily-Borel compactification $\overline{\Gamma\backslash\mathcal{D}}$, and it possesses a natural stratification compatible with that of the Baily-Borel compactification. The strata of toroidal compactifications are torus bundles over abelian

ISSN 0949–5932 / $2.50

schemes. Abelian schemes here are families of abelian varieties parametrized by the locally symmetric space $\Gamma\backslash\mathcal{D}$, which play an important role in number theory and algebraic geometry.

In [14] Satake constructed explicitly abelian schemes over a locally symmetric space of the form $\Gamma\backslash\mathcal{D}$ and torus bundles over such abelian schemes, and he obtained results on the Chern classes and the projectivity of those torus bundles. His construction involves certain automorphy factors on the product $\mathcal{D} \times V_+$ of $\mathcal{D}$ and a complex vector space V_+. In fact, such automorphy factors also play a crucial role for the proof of the algebracity of abelian schemes over which torus bundles reside (see [13, Chapter 4]). In this paper we define Jacobi forms using such automorphy factors and discuss families of modular forms on Hermitian symmetric domains associated to such Jacobi forms. We also prove that these Jacobi forms reduce to the usual Jacobi forms of several variables investigated by Ziegler [16] when the Hermitian symmetric domain is a Siegel upper half space.

2. Torus bundles over abelian schemes

In this section we describe the construction of a torus bundle over an abelian scheme following Satake [14]. Let V be a real vector space of dimension $2m > 0$ defined over $\mathbb{Q}$, and let $\mathrm{Alt}(V)$ denote the space of all alternating bilinear forms on V.

Definition 2.1. A *Hermitian structure* on V is a pair (α, I) consisting of all elements $\alpha \in \mathrm{Alt}(V)$ and $I \in GL(V)$ with $I^2 = -1_V$ such that the bilinear map

$$V \times V \to \mathbb{R}, \quad (v, v') \mapsto \alpha(v, Iv')$$

is symmetric and positive definite. We shall denote by $\mathrm{Herm}(V)$ the space of all Hermitian structures on V.

Given $\alpha \in \mathrm{Alt}(V)$, we denote by $Sp(V, \alpha)$ the symplectic group associated to α, that is,

$$Sp(V, \alpha) = \{g \in GL(V) \mid \alpha(gv, gv') = \alpha(v, v') \text{ for all } v, v' \in V\}.$$

We fix a Hermitian structure $(\alpha_0, I_0) \in \mathrm{Herm}(V)$ on V with $\alpha_0 \in \mathrm{Alt}(V)$ defined over $\mathbb{Q}$. Then the Hermitian symmetric domain associated to the symplectic group $Sp(V, \alpha_0)$ can be identified with the space

$$\mathcal{H} = \mathcal{H}(V, \alpha_0) = \{I \in GL(V) \mid (\alpha_0, I) \in \mathrm{Herm}(V)\} \tag{1}$$

on which $Sp(V, \alpha_0)$ acts by

$$(g, I) \mapsto gIg^{-1}$$

for all $g \in Sp(V, \alpha_0)$ and $I \in \mathcal{H}(V, \alpha_0)$.

Let G be a semisimple Lie group of Hermitian type defined over $\mathbb{Q}$. Thus $G = \mathbb{G}(\mathbb{R})$ for some semisimple linear algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$, and, if K is a maximal compact subgroup of G, the associated Riemannian symmetric space $\mathcal{D} = G/K$ is a Hermitian symmetric domain. We assume that there are a homomorphism $\rho : G \to Sp(V, \alpha_0)$ of Lie groups defined over $\mathbb{Q}$ and a holomorphic map $\tau : \mathcal{D} \to \mathcal{H}$ such that $I_0 \in \tau(\mathcal{D})$ and

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G$ and $z \in \mathcal{D}$. We set

$$U^* = \{\alpha \in \mathrm{Alt}(V) \mid \rho(G) \subset Sp(V, \alpha)\}.$$

Then U^* is a subspace of $\mathrm{Alt}(V)$ defined over $\mathbb{Q}$, and we have $\alpha_0 \in U^*$. Let $U = (U^*)^*$ be the dual space of U^*. Then we obtain an alternating bilinear map $A : V \times V \to U$ defined over $\mathbb{Q}$ by

$$A(v, v')(\alpha) = \alpha(v, v') \tag{2}$$

for all $\alpha \in U^*$ and $v, v' \in V$.

Following Satake (cf. [13, §III.5], [14]), we consider the generalized Heisenberg group $\mathbb{H}$ associated to A consisting of all elements of $V \times U$ together with a multiplication operation given by

$$(v, u) \cdot (v', u') = (v + v', u + u' - A(v, v')/2) \tag{3}$$

for all $(v, u), (v', u') \in V \times U$. Then the group G operates on $\mathbb{H}$ by

$$g \cdot (v, u) = (\rho(g)v, u) \tag{4}$$

for all $g \in G$ and $(v, u) \in \mathbb{H}$, and we can form the semidirect product $G \ltimes \mathbb{H}$ with respect to this operation. Thus $G \ltimes \mathbb{H}$ consists of the elements (g, v, u) of $G \times V \times U$ whose multiplication operation is given by

$$\begin{aligned}(g, v, u) \cdot (g', v', u') &= (gg', (v, u) \cdot (\rho(g)v', u')) \\ &= (gg', v + \rho(g)v', u + u' - A(v, \rho(g)v')/2).\end{aligned} \tag{5}$$

Let $I \in GL(V)$ be an element of $\mathcal{H} = \mathcal{H}(V, \alpha_0)$. We extend the complex structure I on V linearly to the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V, and set

$$V_+(I) = \{v \in V_{\mathbb{C}} \mid Iv = iv\}, \qquad V_-(I) = \{v \in V_{\mathbb{C}} \mid Iv = -iv\}.$$

When I is equal to I_0 considered above, we shall write $V_+ = V_+(I_0)$ and $V_- = V_-(I_0)$.

Lemma 2.2. *To each complex structure I on V there corresponds a unique complex linear map $\xi_I : V_- \to V_+$ satisfying*

$$V_-(I) = (1 + \xi_I)V_-. \tag{6}$$

Furthermore, the map $I \mapsto \xi_I$ determines a bijection between $\mathcal{H} = \mathcal{H}(V, \alpha_0)$ and the set of $\mathbb{C}$-linear maps $\xi : V_- \to V_+$ such that $1 - \xi\overline{\xi}$ is positive definite and $\xi^t = \xi$, where the transpose is taken with respect to the bilinear map α_0.

Proof. This follows from [13, Lemma II.7.2]. ∎

If ξ is an element of $\operatorname{Hom}_{\mathbb{C}}(V_-, V_+)$ in Lemma 2.2 corresponding to an element $I \in \mathcal{H}$, then we shall write $I = I_\xi$.

Lemma 2.3. *For each $\xi \in \operatorname{Hom}_{\mathbb{C}}(V_-, V_+)$ with $I_\xi \in \mathcal{H}$ the map*

$$\Xi_\xi : (V, I_\xi) \to V_+, \quad v \mapsto v_+ - \xi v_-$$

determines an isomorphism of vector spaces over $\mathbb{C}$, where $v = v_+ + v_- \in V \subset V_{\mathbb{C}}$ with $v_\pm \in V_\pm$.

Proof. Given $\xi \in \operatorname{Hom}_{\mathbb{C}}(V_-, V_+)$ with $I_\xi \in \mathcal{H}$, the map Ξ_ξ is linear. Since $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V_+$, it suffices to show that $\operatorname{Ker} \Xi_\xi = \{0\}$. Suppose that $v \in V$ satisfies

$$\Xi_\xi(v) = v_+ - \xi v_- = 0. \tag{7}$$

By (6) there exists $v' \in V_-(I_\xi)$ such that

$$v' = v_- + \xi v_-. \tag{8}$$

¿From (7) and (8) we see that $v' = v_+ + v_- = v \in V$. Since $V \cap V_-(I_\xi) = \{0\}$, we have $v' = v = 0$; hence it follows that $\operatorname{Ker} \Xi_\xi = \{0\}$ ∎

By Lemma 2.2 we may identify the symmetric domain $\mathcal{H}$ in (1) with the set of elements $z \in \operatorname{Hom}_{\mathbb{C}}(V_-, V_+)$ with $z^t = z$ and $1 - z\overline{z} \gg 0$. Then the symplectic group $Sp(V, \alpha_0)$ operates on $\mathcal{H}$ in (1) by

$$g(z) = (az + b)(cz + d)^{-1}$$

for all $z \in \mathcal{H}$ and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(V, \alpha_0);$$

here we wrote $g \in Sp(V, \alpha_0)$ as a 2×2 block matrix with respect to the decomposition $V_{\mathbb{C}} = V_+ + V_-$. The canonical automorphy factor J of $Sp(V, \alpha_0)$ is the map on $Sp(V, \alpha_0) \times \mathcal{H}$ with values in $GL(V_{\mathbb{C}})$ given by

$$J(g, z) = \begin{pmatrix} J_+(g, z) & 0 \\ 0 & J_-(g, z) \end{pmatrix} \tag{9}$$

for all $g \in Sp(V, \alpha_0)$ and $z \in \mathcal{H}$, where

$$J_+(g, z) = a - g(z)c, \quad J_-(g, z) = cz + d. \tag{10}$$

If $\tau : \mathcal{D} \to \mathcal{H}$ is the holomorphic map equivariant with respect to the homomorphism $\rho : G \to Sp(V, \alpha_0)$ as before and if A is as in (2), then we set

$$\begin{aligned} \mathcal{J}((g, r, s), (z, v)) = s &- A(\rho(g)r, J_+(\rho(g), \tau(z))r_z)/2 \\ &- A(\rho(g)v, J_+(\rho(g), \tau(z))v)/2 \\ &- A(\rho(g)r, J_+(\rho(g), \tau(z))v) \end{aligned} \tag{11}$$

for $g \in G$, $(r,s) \in \mathbb{H}$ and $(z,v) \in \mathcal{D} \times V_+$, where

$$r_z = r_+ - zr_- \in V_+ \tag{12}$$

for $r = (r_+, r_-) \in V \subset V_{\mathbb{C}} = V_+ \oplus V_-$. Then it is known (see [13, §III.5]) that the group $G \ltimes \mathbb{H}$ operates on $\mathcal{D} \times V_+ \times U_{\mathbb{C}}$ by

$$(g,r,s) \cdot (z,v,u) = \big(gz, J_+(\rho(g), \tau(z))(v + r_z), u + \mathcal{J}((g,r,s),(z,v))\big). \tag{13}$$

By restricting this action to $\mathcal{D} \times V_+$ we obtain the action of $G \ltimes \mathbb{H}$ on $\mathcal{D} \times V_+$ given by

$$(g,r,s) \cdot (z,v) = \big(gz, J_+(\rho(g), \tau(z))(v + r_z)\big)$$

for $g \in G$, $(r,s) \in \mathbb{H}$ and $(z,v) \in \mathcal{D} \times V_+$.

Proposition 2.4. *The map $\mathcal{J} : (G \ltimes \mathbb{H}) \times (\mathcal{D} \times V_+) \to U_{\mathbb{C}}$ given by (11) satisfies*

$$\mathcal{J}((g',r',s') \cdot (g,r,s),(z,v)) = \mathcal{J}((g',r',s'),(g,r,s) \cdot (z,v)) + \mathcal{J}((g,r,s),(z,v))$$

for all $(g',r',s'), (g,r,s) \in G \ltimes \mathbb{H}$ and $(z,v) \in \mathcal{D} \times V_+$.

Proof. Given $(g',r',s'), (g,r,s) \in G \ltimes \mathbb{H}$ and $(z,v,u) \in \mathcal{D} \times V_+ \times U_{\mathbb{C}}$, we set

$$\begin{aligned}
(z_1, v_1, u_1) &= (g,r,s) \cdot (z,v,u) \\
(z_2, v_2, u_2) &= (g',r',s') \cdot (z_1, v_1, u_1) \\
(z_3, v_3, u_3) &= ((g',r',s') \cdot (g,r,s)) \cdot (z,v,u).
\end{aligned}$$

Then by (13) we see that

$$\begin{aligned}
u_1 &= u + \mathcal{J}((g,r,s),(z,v)) \\
u_2 &= u_1 + \mathcal{J}((g',r',s'),(g,r,s) \cdot (z,v)) \\
u_3 &= u + \mathcal{J}((g',r',s') \cdot (g,r,s),(z,v)).
\end{aligned}$$

Since $G \ltimes \mathbb{H}$ acts on $\mathcal{D} \times V_+ \times U_{\mathbb{C}}$, we have $u_2 = u_3$; hence the proposition follows. ∎

Let $L_{\mathbb{H}}$ be an arithmetic subgroup of $\mathbb{H}$, and set

$$L = p_V(L_{\mathbb{H}}), \quad L_U = p_U(L_{\mathbb{H}}), \tag{14}$$

where $p_V : \mathbb{H} \to V$ and $p_U : \mathbb{H} \to U$ are the natural projection maps. Then L and L_U are lattices in V and U, respectively, and we have $L = L_{\mathbb{H}}/L_U$. Given elements $l, l' \in L$, we have $(l,0), (l',0) \in L_{\mathbb{H}}$; hence by (3) we see that

$$(l,0) \cdot (l',0) = (l + l', -A(l,l')/2) \in L_{\mathbb{H}}.$$

Since $(l + l', 0)^{-1} = (-l - l', 0) \in L_{\mathbb{H}}$, we have

$$(l + l', -A(l,l')/2) \cdot (-l - l', 0) = (0, -A(l,l')/2) \in L_{\mathbb{H}}.$$

Thus it follows that $A(L, L) \subset L_U$. Let γ be a torsion-free arithmetic subgroup of G. Using the isomorphism

$$\Gamma \ltimes L_{\mathbb{H}}/L_U \cong \Gamma \ltimes L,$$

we see that the action of $G \ltimes \mathbb{H}$ on $\mathcal{D} \times V_+ \times U_{\mathbb{C}}$ induces actions of the discrete groups $\Gamma \ltimes L_{\mathbb{H}}$, $\Gamma \ltimes L$ and Γ on the spaces $\mathcal{D} \times V_+ \times U_{\mathbb{C}}$, $\mathcal{D} \times V_+$ and $\mathcal{D}$, respectively. We denote the associated quotient spaces by

$$W = \Gamma \ltimes L_{\mathbb{H}} \backslash \mathcal{D} \times V_+ \times U_{\mathbb{C}}, \quad Y = \Gamma \ltimes L \backslash \mathcal{D} \times V_+, \quad X = \Gamma \backslash \mathcal{D}.$$

Then each of the spaces W, Y and X has a natural structure of a complex manifold, and there are natural projections

$$W \xrightarrow{\pi_1} Y \xrightarrow{\pi_2} X.$$

The complex manifold Y is an abelian scheme over the arithmetic variety X, called a Kuga fiber variety, whose fiber V_+/L has the structure of a polarized abelian variety (see e.g. [8], [9], [10], [13]), and W is a torus bundle over the Kuga fiber variety Y whose fiber is isomorphic to the complex torus $U_{\mathbb{C}}/L_U$.

3. Modular forms and Jacobi forms

Let $\tau : \mathcal{D} \to \mathcal{H}$ be the holomorphic map that is equivariant with respect to the homomorphism $\rho : G \to Sp(V, \alpha_0)$ considered in Section 2.. In this section we define Jacobi forms and modular forms on Hermitian symmetric domains using the map $\mathcal{J}$ in (11) and construct modular forms on $\mathcal{D}$ associated to Jacobi forms.

Let $K_{\mathbb{C}}^{\mathrm{Sp}}$ be the subgroup of $GL(V_{\mathbb{C}})$ given by

$$K_{\mathbb{C}}^{\mathrm{Sp}} = \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in GL(V_{\mathbb{C}}) \;\middle|\; p = (q^t)^{-1} \right\}, \tag{15}$$

where the matrix is written with respect to the decomposition $V_{\mathbb{C}} = V_+ + V_-$. Then it is known that $K_{\mathbb{C}}^{\mathrm{Sp}}$ is the complexification of a maximal compact subgroup K^{Sp} of $Sp(V, \alpha_0)$. Let J, J_+ and J_- be as in (9), and let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(V, \alpha_0)$ and $\zeta \in \mathcal{H}$. Since the matrix $g\zeta = (a\zeta + b)(c\zeta + d)^{-1} \in \mathcal{H}$ is symmetric, by (10) we obtain

$$\begin{aligned} J_+(g, \zeta) &= a - (a\zeta + b)(c\zeta + d)^{-1}c \\ &= a - (\zeta c^t + d^t)^{-1}(\zeta a^t + b^t)c \\ &= (\zeta c^t + d^t)^{-1}(\zeta(c^t a - a^t c) + d^t a - b^t c) \\ &= (\zeta c^t + d^t)^{-1} = (J_-(g, \zeta)^t)^{-1}. \end{aligned} \tag{16}$$

Hence it follows that $J_+(g, \zeta) \in K_{\mathbb{C}}^{\mathrm{Sp}}$.

Let $\sigma : K_{\mathbb{C}}^{\mathrm{Sp}} \to GL(\mathcal{Z})$ be a representation of $K_{\mathbb{C}}^{\mathrm{Sp}}$ in a finite-dimensional complex vector space $\mathcal{Z}$. Given a holomorphic map $f : \mathcal{D} \to \mathcal{Z}$, we set

$$(f \mid_\sigma \gamma)(z) = \sigma(J(\rho(\gamma), \tau(z)))^{-1} f(\gamma z) \tag{17}$$

for $\gamma \in G$ and $z \in \mathcal{D}$. Then it can be shown that

$$f \mid_\sigma \gamma \mid_\sigma \gamma' = f \mid_\sigma \gamma\gamma',$$

for $\gamma, \gamma' \in \Gamma$. Let $\Gamma \subset G$ be a torsion-free arithmetic subgroup as before.

Definition 3.1. A holomorphic map $f : \mathcal{D} \to \mathcal{Z}$ is a *modular form for* Γ *associated to* σ if it satisfies

$$f \mid_\sigma \gamma = f$$

for all $\gamma \in \Gamma$.

Let $\chi : U_\mathbb{C} \to \mathbb{C}^\times$ be a character of $U_\mathbb{C}$ with $\chi(s) = 1$ for all $s \in L_U$, where L_U is as in (14). Then by Proposition 2.4 we see that $\chi \circ \mathcal{J} : (G \ltimes \mathbb{H}) \times (\mathcal{D} \times V_+) \to \mathbb{C}$ is an automorphy factor, that is, it satisfies

$$(\chi \circ \mathcal{J})(\widetilde{g}\widetilde{g}', \widetilde{z}) = (\chi \circ \mathcal{J})(\widetilde{g}, \widetilde{g}'\widetilde{z}) \cdot (\chi \circ \mathcal{J})(\widetilde{g}', \widetilde{z})$$

for all $\widetilde{g}, \widetilde{g}' \in G \ltimes \mathbb{H}$ and $\widetilde{z} \in \mathcal{D} \times V_+$. Given a holomorphic map $F : \mathcal{D} \times V_+ \to \mathcal{Z}$, we set

$$\begin{aligned}(F \mid_{\sigma,\chi} (\gamma, r, s))(z, w) = \chi(-\mathcal{J}((\gamma, r, s), (z, w))) \cdot \sigma(J(\rho(\gamma), \tau(z)))^{-1} \\ \times F(\gamma z, J_+(\rho(\gamma), \tau(z))(w + r_z))\end{aligned} \tag{18}$$

for all $(z, w) \in \mathcal{D} \times V_+$ and $(\gamma, r, s) \in G \ltimes \mathbb{H}$, where r_z is as in (12). Using the fact that $\chi \circ \mathcal{J}$ is an automorphy factor, we see that

$$F \mid_{\sigma,\chi} (\gamma, r, s) \mid_{\sigma,\chi} (\gamma', r', s') = F \mid_{\sigma,\chi} ((\gamma, r, s) \cdot (\gamma', r', s'))$$

for $\gamma, \gamma' \in G$ and $(r, s), (r', s') \in \mathbb{H}$.

Definition 3.2. A holomorphic map $F : \mathcal{D} \times V_+ \to \mathcal{Z}$ is a *Jacobi form for* $\Gamma \ltimes L_\mathbb{H}$ *associated to* σ *and* χ if it satisfies

$$F \mid_{\sigma,\chi} (\gamma, r, s) = F \tag{19}$$

for all $(\gamma, r, s) \in \Gamma \ltimes L_\mathbb{H}$.

We can obtain a family of modular forms on $\mathcal{D}$ parametrized by the rational points of $\mathbb{H}$ as is described in the next theorem.

Theorem 3.3. *Let $F : \mathcal{D} \times V_+ \to \mathcal{Z}$ be a Jacobi form for $\Gamma \ltimes L_\mathbb{H}$ associated to σ and χ, and let $(r, s) \in \mathbb{H}_\mathbb{Q} = V_\mathbb{Q} \times U_\mathbb{Q}$ with $r = (r_+, r_-) \in V \subset V_\mathbb{C} = V_+ \oplus V_-$. If $A : V \times V \to U$ is the bilinear map in (2), we set*

$$f(z) = \chi(-A(r, zr_-)/2) \cdot F(z, r_z) \tag{20}$$

for all $z \in \mathcal{D}$. Then f is a modular form for an arithmetic subgroup $\Gamma' \subset \Gamma$ of G associated to σ.

Proof. Let ε be the identity element of G, and let $(r, s) \in \mathbb{H}_\mathbb{Q}$. Then, for each $z \in \mathcal{D}$, $J(\rho(\varepsilon), \tau(z))$ and $J_+(\rho(\varepsilon), \tau(z))$ are identity matrices, and in particular $\sigma(J(\rho(\varepsilon), \tau(z)))$ is the identity element in $GL(\mathcal{Z})$. Thus, using (11) and (18), we see that

$$\mathcal{J}((\varepsilon, r, s), (z, 0)) = s - A(r, r_z)/2 - A(v, v)/2 - A(r, 0) = s - A(r, r_z)/2,$$

$$(F \mid_{\sigma,\chi} (\varepsilon, r, s))(z, 0) = \chi(-s + A(r, r_z)/2) F(z, r_z)$$

for all $z \in \mathcal{D}$. Hence by (20) we obtain

$$\begin{aligned} f(z) &= \chi(s - A(r, r_+ - zr_-)/2 - A(r, zr_-)/2)(F \mid_{\sigma,\chi} (\varepsilon, r, s))(z, 0) \\ &= \chi(s - A(r, r_+)/2)(F \mid_{\sigma,\chi} (\varepsilon, r, s))(z, 0). \end{aligned}$$

Thus it suffices to show that the function $F_{(r,s)} : \mathcal{D} \to \mathcal{Z}$ given by

$$F_{(r,s)}(z) = (F \mid_{\sigma,\chi} (\varepsilon, r, s))(z, 0)$$

is a modular form for an arithmetic subgroup $\Gamma' \subset \Gamma$ associated to σ. Given an element $\gamma \in \Gamma$, by (17) we have

$$\begin{aligned} (F_{(r,s)} \mid_\sigma \gamma)(z) &= \sigma(J(\rho(\gamma), \tau(z)))^{-1} F_{(r,s)}(\gamma z) \\ &= \sigma(J(\rho(\gamma), \tau(z)))^{-1} (F \mid_{\sigma,\chi} (\varepsilon, r, s))(\gamma z, 0) \\ &= ((F \mid_{\sigma,\chi} (\varepsilon, r, s)) \mid_{\sigma,\chi} (\gamma, 0, 0))(z, 0) \end{aligned}$$

for all $z \in \mathcal{D}$. However, by (5) we see that

$$(\varepsilon, r, s) \cdot (\gamma, 0, 0) = (\gamma, r, s) = (\gamma, 0, 0) \cdot (\varepsilon, \rho(\gamma)^{-1} r, s).$$

Using this and the fact that F is a Jacobi form for Γ, we obtain

$$\begin{aligned} (F_{(r,s)} \mid_\sigma \gamma)(z) &= ((F \mid_{\sigma,\chi} (\gamma, 0, 0)) \mid_{\sigma,\chi} (\varepsilon, \rho(\gamma)^{-1} r, s))(z, 0) \\ &= (F \mid_{\sigma,\chi} (\varepsilon, \rho(\gamma)^{-1} r, s)(z, 0) \\ &= F_{\gamma^{-1} \cdot (r,s)}(z), \end{aligned}$$

where $\gamma^{-1} \cdot (r, s) = (\rho(\gamma)^{-1} r, s)$ by (4). Let $\Gamma_{(r,s)}$ be the subgroup of Γ consisting of the elements $\gamma \in \Gamma$ satisfying

$$\gamma^{-1} \cdot (r, s) = (0, s_1) \cdot (r_2, s_2) \cdot (r, s)$$

with $s_1 \in L_U$ and $(r_2, s_2) \in L_{\mathbb{H}}$. Then $\Gamma_{(r,s)}$ is an arithmetic subgroup, and, if $\gamma \in \Gamma_{(r,s)}$, we have

$$\begin{aligned} (F_{(r,s)} \mid_\sigma \gamma)(z) &= F_{\gamma^{-1} \cdot (r,s)}(z) = F_{(0,s_1) \cdot (r_2, s_2) \cdot (r,s)}(z) \\ &= (F \mid_{\sigma,\chi} (\varepsilon, 0, s_1) \mid_{\sigma,\chi} (\varepsilon, r_2, s_2) \mid_{\sigma,\chi} (\varepsilon, r, s))(z, 0) \\ &= (F \mid_{\sigma,\chi} (\varepsilon, r, s)(z, 0) = F_{(r,s)}(z) \end{aligned}$$

for all $z \in \mathcal{D}$. Hence $F_{(r,s)}$ is a modular form for the arithmetic subgroup $\Gamma_{(r,s)} \subset \Gamma$ associated to σ, and therefore the proof of the theorem is complete. ∎

4. Jacobi forms for symplectic groups

In this section we specialize the equivariant pair (ρ, τ) considered in Section 2. to the case where G is a symplectic group and ρ and τ are identity maps. We then prove that the associated Jacobi forms in the sense of Definition 3.2 coincide with the usual Jacobi forms of several variables (cf. [16]).

We first describe the definition of Jacobi forms of several variables introduced by Ziegler (see [16] for details). If a and b are positive integers and R is a ring, we shall denote by $R^{(a,b)}$ the set of $a \times b$ matrices over R. Let $\mathbb{H}_{\mathbb{R}}^{n,j}$ be the Heisenberg group consisting of the elements

$$(\lambda, \mu, \kappa) \in \mathbb{R}^{(j,n)} \times \mathbb{R}^{(j,n)} \times \mathbb{R}^{(j,j)}$$

such that the matrix $\kappa + \mu\lambda^t$ is symmetric. The multiplication operation in $\mathbb{H}_{\mathbb{R}}^{n,j}$ is given by

$$(\lambda, \mu, \kappa) \cdot (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu, \kappa + \kappa' + \lambda\mu'^t - \mu\lambda'^t),$$

and the symplectic group $Sp(n, \mathbb{R})$ acts on $\mathbb{H}_{\mathbb{R}}^{n,j}$ on the right by

$$(\lambda, \mu, \kappa) \cdot M = ((\lambda, \mu) \cdot M, \kappa)$$

for $M \in Sp(n, \mathbb{R})$ and $(\lambda, \mu, \kappa) \in \mathbb{H}_{\mathbb{R}}^{n,j}$. This action enables us to define the Jacobi group

$$G_{\mathbb{R}}^{n,j} = Sp(n, \mathbb{R}) \ltimes \mathbb{H}_{\mathbb{R}}^{n,j}$$

whose multiplication is given by

$$\begin{aligned}(M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) &= (MM', ((\lambda, \mu, \kappa) \cdot M') \cdot \zeta') \\ &= (MM', (\widetilde{\lambda} + \lambda', \widetilde{\mu} + \mu', \kappa + \kappa' + \widetilde{\lambda}\mu'^t - \widetilde{\mu}\lambda'^t)),\end{aligned}$$

where $(\widetilde{\lambda}, \widetilde{\mu}) = (\lambda, \mu) \cdot M'$. Then $G_{\mathbb{R}}^{n,j}$ acts on $\mathcal{H}_n \times \mathbb{C}^{(j,n)}$ by

$$(M, (\lambda, \mu, \kappa)) \cdot (z, w) = ((az + b)(cz + d)^{-1}, (w + \lambda z + \mu)(cz + d)^{-1})$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(n, \mathbb{R})$.

Let $\eta : GL(n, \mathbb{C}) \to GL(\mathcal{Z})$ be a representation of $GL(n, \mathbb{C})$ in a finite-dimensional complex vector space $\mathcal{Z}$, and let $\mathfrak{M} = (m_{\alpha\beta}) \in \mathbb{R}^{(j,j)}$ be a positive symmetric matrix with $2m_{\alpha\beta}, m_{\alpha\alpha} \in \mathbb{Z}$ for $1 \leq \alpha, \beta \leq j$. Given $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(n, \mathbb{R})$, $\boldsymbol{\xi} = (\lambda, \mu, \kappa) \in \mathbb{H}_{\mathbb{R}}^{n,j}$ and a holomorphic map $\Phi : \mathcal{H}_n \times \mathbb{C}^{(j,n)} \to \mathcal{Z}$, we set

$$\begin{aligned}(\Phi \mid_{\eta,\mathfrak{M}} M)(z, w) &= \eta(cz + d)^{-1} \exp\{-2\pi i \operatorname{Tr}(\mathfrak{M} w(cz + d)^{-1} cw^t)\} \\ &\quad \times \Phi((az + b)(cz + d)^{-1}, (w + \lambda z + \mu)(cz + d)^{-1}), \\ (\Phi \mid_{\mathfrak{M}} \boldsymbol{\xi})(z, w) &= \exp\{-2\pi i \operatorname{Tr}(\mathfrak{M}(\lambda z \lambda^t + 2\lambda w^t + \mu\lambda^t + \kappa))\} \\ &\quad \times \Phi(z, w + \lambda z + \mu),\end{aligned}$$

for all $(z, w) \in \mathcal{H}_n \times \mathbb{C}^{(j,n)}$, where Tr denotes the trace. Let $\mathbb{H}_{\mathbb{Z}}^{n,j}$ be a discrete subgroup of $\mathbb{H}_{\mathbb{R}}^{n,j}$ given by

$$\mathbb{H}_{\mathbb{Z}}^{n,j} = \{(\lambda, \mu, \kappa) \in \mathbb{H}_{\mathbb{R}}^{n,j} \mid \lambda, \mu \in \mathbb{Z}^{(j,n)},\ \kappa \in \mathbb{Z}^{(j,j)}\} \tag{21}$$

and let $\Gamma \subset Sp(n, \mathbb{Z})$ be a subgroup of finite index.

Definition 4.1. Let $\eta : GL(n, \mathbb{C}) \to GL(\mathcal{Z})$ and $\mathfrak{M} \in \mathbb{R}^{(j,j)}$ be as above. A holomorphic map $\Phi : \mathcal{H}_n \times \mathbb{C}^{(j,n)} \to \mathcal{Z}$ is a *Jacobi form of index* $\mathfrak{M}$ *with respect to* η *for* Γ if it satisfies

$$\Phi \mid_{\eta,\mathfrak{M}} M = \Phi, \quad \Phi \mid_{\mathfrak{M}} \boldsymbol{\xi} = \Phi$$

for all $M \in \Gamma$ and $\boldsymbol{\xi} \in \mathbb{H}_{\mathbb{Z}}^{n,j}$.

Thus, if Φ is a Jacobi form of index $\mathfrak{M}$ with respect to η for Γ and if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\boldsymbol{\xi} = (\lambda, \mu, \kappa) \in \mathbb{H}_{\mathbb{Z}}^{n,j}$, we have

$$\begin{aligned} &\Phi((az+b)(cz+d)^{-1}, (w+\lambda z+\mu)(cz+d)^{-1}) \qquad (22)\\ &\qquad = \exp\{2\pi i \operatorname{Tr}(\mathfrak{M}((w+\lambda z+\mu)(cz+d)^{-1}c(w^t+z\lambda^t+\mu^t) \\ &\qquad\qquad - \lambda w^t - (w+\lambda z+\mu)\lambda^t))\} \cdot \eta(cz+d) \cdot \Phi(z,w) \end{aligned}$$

for all $(z, w) \in \mathcal{H}_n \times \mathbb{C}^{(j,n)}$.

Now we specialize the Jacobi forms described in Section 2. to the case of real vector spaces V and U given by

$$V = \mathbb{R}^{(j,n)} \times \mathbb{R}^{(j,n)}, \quad U = \mathbb{R}^{(j,j)} \tag{23}$$

for some positive integer j. We choose a complex structure I_0 on V in such a way that, if $v = (v_1, v_2) \in \mathbb{R}^{(j,n)} \times \mathbb{R}^{(j,n)} = V$, the V_+ and V_- components of v are given by $v_+ = v_2$ and $v_- = -v_1$, respectively, where $V_{\mathbb{C}} = V_+ \oplus V_-$ is the decomposition determined by I_0 (see Section 2.). We define the bilinear maps $\beta, A : V \times V \to U$ by

$$\beta((v_+, v_-), (v'_+, v'_-)) = v'_+ v_-^t - v'_- v_+^t, \tag{24}$$

$$A((v_+, v_-), (v'_+, v'_-)) = \beta((v_+, v_-), (v'_+, v'_-)) + \beta((v_+, v_-), (v'_+, v'_-))^t \tag{25}$$

for all $v_+, v'_+ \in V_+$ and $v_-, v'_- \in V_-$. Then we have

$$\beta((v'_+, v'_-), (v_+, v_-)) = -\beta((v_+, v_-), (v'_+, v'_-))^t;$$

hence we see that A is an alternating bilinear map on V.

If $K_{\mathbb{C}}^{\mathrm{Sp}}$ is the compexification of a maximal compact subgroup of $Sp(n, \mathbb{R})$ as in (15) and if $\sigma : K_{\mathbb{C}}^{\mathrm{Sp}} \to GL(\mathcal{Z})$ is its representation in a finite-dimensional complex vector space $\mathcal{Z}$, then we define a representation $\eta_\sigma : GL(n, \mathbb{C}) \to GL(\mathcal{Z})$ of $GL(n, \mathbb{C})$ by

$$\eta_\sigma(q) = \sigma \begin{pmatrix} (q^t)^{-1} & 0 \\ 0 & q \end{pmatrix} \tag{26}$$

for all $q \in GL(n, \mathbb{C})$. Given $\mathfrak{M} \in \mathbb{R}^{(j,j)}$, let $\chi_{\mathfrak{M}} : U_{\mathbb{C}} \to \mathbb{C}^\times$ be the character on $U_{\mathbb{C}} = \mathbb{C}^{(j,j)}$ given by

$$\chi_{\mathfrak{M}}(u) = \exp(-2\pi i \operatorname{Tr}(\mathfrak{M} \cdot u))$$

for all $u \in \mathbb{C}^{(j,j)}$.

Theorem 4.2. *Let U and V be as in (23), and let $\mathfrak{M} = (m_{\alpha\beta}) \in \mathbb{R}^{(j,j)}$ be a positive symmetric matrix with $2m_{\alpha\beta}, m_{\alpha\beta} \in \mathbb{Z}$ for $1 \le \alpha, \beta \le j$. Let Γ be a subgroup of $Sp(n, \mathbb{Z})$ of finite index, and let $\mathbb{H}_{\mathbb{Z}}^{n,j}$ be as in (21). Then a Jacobi form for $\Gamma \ltimes \mathbb{H}_{\mathbb{Z}}^{n,j}$ associated to a representation $\sigma : K_{\mathbb{C}}^{Sp} \to GL(\mathcal{Z})$ and a character $\chi_{\mathfrak{M}}$ in the sense of Definition 3.2 is a Jacobi form of index $\mathfrak{M}$ with respect to the representation $\eta_\sigma : GL(n, \mathbb{C}) \to GL(\mathcal{Z})$ for Γ in the sense of Definition 4.1.*

Proof. Let $r=(\lambda,\mu)\in\mathbb{R}^{(j,n)}\times\mathbb{R}^{(j,n)}=V$. Then as an element of $V_+\oplus V_-$ it can be written as

$$r=(\mu,-\lambda)\in V_+\oplus V_-=V_{\mathbb{C}}.$$

Let F be a Jacobi form for $\Gamma\ltimes L_{\mathbb{H}}$ associated to σ and $\chi_{\mathfrak{M}}$ in the sense of Definition 3.2, Since we are writing vectors as block row matrices instead of block column matrices, the transformation formula in (19) can be written in the form

$$F(\gamma z,(w+r_z)J_+(\rho(\gamma),\tau(z))^t)=\chi_{\mathfrak{M}}(\mathcal{J}((\gamma,r,s),(z,w)))\sigma(J(\gamma,z))F(z,w)$$

for all $(z,w)\in\mathcal{D}\times V_+$ and $(\gamma,r,s)\in\Gamma\ltimes L_{\mathbb{H}}$ with $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, where

$$\begin{aligned}\mathcal{J}((g,r,s),(z,w))&=s-A(rg^t,r_zJ_+(g,z)^t)/2-A(wg^t,wJ_+(g,z)^t)/2\\&\qquad\qquad-A(rg^t,wJ_+(g,z)^t)\\&=s-[A(rg^t,(w+r_z)J_+(g,z)^t)\\&\qquad\qquad+A((w+r)g^t,wJ_+(g,z)^t)]/2.\end{aligned}\tag{27}$$

by (11). Given an element $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in Sp(n,\mathbb{R})$, we have

$$rg^t=(\mu,-\lambda)\begin{pmatrix}a^t&c^t\\b^t&d^t\end{pmatrix}=(\mu a^t-\lambda b^t,\mu c^t-\lambda d^t).$$

For $z\in\mathcal{H}_n$ we have

$$r_z=r_+-r_-z=\mu+\lambda z\in V_+.$$

By (16) we have $J_+(g,z)^t=J_-(g,z)^{-1}=(cz+d)^{-1}$. We also see that

$$(w+r)g^t=(\mu+w,-\lambda)\begin{pmatrix}a^t&c^t\\b^t&d^t\end{pmatrix}=(\mu a^t+wa^t-\lambda b^t,\mu c^t+wc^t-\lambda d^t).$$

Thus by (24) we have

$$\beta(rg^t,(w+r_z)J_+(g,z)^t)=(w+\lambda z+\mu)(cz+d)^{-1}(c\mu^t-d\lambda^t),$$

$$\begin{aligned}\beta((w+r)g^t,wJ_+(g,z)^t)^t&=(w+\mu)c^t(zc^t+d^t)^{-1}w^t-\lambda d^t(zc^t+d^t)^{-1}w^t\\&=(w+\lambda z+\mu)c^t(zc^t+d^t)^{-1}w^t-\lambda(zc^t+d^t)(zc^t+d^t)^{-1}w^t\\&=(w+\lambda z+\mu)(cz+d)^{-1}cw^t-\lambda w^t,\end{aligned}$$

where we used the fact that $(cz+d)^{-1}c$ is symmetric. Hence we obtain

$$\begin{aligned}&\beta(rg^t,(w+r_z)J_+(g,z)^t)+\beta((w+r)g^t,wJ_+(g,z)^t)^t\\&\qquad=(w+\lambda z+\mu)(cz+d)^{-1}(c(w^t+z\lambda^t+\mu^t)-(cz+d)\lambda^t)-\lambda w^t\\&\qquad=(w+\lambda z+\mu)(cz+d)^{-1}c(w^t+z\lambda^t+\mu^t)-\lambda w^t-(w+\lambda z+\mu)\lambda^t.\end{aligned}$$

Using this, (27) and (25), we have

$$\begin{aligned}-2\mathcal{J}((g,r,s),(z,w))&=-2s+\beta(rg^t,(w+r_z)J_+(g,z)^t)+\beta((w+r)g^t,wJ_+(g,z)^t)^t\\&\qquad+\beta(rg^t,(w+r_z)J_+(g,z)^t)^t+\beta((w+r)g^t,wJ_+(g,z)^t)\\&=-2s+2(w+\lambda z+\mu)(cz+d)^{-1}c(w^t+z\lambda^t+\mu^t)\\&\qquad-2\lambda w^t-2w\lambda^t-2\lambda z\lambda^t-\mu\lambda^t-\lambda\mu^t.\end{aligned}$$

Since $(\lambda,\mu,s)\in \mathbb{H}_{\mathbb{Z}}^{n,j}\subset \mathbb{H}_{\mathbb{R}}^{n,j}$ and

$$\mathbb{H}_{\mathbb{R}}^{n,j}=\{(\lambda,\mu,s)\in \mathbb{R}^{(j,n)}\times\mathbb{R}^{(j,n)}\times\mathbb{R}^{(j,j)}\mid (s+\mu\lambda^t)^t=s+\mu\lambda^t\},$$

we see that $\mu\lambda^t+\lambda\mu^t=2\mu\lambda^t+s-s^t$. Since $\operatorname{Tr}(\mathfrak{M}(-2s-s+s^t))\in\mathbb{Z}$, we obtain

$$\chi_{\mathfrak{M}}(\mathcal{J}((\gamma,r,s),(z,w)))=\exp(2\pi i\operatorname{Tr}(\mathfrak{M}\cdot((w+\lambda z+\mu)(cz+d)^{-1}c(w^t+z\lambda^t+\mu^t)\\ -\lambda w^t-(w+\lambda z+\mu)\lambda^t))).$$

On the other hand, by (9), (16) and (26) we have

$$\sigma(J(\gamma,z))=\eta_\sigma(J_-(\gamma,z))=\eta_\sigma(cz+d).$$

Hence the transformation formula (19) reduces to (22) with $\eta=\eta_\sigma$, and therefore the proof of the theorem is complete. ■

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Min Ho Lee
Department of Mathematics
University of Northern Iowa
Cedar Falls, Iowa 50614
U. S. A.
lee@math.uni.edu

Received August 4, 2000
and in final form October 18, 2000