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Mixed Models for Reductive Dual Pairs and Siegel Domains for Hermitian Symmetric Spaces

C. S. Leslie[1]

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Abstract. Let (G, G') be the reductive dual pair $(Sp(n,\mathbb{R}), O(k))$ or $(U(p,q), U(k))$, and let K be a maximal compact subgroup of the noncompact group G. Then for the representations π of $\widetilde{G}$ which occur in the Howe duality correspondence for (G, G'), we construct explicit intertwining maps between mixed models of π and spaces of holomorphic sections of vector bundles over the hermitian symmetric space G/K, where G/K is embedded in its holomorphic tangent space as a type III Siegel domain. This result provides a link between the original construction of these representations using tube domain and type II Siegel domain realizations of G/K and more recent constructions using the bounded domain realization of G/K.

1. Introduction

Let (G, G') be the reductive dual pair $(Sp(n,\mathbb{R}), O(k))$ or $(U(p,q), U(k))$. Then the noncompact group G is of hermitian type, meaning that G/K is a hermitian symmetric space for K maximal compact in G, and the representations π of $\widetilde{G}$ (a suitable covering group for G) that occur in the Howe correspondence are all unitarizable highest weight representations. A natural way to try to geometrically realize these representations of $\widetilde{G}$ is to find appropriate invariant subspaces of sections of vector bundles over G/K. In this paper, we obtain such a realization by constructing a set of new and explicit intertwining maps between mixed models [11] of these representations and spaces of holomorphic functions on unbounded realizations of G/K known as type III Siegel domains.

Our results rely on a method due to Davidson and Fabec [3] that produces an intertwining map between an abstract unitary highest weight representation for a hermitian linear group G and a multiplier representation on a function space on the bounded domain realization $\mathcal{D}$ of the hermitian symmetric space G/K. In particular, their method produces a G-intertwining map between Fock models for the representations we consider and reproducing kernel spaces of holomorphic

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vector-valued functions on $\mathcal{D}$. By using the geometric framework of Siegel domains and the Davidson-Fabec construction, we obtain a new set of intertwining maps corresponding to mixed models of the oscillator representation. First, we fix partial Cayley transforms c_b, $1 \leq b \leq r$, where r is the real rank of G, and consider the image $\mathcal{S}_b$ under c_b of the bounded domain $\mathcal{D}$ via the Harish-Chandra embedding of G/K in its holomorphic tangent space. For $b \neq r$, these domains are type III Siegel domains in the sense of Piatetski-Shapiro [17]. Second, we define partial Bargmann transforms B_b which intertwine mixed models and Fock models of ω. We show how the action of the inverse partial Bargmann transform on the space of G'-invariant polynomials and the space of G'-harmonic polynomials can be reinterpreted in terms of the geometric action of the partial Cayley transform. As a result, in Theorem 8.8 and Theorem 13.5, we obtain explicit intertwining maps between mixed models of highest weight representations and multiplier representations of holomorphic functions on S_b. For the case $b = r$, we retrieve the intertwining maps found in Kashiwara-Vergne [14]. The structure of the different Siegel domain realizations S_b is closely connected to the boundary component theory of the bounded domain $\mathcal{D}$ [22]. Thus we obtain a new connection between the polarization of models of the oscillator representation and the boundary component theory of $\mathcal{D}$, which could be of interest in the study of automorphic forms.

In Sections 2 and 3 we outline a (straightforward) generalization of the Davidson-Fabec construction [3] for arbitrary covers of linear hermitian groups. Sections 4 and 5 describe the dual pair $(Sp(n,\mathbb{R}), O(k))$ and the Fock model for the representations of $\widetilde{Sp}(n,\mathbb{R})$ which occur, while in Section 6 to Section 8, we describe the domains S_b and partial Bargmann transforms, and we construct our new $\widetilde{Sp}(n,\mathbb{R})$-intertwining maps for mixed models. Background on the Fock model for the second dual pair $(U(p,q), U(k))$ is given in Sections 9 and 10, while in Sections 11 to 13 we develop our intertwining maps for mixed models of the $U(p,q)$-representations in the Howe correspondence.

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2. Lifting Factors of Automorphy

As we will be working with non-linear Lie groups, we need to define various notions in the geometry of hermitian symmetric domains for arbitrary covers of linear groups. In particular, we need to describe how to lift the factor of automorphy for the linear group to a covering group. These ideas are well-known and can be found in a number of references [20, 10, 21]. For convenience, we follow our exposition in [15].

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We assume that $[\mathfrak{k}, \mathfrak{k}] \neq \mathfrak{k}$. Then $\mathfrak{k}$ has one dimensional center $\mathfrak{z}(\mathfrak{k})$, and we can find an element $H \in \mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(H)$ gives a complex structure on $\mathfrak{p}$. This gives the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-$, where $\mathfrak{p}_{\pm}$ are the $\pm i$ eigenspaces of $\mathrm{ad}(H)$ on $\mathfrak{p}_{\mathbb{C}}$ and $\overline{\mathfrak{p}}_+ = \mathfrak{p}_-$ under conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$.

Let $G_{\mathbb{C}}$ be the connected, simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Then $G_{\mathbb{C}}$ is a complex simple Lie group. Let $K_{\mathbb{C}}$ be the connected subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{k}_{\mathbb{C}}$. Let $P_+ = \exp(\mathfrak{p}_+)$, $P_- = \exp(\mathfrak{p}_+)$. Then the map $\mathfrak{p}_+ \times K_{\mathbb{C}} \times \mathfrak{p}_- \to G_{\mathbb{C}}$ given by $(Z, k, W) \mapsto \exp(Z)\, k \exp(W)$ is a holomorphic diffeomorphism onto a dense open subset $\Omega = P_+ K_{\mathbb{C}} P_-$ of $G_{\mathbb{C}}$. We can write $g \in \Omega$ uniquely as $g = (g)_+ k(g)(g)_-$, $(g)_\pm \in P_\pm$, $k(g) \in K_{\mathbb{C}}$. Then $g \mapsto k(g)$ is a holomorphic map $\Omega \to K_{\mathbb{C}}$.

Let $G \subset G_{\mathbb{C}}$ be the connected subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. Then G is a linear real Lie group, $G \subset P_+ K_{\mathbb{C}} P_-$ and $G \cap K_{\mathbb{C}} P_- = K$, where K is the connected subgroup of G corresponding to $\mathfrak{k}$. Finally, G/K is a noncompact hermitian symmetric space, and we say G (or $\mathfrak{g}$) is of hermitian type.

The map $\zeta : \Omega \to \mathfrak{p}_+$ given by $\exp(\zeta(g)) = (g)_+$ induces a holomorphic diffeomorphism of G/K onto the domain $\mathcal{D} = \zeta(G) \subset \mathfrak{p}_+$. The set $\mathcal{D}$ is a bounded domain in $\mathfrak{p}_+$, and the map $G/K \to \mathcal{D} \subset \mathfrak{p}_+$ is known as the Harish-Chandra embedding.

For $(g, Z) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ such that $g \exp(Z) \in \Omega$, we define $g(Z) \in \mathfrak{p}_+$, $j(g, Z) \in K_{\mathbb{C}}$ by the formulas

$$\begin{aligned} \exp(g(Z)) &= (g \exp(Z))_+, \\ j(g, Z) &= k(g \exp(Z)). \end{aligned}$$

Now consider a covering map $\nu : \widetilde{G} \to G$ of Lie groups, with $\widetilde{K}$ the maximal compact subgroup of $\widetilde{G}$ such that $\nu(\widetilde{K}) = K$. We can define an action of $\widetilde{G}$ on $\mathcal{D}$ by $\widetilde{g} \cdot Z = \nu(\widetilde{g})(Z)$. Thus $\mathcal{D}$ can be identified with $\widetilde{G}/\widetilde{K}$.

The maximal compact subgroup $\widetilde{K} \subset \widetilde{G}$ is a linear group and therefore has a complexification $\widetilde{K}_{\mathbb{C}}$ corresponding to $\mathfrak{k}_{\mathbb{C}}$. We can extend the covering map $\nu : \widetilde{K} \to K$ to a covering map $\overline{\nu} : \widetilde{K}_{\mathbb{C}} \to K_{\mathbb{C}}$. When we have $Z \in \mathfrak{p}_+$ for which $j(g, Z)$ is defined for all $g \in G$, we can define a map which we will also call j by $(\widetilde{g}, Z) \mapsto j(\nu(\widetilde{g}), Z)$ for $\widetilde{g} \in \widetilde{G}$. We need to lift this map to $\widetilde{K}_{\mathbb{C}}$.

The following statement comes from [15, Proposition 1.1.1]:

Proposition 2.1. *Let $\mathcal{U}$ be any connected open domain in $\mathfrak{p}_+$ on which the function $j : \widetilde{G} \times \mathcal{U} \to K_{\mathbb{C}}$ is defined. Fix a basepoint $U_0 \in \mathcal{U}$. Then there exists a unique holomorphic map $\widetilde{j} : \widetilde{G} \times \mathcal{U} \to \widetilde{K}_{\mathbb{C}}$ such that $\widetilde{j}(1_{\widetilde{G}}, U_0) = 1_{\widetilde{K}_{\mathbb{C}}}$ and $\overline{\nu} \circ \widetilde{j} = j$.*

In particular, we have a unique holomorphic lift:

$$\widetilde{j} : \widetilde{G} \times \mathcal{D} \to \widetilde{K}_{\mathbb{C}}, \qquad \widetilde{j}(1_{\widetilde{G}}, 0) = 1_{\widetilde{K}_{\mathbb{C}}}$$

Corollary 2.2. *The map $\widetilde{j} : \widetilde{G} \times \mathcal{D} \to \widetilde{K}_{\mathbb{C}}$ satisfies the cocyle relation*

$$\widetilde{j}(\widetilde{g}_1 \widetilde{g}_2, Z) = \widetilde{j}(\widetilde{g}_1, \widetilde{g}_2 \cdot Z)\widetilde{j}(\widetilde{g}_2, Z), \quad \widetilde{g}_1, \widetilde{g}_2 \in \widetilde{G}, Z \in \mathcal{D} \tag{1}$$

and the property

$$\widetilde{j}(\widetilde{k}, Z) = \widetilde{k}. \tag{2}$$

Finally, the canonical kernel function $q : \mathcal{D} \times \mathcal{D} \to K_{\mathbb{C}}$ is defined by

$$q(Z, W) = j(\exp(\overline{W})^{-1}, Z)^{-1}$$

and satisfies

$$q(0,Z) = q(Z,0) = 1, \quad q(W,Z) = \overline{q(Z,W)}^{-1},$$
$$q(g(Z), g(W)) = j(g,Z)q(Z,W)\overline{j(g,W)}^{-1}.$$

Since $\mathcal{D}$ is simply connected, we obtain a unique holomorphic lift

$$\widetilde{q} : \mathcal{D} \times \mathcal{D} \to \widetilde{K}_{\mathbb{C}}, \qquad \widetilde{q}(0,0) = 1_{\widetilde{K}_{\mathbb{C}}}$$

such that

$$\widetilde{q}(g(Z), g(W)) = \widetilde{j}(g,Z)\widetilde{q}(Z,W)\overline{\widetilde{j}(g,W)}^{-1}. \tag{3}$$

3. The Davidson-Fabec Construction

Let G be a linear group of Hermitian type and ω an irreducible unitary highest weight representation of G on a Hilbert space $\mathbb{H}$. In [3], Davidson and Fabec give a geometric construction of an intertwining map between $(\omega, \mathbb{H})$ and a space of holomorphic sections of a homogeneous vector bundle over G/K with G acting by translation. The latter space is realized as a space of holomorphic vector-valued functions on the bounded domain $\mathcal{D}$.

We will need to extend the result in [3] to arbitrary covers $\nu : \widetilde{G} \to G$ of the linear group G. We will outline the original argument, making the necessary reformulations to handle our more general setting.

Let ω be a nontrivial irreducible highest weight representation of $\widetilde{G}$ on a Hilbert space $\mathbb{H}$ with inner product $\langle \cdot | \cdot \rangle$. Let $\mathcal{H} \subset \mathbb{H}$ be the $\widetilde{K}$-span of the highest weight vector. Then $\mathcal{H}$ is an irreducible $\widetilde{K}$-space, and we let τ denote the restriction of ω to $V_\tau = \mathcal{H}$. Since $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\pm}] \subset \mathfrak{p}_{\pm}$, we have $T \cdot v = 0$ for all $T \in \mathfrak{p}_+$ and $v \in \mathcal{H}$.

Extend τ to a holomorphic representation of $\widetilde{K}_{\mathbb{C}}$, the complexification of $\widetilde{K}$. Then we define:

$$J : \widetilde{G} \times \mathcal{D} \to \mathrm{Aut}(V_\tau), \qquad J(\widetilde{g}, T) = \tau(\widetilde{j}(\widetilde{g}, T)).$$

Let $T \in \mathfrak{p}_+$. Define $q_T : \mathcal{H} \to \mathbb{H}$ formally by $q_T v = \sum_{n=0}^{\infty} \frac{\overline{(T)}^n v}{n!}$ for $v \in \mathcal{H}$. By [3, Theorem 5.1] we have:

Theorem 3.1. *Let v be a nonzero vector in $\mathcal{H}$ and $T \in \mathfrak{p}_+$. Then the series which defines $q_T v$ converges if and only if $T \in \mathcal{D}$.*

The main result is a reformulation of [3, Theorem 6.1]:

Theorem 3.2. *Let $v \in \mathcal{H}$ and $\widetilde{g} \in \widetilde{G}$. Then $\omega(\widetilde{g})v = q_{\nu(\widetilde{g})(0)} J(\widetilde{g}, 0)^{*-1} v$.*

Theorem 3.2 leads easily to the following Proposition by the proof of [3, Proposition 7.1].

Proposition 3.3. *Let $\widetilde{g} \in \widetilde{G}$, $T \in \mathcal{D}$, and $v \in \mathcal{H}$. Then*

$$\omega(\widetilde{g}) q_T v = q_{\nu(\widetilde{g})(T)} J(\widetilde{g}, T)^{*-1} v.$$

Now the previous Proposition and the irreducibility of ω give:

Proposition 3.4. *The span of the vectors $q_T v$, $T \in \mathcal{D}$, $v \in \mathcal{H}$, is dense in the Hilbert space $\mathbb{H}$.*

Thus it will be sufficient to determine the intertwining operator on the set of vectors $q_T v$. We first define a kernel function on $\mathcal{D}$:

Theorem 3.5. *There is a positive definite operator-valued kernel function $Q : \mathcal{D} \times \mathcal{D} \to \mathrm{Aut}(V_\tau)$ such that*

$$\langle q_T v | q_S w \rangle = \langle Q(S,T) v | w \rangle. \tag{4}$$

Following [3], as Q is a positive definite operator-valued kernel function, we use a result of Kunze [13] to construct a Hilbert space $H(\mathcal{D},\tau)$ of continuous functions $f : \mathcal{D} \to \mathcal{H}$ such that:

(a) The span of $\{S \mapsto Q(S,T)v : T \in \mathcal{D}, v \in \mathcal{H}\}$ is dense in $H(\mathcal{D},\tau)$;
(b) For $S \in \mathcal{D}$, the map $e_S : f \to f(S)$ is a continuous map $H(\mathcal{D},\tau) \to \mathcal{H}$;
(c) $(S,T) = e_S e_T^*$ for all $S,T \in \mathcal{D}$;
(d) The inner product on $H(\mathcal{D},\tau)$ is given by
$$(Q(\cdot,T)v | Q(\cdot,S)w) = \langle Q(S,T)v|w\rangle_{V_\tau}.$$

Using property (3) of $\widetilde{q}$ and the fact that $\tau(\overline{\overline{\widetilde{k}}}) = \tau(\widetilde{k})^{*^{-1}}$ for $\widetilde{k} \in \widetilde{K}_{\mathbb{C}}$, we establish:

Proposition 3.6. *Let $S,T \in \mathcal{D}$, $\widetilde{g} \in \widetilde{G}$. Then*

$$Q(\widetilde{g}\cdot S, \widetilde{g}\cdot T) = J(\widetilde{g},S)Q(S,T)J(\widetilde{g},T)^{*^{-1}}$$

It follows, using arguments of Kunze [13] that $H(\mathcal{D},\tau)$ is a representation space for $\widetilde{G}$:

Proposition 3.7. *The formula $T(\widetilde{g})f(S) = J(\widetilde{g}^{-1},S)^{-1} f(\widetilde{g}^{-1}\cdot S)$ where $f \in H(\mathcal{D},\tau)$, defines a strongly continuous unitary representation of $\widetilde{G}$ on $H(\mathcal{D},\tau)$.*

Finally, we can write down the intertwining map as in [3]:

Theorem 3.8. *The map $q_T v \mapsto Q(\cdot,T)v$ extends to an intertwining operator Ξ between $(\omega,\mathbb{H})$ and $(T, H(\mathcal{D},\tau))$. The intertwining map can be defined globally by $\Xi f(S) = q_S^* f$ for $f \in \mathbb{H}$.*

4. The Fock Model for $Sp(n,\mathbb{R})$

We let $G_{\mathbb{C}} = Sp(n,\mathbb{C})$ be the group of automorphisms of $\mathbb{C}^{2n}$ preserving a nondegenerate antisymmetric bilinear form J. Fix a basis with respect to which $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, and define an indefinite hermitian form on $\mathbb{C}^{2n}$ by $\langle u,v\rangle = -\sum_{j=1}^n u_j \overline{v}_j + \sum_{j=1}^n u_{n+j}\overline{v}_{n+j}$ with respect to the same basis. We have a real form G of $Sp(n,\mathbb{C})$, isomorphic to $Sp(n,\mathbb{R})$, defined by

$$G = Sp(n,\mathbb{C}) \cap U(n,n) = \left\{ \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \ : \ AA^* - BB^* = 1_n,\ {}^tAB = {}^tBA \right\}$$

with maximal compact subgroup $K \cong U(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : A \in U(n) \right\}$ and Harish-Chandra decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-$, where

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}, Z \in \mathrm{Sym}_n(\mathbb{C}) \right\}$$

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} X & 0 \\ 0 & -{}^tX \end{pmatrix} : X \in \mathfrak{gl}(n, \mathbb{C}) \right\}$$

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, Y \in \mathrm{Sym}_n(\mathbb{C}) \right\}.$$

Let $P_\pm = \exp(\mathfrak{p}_\pm)$. Then $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to the dense open subset $\Omega = P_+ K_{\mathbb{C}} P_- \subset G_{\mathbb{C}}$ if and only if D is nonsingular [19, Lemma 7.3], and for $g \in \Omega$ we have a unique decomposition $g = (g)_+ k(g) (g)_-$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^tD^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 0 \end{pmatrix}.$$

We identify $K_{\mathbb{C}}$ with $\mathfrak{gl}(n, \mathbb{C})$ via $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \leftrightarrow A$. Then for $(g, Z) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ such that $g \cdot \exp(Z) \in \Omega$, the automorphy factor is given by

$$j(g, Z) = k(g \cdot \exp(Z)) = {}^t(CZ + D)^{-1}$$

and the action of g on $\mathfrak{p}_+$ defined by $\exp(g(Z)) = (g \cdot \exp(Z))_+$ is given by

$$g(Z) = (AZ + B)(CZ + D)^{-1}.$$

The image of the Harish-Chandra embedding $G/K \to \mathcal{D} \subset \mathfrak{p}_+$ is given by the generalized Siegel disk

$$\mathcal{D} = \{n\text{-planes } p_Z : \langle \cdot, \cdot \rangle|_{p_Z} \gg 0\} = \{Z \in \mathrm{Sym}_n(\mathbb{C}) : 1_n - Z^* Z \gg 0\}$$

where we use the notation that $W \gg 0$ if W is positive definite.

Fix $k \geq 1$ and let $M = M_{n,k}(\mathbb{C})$ with inner product $(w|w') = \mathrm{Tr}(ww'^*)$. We realize the Heisenberg group $H = M \times \mathbb{R}$ with group law

$$(w; t)(w'; t') = (w + w'; t + t' + \mathrm{Im}(w|w')).$$

Let $\mathcal{F}$ be the Hilbert space of holomorphic functions on M which are square-integrable with respect to the Gaussian measure $d\mu(w) = e^{-\pi(w|w)} dw$. Note that $\mathcal{F}$ has a reproducing kernel K given by $K(z, w) = e^{\pi(z|w)}$, so that we have

$$f(z) = \int_M e^{\pi(z|w)} f(w) e^{-\pi(w|w)} dw$$

for $f \in \mathcal{F}$ and $z \in M$.

As is well known, we obtain an irreducible unitary representation ρ_0 of H on $\mathcal{F}$ with central character $\chi(t) = e^{-\pi i t}$ by

$$\rho_0(w; t) f(w') = e^{-\pi i t} e^{\pi(w'|w)} e^{-\frac{\pi}{2}(w|w)} f(w' - w).$$

We define an action of G on M by $g\cdot w = Aw - iB\overline{w}$ for $g=\begin{pmatrix} A & B \\ \overline{B} & \overline{A}\end{pmatrix}$ and $w\in M$. If we identify w with the $2n\times k$ matrix $\begin{pmatrix} w \\ -i\overline{w}\end{pmatrix}$, then the action $g\cdot w$ simply becomes matrix multiplication.

Via the action $w\mapsto g\cdot w$, we can identify G as a group of real-linear automorphisms of M that preserve the real symplectic form $\operatorname{Im}(\cdot\,|\,\cdot)$. It follows that G acts as a group of automorphisms of H preserving the center $0\times\mathbb{R}$ of H, and $g\in G$ leads to a unitary representation $\rho_0^g(w;t)=\rho_0(g\cdot w,t)$ with the same central character χ. Thus by the Stone-von Neumann theorem, ρ_0^g and ρ_0 are unitarily equivalent, and we obtain a projective representation $\omega : G\to U(\mathcal{F})/U(1)$ defined by

$$\omega(g)\rho_0(w;t)=\rho_0(g\cdot w;t)\omega(g). \tag{5}$$

By straightforward calculations as in [6, Proposition 4.31], we obtain the following description of the action of G.

Theorem 4.1. *For $g=\begin{pmatrix} A & B \\ \overline{B} & \overline{A}\end{pmatrix}\in G$, define the operator $\overline{\omega}(g)$ modulo ±1 on $\mathcal{F}$ by*

$$\overline{\omega}(g)f(z)=\int_M K_g(z,w)f(w)e^{-\pi(w|w)}dw, \tag{6}$$

$$K_g(z,w)=(\det{}^{-\frac{k}{2}}A)e^{\frac{i\pi}{2}\operatorname{Tr}({}^tz\overline{B}A^{-1}z)}e^{\pi\operatorname{Tr}({}^t\overline{w}A^{-1}z)}e^{\frac{i\pi}{2}\operatorname{Tr}({}^t\overline{w}A^{-1}B\overline{w})}$$

where the sign of $\det^{-\frac{k}{2}}A$ is left undetermined. Then $\overline{\omega}(g)$ is a unitary operator modulo $\pm$ satisfying (5) and moreover $\overline{\omega}(g_1)\overline{\omega}(g_2)=\pm\overline{\omega}(g_1g_2)$.

For $T\in\mathfrak{p}_+$ we can define an entire function on M by

$$q_T(z)=e^{\frac{i\pi}{2}(z|T\overline{z})}=e^{\frac{i\pi}{2}\operatorname{Tr}(z^tzT^*)}=e^{\frac{i\pi}{2}\operatorname{Tr}({}^tz\overline{T}z)}.$$

Now, by a generalization of of the proof for the case $k=1$ given in [6, Proposition 4.69], we obtain:

Lemma 4.2. *The function q_T lies in $\mathcal{F}$ if and only if $T\in\mathcal{D}$.*

We use the q_T notation and also our expression for the reproducing kernel K on $\mathcal{F}$ to obtain:

Lemma 4.3. *The operator $\overline{\omega}(g)$ given by (6) can be expressed as*

$$\overline{\omega}(g)f(z)=(\det{}^{-\frac{k}{2}}A)q_{B\overline{A}^{-1}}(z)\int_M\overline{q}_{-A^{-1}B}(w)K(A^{-1}z,w)f(w)e^{-\pi(w|w)}dw. \tag{7}$$

We wish to construct from $\overline{\omega}$ a representation of the the double cover $\widetilde{G}$ of G. We use the following description of $\widetilde{G}$. Consider the function $d: G\times\mathcal{D}\to\mathbb{C}-\{0\}$ given by $d(g,Z)=\det(j(g,Z))$. Since $\mathcal{D}$ is simply connected, a branch of the square root function is uniquely determined by its value at a single point.

Given $g \in G$, we identify the choice of branch of $d^{\frac{1}{2}}(g,Z)$ by its value at $Z=0$, and we obtain a pair $(g, d^{\frac{1}{2}}(g,Z)) \leftrightarrow (g,t)$ where $t^2 = d(g,0) = \det(k(g))$. We define $\widetilde{G}$ as the set $\left\{(g, d^{\frac{1}{2}}(g,Z))\right\}$ with multiplication

$$(g_1, d_1^{\frac{1}{2}}(g_1,Z))(g_2, d_2^{\frac{1}{2}}(g_2,Z)) = (g_1 g_2, d_1^{\frac{1}{2}}(g_1, g_2(Z)) d_2^{\frac{1}{2}}(g_2,Z)). \tag{8}$$

By the cocycle relation $d(g_1 g_2, Z) = d(g_1, g_2(Z)) d(g_2, Z)$, we see that the group law is well-defined. The identity element is $\widetilde{1} = (1_G, 1)$, that is, we take the root of $d(1_G, Z) = \det(1_{K_{\mathbb{C}}}) = 1$ whose value is identically 1.

We now define a lift $\widetilde{G} \to U(\mathcal{F})$ of the map $\varpi : G \to U(\mathcal{F})/\mathbb{Z}_2$ given by (6). We will call this lift ω. For $(g,t) \in \widetilde{G}$, $g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$ we define an operator in $U(\mathcal{F})$ by

$$\omega(g,t) f(z) = (\overline{t}^{-1})^{-k} q_{B\overline{A}^{-1}}(z) \int_M \overline{q}_{-A^{-1}B}(w) K(A^{-1}z, w) f(w) e^{-\pi(w|w)} dw. \tag{9}$$

Note that for $(g,t) \in \widetilde{G}$, $t^2 = \det(\mathrm{k(g)}) = \det({}^t\overline{A}^{-1})$, so $\overline{t}^{-1}$ agrees with $\det^{\frac{1}{2}} A$ up to sign. More precisely, consider the covering maps $\nu : \widetilde{G} \to G$ and $p : U(\mathcal{F}) \to U(\mathcal{F})/\mathbb{Z}_2$. The composition $\varpi \circ \nu$ gives a map $\widetilde{G} \to U(\mathcal{F})/\mathbb{Z}_2$. We can see:

Lemma 4.4. *The map $\omega : \widetilde{G} \to U(\mathcal{F})$ defines a continuous lift of $\varpi \circ \nu$ with respect to the covering map $p : U(\mathcal{F}) \to U(\mathcal{F})/\mathbb{Z}_2$.*

Now as in [15, Theorem 2.2.5] we have:

Theorem 4.5. *The lift $\omega : \widetilde{G} \to U(\mathcal{F})$ defines a unitary representation.*

5. The Dual Pair $(Sp(n,\mathbb{R}), O(k))$

Let $G' = O(k)$ act on $\mathcal{F}(M)$ by right multiplication:

$$c \cdot f(z) = f(zc)$$

for $c \in O(k)$ and $f \in \mathcal{F}$. Clearly the action of G' commutes with ω. In fact, $(G, G') = (Sp(n,\mathbb{R}), O(k))$ is a dual pair inside $Sp(M) = Sp(nk,\mathbb{R})$, the real symplectic group on the symplectic vector space $(M, \mathrm{Im}(\cdot\,|\,\cdot))$, and ω is the restriction to $\widetilde{G}$ of the oscillator representation on $\widetilde{Sp}(M)$.

Following [2, Chapter 7], we define a map

$$\theta : M \to \mathfrak{p}_+, \qquad \theta(z) = z \cdot {}^t z.$$

Let $\mathcal{P} = \mathcal{P}(M)$ be the subspace of polynomial functions in $\mathcal{F}$. Then $\mathcal{P}$ is dense in $\mathcal{F}$. Let I be the space of polynomials invariant under the action of G'. Then I is generated as an algebra by the matrix entries of $z \to \theta(z)$ and the constant functions [2, Chapter 7]. Let $\mathcal{I}$ be the ideal in $\mathcal{P}$ generated by the matrix entries of $z \to \theta(z)$ and let $\mathcal{H}$ be the orthogonal complement of $\mathcal{I}$. We call $\mathcal{H}$ the space of harmonic polynomials.

We have [2, Chapter 7]:

Proposition 5.1. *Let $h \in \mathcal{P}$. Then the following are equivalent:*

(i) $h \in \mathcal{H}$

(ii) $g(\partial)h = 0$ *for all matrix entries* g *of* $z \to \theta(z)$

(iii) $\Delta_{i,j}h(z) = 0$, $1 \leq i \leq j \leq n$ *where* $\Delta_{i,j} = \sum_{\nu=1}^{k} \frac{\partial^2}{\partial z_{i\nu}\partial z_{j\nu}}$.

Let $\widetilde{K} \subset \widetilde{G}$ be the inverse image under ν of K. It is straightforward to check that $\widetilde{K}$ is just the linear group $\{(u,t) \in U(n) \times \mathbb{C} : \det(u) = t^2\}$. A simple calculation shows:

Lemma 5.2. *Let $f \in \mathcal{F}$ and $(u,t) \in \widetilde{K}$. Then*

$$\omega(u,t)f(z) = t^{-k}f(u^{-1}z) \tag{10}$$

Let $\widehat{O(k)}$ denote the unitary dual of $O(k)$. We write (λ, V_λ) for an irreducible unitary representation of $O(k)$ on a space V_λ, and we let λ' be the contragredient representation of λ on the dual space V'_λ. We now have sufficient background to state the following result of Kashiwara-Vergne [14]:

Theorem 5.3. *The space $\mathcal{H}$ is $\widetilde{K} \times G'$-invariant and decomposes as a multiplicity free orthogonal direct sum of unitary $\widetilde{K} \times G'$ representations $\tau(\lambda) \otimes \lambda'$.*

For later use, we wish to make the correspondence $\lambda \mapsto \tau(\lambda)$ more concrete. Since $G' = O(k)$ commutes with the representation ω, we get the decomposition

$$\mathcal{F}(M) = \oplus_{\lambda \in \widehat{O(k)}} \mathcal{F}(M)_\lambda$$

where $\mathcal{F}(M)_\lambda$ denotes the isotypic component of type λ. We write $(\cdot,\cdot)$ for the canonical bilinear pairing between a vector space V and its dual V'. Now for (λ, V_λ) in $\widehat{O(k)}$, we let $\mathcal{F}(M;\lambda)$ be the subspace of functions f from M to V_λ such that $z \mapsto (f(z), \phi) \in \mathcal{F}$ for every $\phi \in V'_\lambda$ and $f(zc) = \lambda(c)^{-1}f(z)$ for all $c \in O(k)$. The group $\widetilde{G}$ acts on $\mathcal{F}(M;\lambda)$ by the same formula as ω; we will call this representation ω_λ. Then $\mathcal{F}(M)_\lambda$ is isomorphic to $\mathcal{F}(M;\lambda') \otimes V_\lambda$ by $(h \otimes \psi)(z) = (\psi, h(z))$ for $h \in \mathcal{F}(M;\lambda')$, $\psi \in V_\lambda$. Thus we get

$$\mathcal{F}(M) = \oplus_{\lambda \in \widehat{O(k)}} \mathcal{F}(M;\lambda) \otimes V'_\lambda$$

as a representation of $\widetilde{G} \times G'$.

Similarly we have the decomposition

$$\mathcal{H}(M) = \oplus_{\lambda \in \widehat{O(k)}} \mathcal{H}(M;\lambda) \otimes V'_\lambda$$

as a representation of $\widetilde{K} \times G'$. Here $\mathcal{H}(M;\lambda) \subset \mathcal{F}(M;\lambda)$, and given λ, we identify the representation $\tau(\lambda)$ in the theorem as the irreducible unitary $\widetilde{K}$ representation $\mathcal{H}(M;\lambda)$, where $\widetilde{K}$ acts as (10). From [2, Proposition 7.12], we obtain:

Proposition 5.4. *Let $\lambda \in \widehat{O(k)}$ be such that $\mathcal{F}(M;\lambda) \neq 0$. Then $\mathcal{F}(M;\lambda)$ is an irreducible $\widetilde{G}$ representation.*

Fix $\lambda \in \widehat{O(k)}$ such that the space $\mathcal{H}(M;\lambda)$ is non-zero. As outlined in Section 3, we will construct a $\widetilde{G}$-intertwining map between $(\omega, \mathcal{F}(M;\lambda))$ and sections of the homogeneous vector bundle over $\mathcal{D}$ associated to the $\widetilde{K}$-module $\mathcal{H}(M;\lambda)$.

We write $\tau = \tau(\lambda)$ for the $\widetilde{K}$-representation given by $\omega_\lambda|_{\widetilde{K}}$ acting on $\mathcal{H}(M;\lambda)$. We can complexify $\widetilde{K}$ as follows. Let

$$\widetilde{K}_{\mathbb{C}} = \left\{(l,t) \in GL(n,\mathbb{C}) \times \mathbb{C} : \det(l) = t^2\right\}.$$

Then $\widetilde{K}$ is a real form of $\widetilde{K}_{\mathbb{C}}$, and we can extend τ holomorphically to $\widetilde{K}_{\mathbb{C}}$ by $\tau(l,t)h(z) = t^{-k}h(l^{-1}z)$ for $(l,t) \in \widetilde{K}_{\mathbb{C}}$, $h \in \mathcal{H}(M;\lambda)$.

Let σ be the real group automorphism of $K_{\mathbb{C}}$ whose differential is conjugation with respect to the real form $\mathfrak{k}$ of $\mathfrak{k}_{\mathbb{C}}$. If we identify K with $U(n)$ and $K_{\mathbb{C}}$ with $GL(n,\mathbb{C})$, then for $l \in GL(n,\mathbb{C})$, we have $\sigma(l) = l^{*-1}$, where l^* is the matrix operation of conjugate transpose. We can lift σ to a real group automorphism $\widetilde{\sigma}$ of $\widetilde{K}_{\mathbb{C}}$ fixing $\widetilde{K}$ in the obvious way: $\widetilde{\sigma}(l,t) = (l^{*-1}, \overline{t}^{-1})$. Then $\tau(\widetilde{\sigma}(l,t)) = \tau(l,t)^{*-1}$.

The automorphy factor gives a map $j : \widetilde{G} \times \mathcal{D} \to K_{\mathbb{C}}$ defined by $j((g,t),Z) = k(g \cdot \exp(Z))$. For $(g,t) \in \widetilde{G}$, t corresponds to a choice of branch of the square root of the function $d(g,Z) = \det(j(g,Z))$ via $t = d^{\frac{1}{2}}(g,0)$. Let $t_Z = d^{\frac{1}{2}}(g,Z)$, where we are using the branch of the root function determined by t. Now let $\widetilde{j}((g,t),Z) = (j(g,Z), t_Z)$. Clearly $(t_Z)^2 = \det(j(g,Z))$, so we have a well-defined continuous lift of j which satisfies $\widetilde{j}(1_G, 0) = 1_{\widetilde{K}_{\mathbb{C}}}$.

We write $J((g,t),Z) = \tau(\widetilde{j}((g,t),Z))$.

Lemma 5.5. *Let $h \in \mathcal{H}(M;\lambda)$. Then $\omega_\lambda(g,t)h(z) = q_{g(0)}(z)(J((g,t),0)^{*-1}h)(z)$.*

Proof. Recall from (9) that we have for $g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix}$

$$\omega_\lambda(g,t)h(z) = (\overline{t}^{-1})^{-k} q_{B\overline{A}^{-1}}(z) \int_M \overline{q}_{-A^{-1}B}(w) K(A^{-1}z, w) h(w) e^{-\pi(w|w)} dw.$$

Now $q_{-A^{-1}B} = 1 + \phi$ where ϕ belongs to the closure in $\mathcal{F}$ of the subalgebra of I consisting of $O(k)$-invariants with zero constant coefficient. Thus the function $w \mapsto \phi(w)K(w, A^{-1}z)$ belongs to the closure of the ideal $\mathcal{I}$ in $\mathcal{F}$. Since h is harmonic, h is orthogonal to $\overline{\mathcal{I}}$, and we obtain:

$$\begin{aligned} \omega_\lambda(g,t)h(z) &= (\overline{t}^{-1})^{-k} q_{B\overline{A}^{-1}}(z) \int_M K(A^{-1}z, w) h(w) e^{-\pi(w|w)} dw \\ &= (\overline{t}^{-1})^{-k} q_{B\overline{A}^{-1}}(z) h(A^{-1}z). \end{aligned}$$

Now $\widetilde{j}((g,t),0) = (j(g,0),t) = ({}^t\overline{A}^{-1})$, $J((g,t),0)^{*-1} = \tau(\widetilde{\sigma}(\widetilde{j}((g,t),0))) = \tau(A, \overline{t}^{-1})$, and $g(0) = B\overline{A}^{-1}$. ∎

Define the operator $q_T : \mathcal{H} \to \mathcal{F}$, $T \in \mathcal{D}$, from Section 3 by $(q_T h)(z) = q_T(z)h(z)$, that is, multiplication by the function $q_T(z)$. Then just as in Proposition 3.3, Lemma 5.5 implies:

Corollary 5.6. *Let $T \in \mathcal{D}$ and $h \in \mathcal{H}(M;\lambda)$. Then:*

$$\omega_\lambda(g,t)q_T h = q_{g(T)}J((g,t),T)^{*-1}h.$$

Let $V_\tau = \mathcal{H}(M;\lambda)$, and take $v, w \in V_\tau$. Exactly as in Section 3, we can construct a positive definite operator-valued kernel function $Q : \mathcal{D}\times\mathcal{D} \to \mathrm{Aut}(V_\tau)$ by: $\langle q_T v | q_S w\rangle = \langle Q(S,T)v|w\rangle$ and a calculation shows that as in Theorem 3.5: $Q(S,T) = \tau(\widetilde{q}(S,T))$. Q gives the reproducing kernel for a space of V_τ-valued holomorphic functions on $\mathcal{D}$, denoted by $H(\mathcal{D},\tau)$, on which $\widetilde{G}$ acts by translation T_τ, where we recall

$$T_\tau(g,t)F(T) = J((g,t)^{-1},T)^{-1}F(g^{-1}(T))$$

for $(g,t) \in \widetilde{G}$, $T \in \mathcal{D}$, and $F \in H(\mathcal{D},\tau)$.

Finally, the intertwining map $\Xi_\lambda : \mathcal{F}(M;\lambda) \to H(\mathcal{D},\tau)$ is given on a dense subset of $\mathcal{F}(M;\lambda)$ by $q_T v \mapsto Q(\cdot,T)v$, and is globally defined by $\Xi_\lambda f(S) = q_S^* f$ for $f \in \mathcal{F}$. In the last statement, q_S is regarded as an operator $\mathcal{H}(M;\lambda) \to \mathcal{F}(M;\lambda)$ given by multiplication by the function q_S.

We want to establish one more form of the intertwining map which will be best suited to our situation:

Proposition 5.7. *Let $f \in \mathcal{F}(M;\lambda)$ and let $w \mapsto I_\lambda(w)$ be the Hom(V_τ, V_λ)-valued polynomial function on M defined by $I_\lambda(w)h = h(w)$ for $h \in V_\tau = \mathcal{H}(M;\lambda)$. Then:*

$$\Xi_\lambda f(T) = \int_M \overline{q_T(w)} I_\lambda(w)^* f(w) e^{-\pi(w|w)}dw.$$

Proof. The map Ξ_λ is defined by $\langle \Xi_\lambda f(T)|h\rangle_{\mathcal{H}(M;\lambda)} = \langle f|q_T h\rangle_{\mathcal{F}(M;\lambda)}$ for $f \in \mathcal{F}(M;\lambda)$ and $h \in \mathcal{H}(M;\lambda)$. Now we calculate:

$$\begin{aligned}
&\langle \int_M \overline{q_T(w)} I_\lambda(w)^* f(w) e^{-\pi(w|w)}dw | h\rangle_{\mathcal{H}(M;\lambda)} \\
= &\int_M \overline{q_T(w)} \langle I_\lambda(w)^* f(w)|h\rangle_{\mathcal{H}(M;\lambda)} e^{-\pi(w|w)}dw \\
= &\int_M \overline{q_T(w)} \langle f(w)|h(w)\rangle_{V_\lambda} e^{-\pi(w|w)}dw \\
= &\int_M \langle f(w)|q_T(w)h(w)\rangle_{V_\lambda} e^{-\pi(w|w)}dw = \langle f|q_T h\rangle_{\mathcal{F}(M;\lambda)}.
\end{aligned}$$

∎

6. Cayley Transforms and Unbounded Domains

Let $\mathfrak{t}_\mathbb{C}$ be a compact Cartan subalgebra for $\mathfrak{g}_\mathbb{C}$, $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ the set of roots, and let Δ_c and Δ_{nc} be the subset of compact and noncompact roots, respectively. We can choose an ordering on Δ such that

$$\mathfrak{p}_+ = \sum_{\phi\in\Delta_{nc}^+} \mathfrak{g}_\mathbb{C}^\phi, \qquad \mathfrak{p}_- = \sum_{\phi\in\Delta_{nc}^-} \mathfrak{g}_\mathbb{C}^\phi.$$

For $\phi \in \Delta$, define $h_\phi \in i\mathfrak{t}$ by $\frac{2\phi(h)}{\langle \phi,\phi\rangle} = \langle h_\phi, h\rangle$ for $h \in \mathfrak{t}$. We choose root vectors $e_\phi \in \mathfrak{g}_{\mathbb{C}}^{\phi}$ normalized so that $[e_\phi, e_{-\phi}] = h_\phi$ and $\theta(\overline{e}_\phi) = -e_{-\phi}$. Then $\mathfrak{p}$ will have a real basis consisting of $x_\phi = e_\phi + e_{-\phi}$, $y_\phi = i(e_\phi - e_{-\phi})$, $\phi \in \Delta_{nc}^{+}$.

Let r be the real rank of $\widetilde{G}$. We inductively construct a maximal set Ψ of strongly orthogonal noncompact roots by taking ψ_1 to be the largest element of Δ_{nc}^{+}, ψ_j the largest root in Δ_{nc}^{+} strongly orthogonal to $\psi_1, \dots, \psi_{j-1}$. Then $|\Psi| = r$. For $\Gamma \subset \Psi$, we define the partial Cayley transform c_Γ [22] by

$$c_\Gamma = \prod_{\Gamma} c_\gamma, \qquad c_\gamma = \exp\left(\frac{\pi}{4} y_\gamma\right).$$

Now we have $c_\Gamma \cdot \exp Z \in P_+ K_{\mathbb{C}} P_-$ for every $Z \in \mathcal{D}$ [19, Chapter III, Section 7], so we can define a domain

$$\mathcal{S}_\Gamma = c_\Gamma\left(\mathcal{D}\right) = \{c_\Gamma(Z) : Z \in \mathcal{D}\} \subset \mathfrak{p}_+.$$

Then $\mathcal{S}_\Gamma$ is the image under the Harish-Chandra embedding ζ of the translated orbit $c_\Gamma \cdot G(x_0) \subset G_{\mathbb{C}}/K_{\mathbb{C}}P_+$, where the basepoint $x_0 = 1K_{\mathbb{C}}P_+$. For Γ nonempty, $\mathcal{S}_\Gamma$ gives an unbounded realization of the symmetric domain $\mathcal{D}$ [19, 23].

We now specialize to the example $G = Sp(n,\mathbb{C}) \cap U(n,n)$. We take compact Cartan subalgebra for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(n,\mathbb{C})$ to be

$$\mathfrak{t}_{\mathbb{C}} = \left\{\operatorname{diag}(b_1 \dots b_n, -b_1 \dots - b_n) : b_1 \dots b_n \in \mathbb{C}\right\}.$$

Write ω_i for the linear form on $\mathfrak{t}_{\mathbb{C}}$ given by $\omega_i(\operatorname{diag}(b_1 \dots b_n, -b_1 \dots - b_n)) = b_i$. Then the set of roots is

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm 2\omega_i, 1 \le i \le n;\ \pm\omega_i \pm \omega_j, 1 \le i \ne j \le n\}$$

with simple system $S = \{\alpha_1 \dots \alpha_n\}$ where $\alpha_i = \omega_i - \omega_{i+1}, 1 \le i \le n-1$, $\alpha_n = 2\omega_n$. Write Δ^+ for the system of positive roots determined by S, $\Delta^- = -\Delta^+$. Let $\psi_1 = 2\omega_1, \dots, \psi_n = 2\omega_n$ and let $\Psi = \{\psi_1 \dots \psi_n\}$. Then $\Psi \subset \Delta_{nc}^{+}$ is a maximal set of strongly orthogonal noncompact positive roots.

For $\psi_i \in \Psi$, the element h_{ψ_i} defined above is given by $h_{\psi_i} = E_{i,i} - E_{n+i,n+i} \in i\mathfrak{t}$. Taking $e_{\psi_i} = -iE_{i,n+i} \in \mathfrak{g}_{\mathbb{C}}^{\psi_i}$ and $e_{-\psi_i} = iE_{n+i,i} \in \mathfrak{g}_{\mathbb{C}}^{-\psi_i}$ gives $[e_{\psi_i}, e_{-\psi_i}] = h_{\psi_i}$ and defines elements $x_{\psi_i} = e_{\psi_i} + e_{-\psi_i}$ and $y_{\psi_i} = i(e_{\psi_i} - e_{-\psi_i})$ in $\mathfrak{p}$. Now for $b = 1 \dots n$, define $\Gamma_b = \{\psi_1, \dots, \psi_b\} \subset \Psi$ and let $x_b = \sum_{\psi \in \Gamma_b} x_\psi, y_b = \sum_{\psi \in \Gamma_b} y_\psi \in \mathfrak{p}$. We have:

$$x_b = \begin{pmatrix} 0_b & 0 & -i\cdot 1_b & 0 \\ 0 & 0_{n-b} & 0 & 0 \\ i\cdot 1_b & 0 & 0_b & 0 \\ 0 & 0 & 0 & 0_{n-b} \end{pmatrix}, \qquad y_b = \begin{pmatrix} 0_b & 0 & 1_b & 0 \\ 0 & 0_{n-b} & 0 & 0 \\ 1_b & 0 & 0_b & 0 \\ 0 & 0 & 0 & 0_{n-b} \end{pmatrix}.$$

We define the b^{th} partial Cayley transform by $c_b = c_{\Gamma_b} = \prod_{\psi \in \Gamma_b} \exp(\frac{\pi i}{4} y_\psi)$. Then:

$$c_b = \exp\left(\frac{\pi i}{4} y_b\right) = \begin{pmatrix} \frac{1}{\sqrt{2}}\cdot 1_b & 0 & \frac{i}{\sqrt{2}}\cdot 1_b & 0 \\ 0 & 1_{n-b} & 0 & 0 \\ \frac{i}{\sqrt{2}}\cdot 1_b & 0 & \frac{1}{\sqrt{2}}\cdot 1_b & 0 \\ 0 & 0 & 0 & 1_{n-b} \end{pmatrix}.$$

For $T = \begin{pmatrix} T_{11} & T_{12} \\ {}^tT_{12} & T_{22} \end{pmatrix} \in \mathcal{D}$, we calculate

$$c_b(T) = \begin{pmatrix} (T_{11}+i)(iT_{11}+1)^{-1} & \sqrt{2}(iT_{11}+1)^{-1}T_{12} \\ \sqrt{2}\,{}^tT_{12}(iT_{11}+1)^{-1} & T_{22} - i\,{}^tT_{12}(iT_{11}+1)^{-1}T_{12} \end{pmatrix}$$

with the factor of automorphy given by

$$j(c_b, T) = \begin{pmatrix} \sqrt{2}(iT_{11}+1)^{-1} & 0 \\ -i\,{}^tT_{12}(iT_{11}+1)^{-1} & 1_{n-b} \end{pmatrix}.$$

Write $\mathcal{S}_b = c_b(\mathcal{D}) \subset \mathfrak{p}_+$ for the unbounded realizations of G/K obtained as above. We will write $c = c_n$ for the full Cayley transform and $\mathcal{S} = c(\mathcal{D}) \subset \mathfrak{p}_+$. As G^{c_b} preserves the form

$$\langle u, v\rangle_b = u^* c_b \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix} c_b^* v, \quad u, v \in \mathbb{C}^{2n}$$

we see that

$$\begin{aligned} \mathcal{S}_b &= \{n\text{-planes } p_W : \langle\cdot,\cdot\rangle_b|_{p_W} \gg 0\} \\ &= \left\{ W \in \mathrm{Sym}_n(\mathbb{C}) : \begin{pmatrix} -i(W_{11} - \overline{W}_{11}) - \overline{W}_{12}{}^tW_{12} & -\overline{W}_{12}W_{22} - iW_{12} \\ -\overline{W}_{22}{}^tW_{12} + i\,{}^t\overline{W}_{12} & 1 - W_{22}^*W_{22} \end{pmatrix} \gg 0 \right\}. \end{aligned} \tag{11}$$

The domains $\mathcal{S}_b$ are type III Siegel domains [17, 19]. In particular, the domain $\mathcal{S}$ is the Siegel upper half-space:

$$\mathcal{S} = \{W \in \mathfrak{p}_+ \cong \mathrm{Sym}_n(\mathbb{C}) : \mathrm{Im}(W) \gg 0\}.$$

We will need the following easy observation:

Lemma 6.1. *Let* $W = \begin{pmatrix} W_{11} & W_{12} \\ {}^tW_{12} & W_{22} \end{pmatrix} \in \mathcal{S}_b$. *Then* $\mathrm{Im}\, W_{11} \gg 0$.

Proof. Taking the $b \times b$ principal submatrix of the hermitian matrix in (11) defining $\mathcal{S}_b$ gives

$$-i(W_{11} - \overline{W}_{11}) - \overline{W}_{12}{}^tW_{12} \gg 0 \Rightarrow 2\,\mathrm{Im}\, W_{11} \gg \overline{W}_{12}{}^tW_{12} \geqslant 0.$$

We will also need formulas for the inverse Cayley transforms. For $W \in \mathcal{S}_b$, we have $c_b^{-1} \cdot \exp(W) \in P_+K_{\mathbb{C}}P_-$, and we obtain

$$c_b^{-1}(W) = \begin{pmatrix} (W_{11}-i)(-iW_{11}+1)^{-1} & \sqrt{2}(-iW_{11}+1)^{-1}W_{12} \\ \sqrt{2}\,{}^tW_{12}(-iW_{11}+1)^{-1} & W_{22} + i\,{}^tW_{12}(-iW_{11}+1)^{-1}W_{12} \end{pmatrix}$$

with factor of automorphy

$$j(c_b^{-1}, W) = \begin{pmatrix} \sqrt{2}(-iW_{11}+1)^{-1} & 0 \\ i\,{}^tW_{12}(-iW_{11}+1)^{-1} & 1_{n-b} \end{pmatrix}.$$

■

7. Partial Bargmann Transforms

We begin by introducing the Schrödinger model for irreducible unitary representations of the Heisenberg group. Let $\mathcal{M} = M_{n,k}(\mathbb{R})$ with real inner product $\langle x|y\rangle = \mathrm{Tr}(x \cdot {}^t y)$. We have a real symplectic form on $M_{2n,k}(\mathbb{R}) = \mathcal{M} \times \mathcal{M}$ given by:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right) \mapsto \mathrm{Tr}\left(J\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} {}^t x' & {}^t y' \end{pmatrix}\right) = \langle x|y'\rangle - \langle y|x'\rangle. \tag{12}$$

In this section only, we let $\mathcal{H}$ denote the Heisenberg group $\mathcal{H} = \mathcal{M}\times\mathcal{M}\times\mathbb{R}$ with group law

$$(x, y; t)(x', y'; t') = (x + x', y + y'; t + t' + \frac{1}{2}(\langle x|y'\rangle - \langle y|x'\rangle)).$$

The group $\mathcal{H}$ is of course isomorphic to the Heisenberg group $H = M \times \mathbb{R}$, $M = M_{n,k}(\mathbb{C})$, which we used for the Fock model. We express this isomorphism in terms of the full Cayley transform c as follows. For $w \in M$, identify $w \leftrightarrow \begin{pmatrix} w \\ -i\overline{w} \end{pmatrix}$. Define a map $\mathcal{M}\times\mathcal{M} \to M$ sending $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto w$ where $\begin{pmatrix} w \\ -i\overline{w} \end{pmatrix} = c^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow w = \frac{1}{\sqrt{2}}(x - iy)$.

Lemma 7.1. *The map $\mathcal{H} \to H$ defined by $(x, y; t) \mapsto (\frac{1}{\sqrt{2}}(x - iy); t)$ is a group homomorphism.*

Proof. The result follows from the fact that

$$\mathrm{Tr}\left(J\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} {}^t x' & {}^t y' \end{pmatrix}\right) = \mathrm{Tr}\left(c^* J c\begin{pmatrix} w \\ -i\overline{w} \end{pmatrix} \cdot \begin{pmatrix} w'^* & i\overline{w}'^* \end{pmatrix}\right) = 2\ \mathrm{Im}(w|w'). \qquad \blacksquare$$

We define an irreducible unitary representation ρ of $\mathcal{H}$ on $L^2(\mathcal{M})$ with central character $\chi(t) = e^{-\pi i t}$ by

$$\rho(x_0, y_0; t) f(x) = e^{-\pi i t} e^{-\pi i \langle x_0|x\rangle} e^{\frac{\pi i}{2}\langle x_0|y_0\rangle} f(x - y_0).$$

The representation $(\rho, L^2(\mathcal{M}))$ of $\mathcal{H}$ is called the Schrödinger model. We now wish to establish the connection between the representation ρ of $\mathcal{H}$ and the Fock model ρ_0 of H.

Following [6], define the Fourier-Wigner transform $V : L^2(\mathcal{M}) \times L^2(\mathcal{M}) \to L^\infty(\mathcal{M} \times \mathcal{M})$ by

$$V(f, g)(p, q) = \langle \rho(p, q) f, g\rangle_{L^2(\mathcal{M})}$$

for $(p, q) \in \mathcal{M} \times \mathcal{M}$. We think of $V(f, g)$ as a matrix coefficient for ρ where the dependence on the variable t is being ignored. One can easily verify:

$$V(\rho(x, y) f, g)(p, q) = e^{-\frac{\pi i}{2}(\langle p|y\rangle - \langle q|x\rangle)} V(f, g)(p + x, q + y). \tag{13}$$

Now we let $\phi_0(x) = e^{-\frac{\pi}{2}\langle x|x\rangle} \in L^2(\mathcal{M})$ and calculate:

$$\begin{aligned} V(f,\phi_0)(p,q) &= \langle f|\rho(-p,-q)\phi_0\rangle \\ &= \int_{\mathcal{M}} f(x)e^{-\pi i\langle p-iq|x\rangle}e^{\frac{\pi}{4}\langle p-iq|p-iq\rangle}e^{-\frac{\pi}{4}(\langle p|p\rangle+\langle q|q\rangle)}e^{-\frac{\pi}{2}\langle x|x\rangle} \\ &= e^{-\frac{\pi}{2}|z|^2}\cdot e^{\frac{\pi}{2}\langle z|z\rangle}\int_{\mathcal{M}} f(x)e^{\pi i\sqrt{2}\langle z|x\rangle}e^{-\frac{\pi}{2}\langle x|x\rangle}dx \end{aligned}$$

where we have made the change of variables $z = -\frac{1}{\sqrt{2}}(p-iq)$ and we write $|z|^2$ for $\mathrm{Tr}(z\cdot z^*)$.

Definition 7.2. The (full) Bargmann transform Bf of a function $f \in L^2(\mathcal{M})$ is given by

$$Bf(z) = e^{\frac{\pi}{2}|z|^2}V(f,\phi_0)(p,q) = e^{\frac{\pi}{2}\langle z|z\rangle}\int_{\mathcal{M}} f(x)e^{\pi i\sqrt{2}\langle z|x\rangle}e^{-\frac{\pi}{2}\langle x|x\rangle}dx$$

where $z = -\frac{1}{\sqrt{2}}(p-iq)$.

Calculations as in [6] show:

Lemma 7.3. *The Bargmann transform $f \mapsto Bf$ is an isometry from $L^2(\mathcal{M})$ to $\mathcal{F}(M)$.*

By property (13) of the Fourier-Wigner transform, we obtain:

Lemma 7.4. *Let $(x,y) \in \mathcal{M}\times\mathcal{M}$, $f \in L^2(\mathcal{M})$. Then if we take $w = \frac{1}{\sqrt{2}}(x-iy)$, we obtain:*

$$\rho_0(w)Bf(z) = (B\rho(x,y)f)(z).$$

We will also need the formula for the inverse Bargmann transform [6].

Lemma 7.5. *The (full) inverse Bargmann transform $B^{-1}: \mathcal{F}(M) \to L^2(\mathcal{M})$ is given by*

$$B^{-1}g(x) = e^{-\frac{\pi}{2}\langle x|x\rangle}\int_{M} e^{-\pi i\sqrt{2}\langle\overline{z}|x\rangle}e^{\frac{\pi}{2}\langle\overline{z}|\overline{z}\rangle}e^{-\pi|z|^2}g(z)dz$$

for $g \in \mathcal{F}(M)$, provided this integral converges absolutely.

Now fix $b \in \{0\ldots n\}$ and define the mixed polarization model of the Heisenberg group as $\mathcal{H}_b = \mathcal{M}_b \times M_{n-b} \times \mathcal{M}_b \times \mathbb{R}$, where $\mathcal{M}_b = M_{b\times k}(\mathbb{R})$ and $M_{n-b} = M_{(n-b)\times k}(\mathbb{C})$, with group multiplication

$$\begin{aligned} &(x,w,y;t)(x',w',y';t') = \\ &\qquad (x+x',w+w',y+y';t+t'+\frac{1}{2}(\langle x|y'\rangle - \langle y|x'\rangle) + \mathrm{Im}(w|w')). \end{aligned}$$

If we make the identification $(x,w,y) \in \mathcal{M}_b \times M_{n-b} \times \mathcal{M}_b \leftrightarrow \gamma = \begin{pmatrix} x \\ w \\ y \\ -i\overline{w} \end{pmatrix}$ then we get a real symplectic form on $\mathcal{M}_b \times M_{n-b} \times \mathcal{M}_b$ via

$$(\gamma,\gamma') \mapsto \mathrm{Tr}\left(c_b^* J c_b \gamma \cdot \gamma'^*\right) = \langle x|y'\rangle - \langle y|x'\rangle + 2\,\mathrm{Im}(w|w'). \tag{14}$$

We let $L^2(\mathcal{M}_b, \mathcal{F})$ be the space of L^2 functions on $\mathcal{M}_b$ with values in $\mathcal{F}(M_{n-b})$. Alternatively, $L^2(\mathcal{M}_b, \mathcal{F})$ is the Hilbert space of classes of measurable functions f on $\mathcal{M}_b \times M_{n-b}$ such that

$$\int_{\mathcal{M}_b} \int_{M_{n-b}} f(x,v) e^{-\pi(v|v)} dv\, dx < \infty$$

and such that $v \mapsto f(x,v)$ is a holomorphic function on M_{n-b}.

We obtain an irreducible unitary representation ρ of $\mathcal{H}_b$ on $L^2(\mathcal{M}_b, \mathcal{F})$ with central character $\chi(t) = e^{-\pi i t}$ by

$$\rho(x_0, w_0, y_0; t) f(x,v) = \\ e^{-\pi i t} e^{\pi(v|w_0)} e^{-\frac{\pi}{2}(w_0|w_0)} e^{-\pi i \langle x_0|x\rangle} e^{\frac{\pi i}{2}\langle x_0|y_0\rangle} f(x - y_0, v - w_0).$$

The case $b = n$ gives the Schrödinger model of ρ as above, $b = 0$ gives the Fock model, and the values $0 < b < n$ correspond to mixed models [11].

Definition 7.6. For $z \in M$, write $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ where $z_1 \in M_b$ and $z_2 \in M_{n-b}$. The partial Bargmann transform $B_b f$ of a function $f \in L^2(\mathcal{M}_b, \mathcal{F})$ is given by

$$B_b f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{\frac{\pi}{2}\langle z_1|z_1\rangle} \int_{\mathcal{M}_b} f(x, z_2) e^{\pi i \sqrt{2}\langle z_1|x\rangle} e^{-\frac{\pi}{2}\langle x|x\rangle} dx.$$

The following Lemmas follow immediately from Lemmas 7.3, 7.4 and 7.5:

Lemma 7.7. *The partial Bargmann transform $f \mapsto B_b f$ is an isometry from $L^2(\mathcal{M}_b, \mathcal{F})$ to $\mathcal{F}(M)$.*

Lemma 7.8. *Let $(x,y) \in \mathcal{M}_b \times \mathcal{M}_b$, $f \in L^2(\mathcal{M}_b, \mathcal{F})$. Then if we take $w_1 = \frac{1}{\sqrt{2}}(x - iy)$, we obtain:*

$$\rho_0 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} B_b f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (B_b \rho(x, w_2, y) f) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Lemma 7.9. *The inverse partial Bargmann transform B_b^{-1} mapping $\mathcal{F}(M)$ to $L^2(\mathcal{M}_b, \mathcal{F})$ is given by*

$$B^{-1} g \begin{pmatrix} x \\ z_2 \end{pmatrix} = e^{-\frac{\pi}{2}\langle x|x\rangle} \int_{M_b} e^{-\pi i \sqrt{2}\langle \bar{z}_1|x\rangle} e^{\frac{\pi}{2}\langle \bar{z}_1|\bar{z}_1\rangle} g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} e^{-\pi|z_1|^2} dz_1$$

for $g \in \mathcal{F}(M)$, provided that this integral converges absolutely.

8. Intertwining Maps for Mixed Models

We have set up our partial Bargmann transform so that

$$\rho_0(w) B_b f(z) = (B_b \rho(c_b(w)) f)(z) \tag{15}$$

where we are identifying $c_b(w) = c_b\begin{pmatrix} w \\ -i\overline{w} \end{pmatrix} = \begin{pmatrix} x \\ w_2 \\ y \\ -i\overline{w}_2 \end{pmatrix} \leftrightarrow (x, w_2, y)$. Under this identification, the group $G^{c_b} = c_b G c_b^{-1}$ acts on $\mathcal{M}_b \times M_{n-b} \times \mathcal{M}_b$ by matrix multiplication. We define $\overline{\omega} : G^{c_b} \to U(L^2(\mathcal{M}_b, \mathcal{F}))/\mathbb{Z}_2$ by $\overline{\omega}(c_b g c_b^{-1}) = B_b^{-1} \circ \overline{\omega}(g) \circ B_b$ for $g \in G$ and by (15), $\overline{\omega}$ will satisfy

$$\overline{\omega}(g')\rho(x, w, y) = \rho(g' \cdot (x, w, y))\overline{\omega}(g'), \text{ where } g' = c_b g c_b^{-1}.$$

We define the double cover $\widetilde{G}^{c_b}$ of G^{c_b} to be the set $\left\{(g', d^{\frac{1}{2}}(g', W))\right\}$, $g' \in G^{c_b}$, where now we take $W \in \mathcal{S}_b$, so $d(g', W) = \det(j(g', W))$ is a function $d : G^{c_b} \times \mathcal{S}_b \to \mathbb{C} - \{0\}$. Here we can identify a choice of branch of $\det(j(g', W))$ with its value at $W = e_b = c_b(0) = \left(\begin{smallmatrix} i \cdot 1_b & 0 \\ 0 & 0_{n-b} \end{smallmatrix}\right)$.

We lift the functions $j(c_b, \cdot) : \mathcal{D} \to K_{\mathbb{C}}$ and $j(c_b^{-1}, \cdot) : \mathcal{S}_b \to K_{\mathbb{C}}$, now choosing $d^{\frac{1}{2}}(c_b^{-1}, e_b) = \left(\frac{1}{\sqrt{2}}\right)^{\frac{b}{2}}$ to be positive. Thus we obtain a map $\widetilde{G} \xrightarrow{c_b} \widetilde{G}^{c_b}$ by $(g, t) \xmapsto{c_b} (g', t')$, where $g' = c_b g c_b{}^{-1}$ and $t' = s_2\, t\, s_1$ with s_1, s_2 defined by $\widetilde{j}(c_b^{-1}, e_b) = (j(c_b^{-1}, e_b), s_1)$ and $\widetilde{j}(c_b, g(0)) = (j(c_b, g(0)), s_2)$. Then we get:

$$\omega(g', t') = B_b^{-1} \circ \omega(g, t) \circ B_b.$$

Remark 8.1. When $b = n$, this realization of ω gives the Schrödinger model of the oscillator representation of $\widetilde{G}^c$, while the cases $0 < b < n$ correspond to mixed models. For explicit formulas for the action of ω in these models for a generating set of elements of G^{c_b}, see [15] or [16].

Finally, we lift $j : \widetilde{G}^{c_b} \times \mathcal{S}_b \to K_{\mathbb{C}}$ to obtain:

$$\widetilde{j}((g', t'), W) = \widetilde{j}(c_b, g(T))\widetilde{j}((g, t), T)\widetilde{j}(c_b^{-1}, c_b(T))$$

for $T \in \mathcal{D}$, $c_b(T) \in \mathcal{S}_b$.

For $\lambda \in \widehat{O(k)}$, we let $L^2(\mathcal{M}_b, \mathcal{F}; \lambda)$ be the subspace of functions f in $L^2(\mathcal{M}_b, \mathcal{F}) \otimes V_\lambda$ satisfying $f(xc, vc) = \lambda(c)^{-1} f(x, v)$ for all $c \in O(k)$. The partial Bargmann transform B_b maps $L^2(\mathcal{M}_b, \mathcal{F}; \lambda) \to \mathcal{F}(M; \lambda)$, so once again we can replace ω with ω_λ in the above discussion.

We will now construct a $\widetilde{G}^{c_b}$ intertwining map between $L^2(\mathcal{M}_b, \mathcal{F}; \lambda)$ and a space of V_τ-valued functions on $\mathcal{S}_b$, where $V_\tau = \mathcal{H}(M; \lambda)$ as before.

The idea is to apply the partial inverse Bargmann transform to the equation:

$$\omega_\lambda(g, t) q_T h = q_{g(T)} J((g, t), T)^{*-1} h, \; (g, t) \in \widetilde{G}, \; h \in \mathcal{H}(M; \lambda) \tag{16}$$

to obtain a similar expression for $\widetilde{G}^{c_b}$, from which we can determine the intertwining map.

Let $W = \left(\begin{smallmatrix} W_{11} & W_{12} \\ {}^tW_{12} & W_{22} \end{smallmatrix}\right) \in \mathcal{S}_b$ with block decomposition as defined in Section 6.

Proposition 8.2. *For $W \in \mathcal{S}_b$, define a function on $\mathcal{M}_b \times M_{n-b}$ by*

$$q'_W(x, v) = e^{-\frac{\pi i}{2} \operatorname{Tr}({}^tx\overline{W}_{11}x)} e^{\pi \operatorname{Tr}({}^tx\overline{W}_{12}v)} e^{\frac{\pi i}{2} \operatorname{Tr}({}^tv\overline{W}_{22}v)}.$$

Then $q'_W \in L^2(\mathcal{M}_b, \mathcal{F})$.

Proof. Let $T = c_b^{-1}(W) \in \mathcal{D}$. We will show that $q'_W = \alpha B_b^{-1} q_T$, α a constant. Since $q_T \in \mathcal{F}(M)$ and $|q_T(z)| < e^{C\pi|z|^2} = e^{C\pi(|z_1|^2+|z_2|^2)}$, the integral defining $B_b^{-1}q_T$ converges absolutely and gives a function in $L^2(\mathcal{M}_b, \mathcal{F})$.

We will need the following Lemma on Gaussian integrals [6, Appendix A]:

Lemma 8.3. *Let $A, D \in \mathrm{Sym}_N(\mathbb{C})$ with $\|A\| \leq 1$, $\|D\| \leq 1$, and $\|A\|\|D\| < 1$. Then for any $u, v \in M_{N,k}(\mathbb{C})$*

$$\begin{aligned}
&\int_{M_{N,k}(\mathbb{C})} e^{\frac{\pi}{2}\operatorname{Tr}({}^t w A w)} e^{\frac{\pi}{2}\operatorname{Tr}({}^t \overline{w} D \overline{w})} e^{\pi \operatorname{Tr}({}^t u w)} e^{\pi \operatorname{Tr}({}^t v \overline{w})} e^{-\pi(w|w)} dw \\
= \; & \det{}^{-\frac{k}{2}}(1-AD) e^{\frac{\pi}{2}\operatorname{Tr}({}^t u D(1-AD)^{-1}u)} e^{\pi \operatorname{Tr}({}^t v(1-AD)^{-1}u)} e^{\frac{\pi}{2}\operatorname{Tr}({}^t v A(1-DA)^{-1}v)}.
\end{aligned}$$

Now write $T = \left(\begin{smallmatrix} T_{11} & T_{12} \\ {}^tT_{12} & T_{22}\end{smallmatrix}\right)$ and calculate:

$$\begin{aligned}
& B_b^{-1} q_T \begin{pmatrix} x \\ z_2 \end{pmatrix} \\
= \; & e^{-\frac{\pi}{2}\operatorname{Tr}({}^t x x)} e^{\frac{\pi i}{2}\operatorname{Tr}({}^t z_2 \overline{T}_{22} z_2)} \cdot \\
& \int_{M_b} e^{-\pi i\sqrt{2}\operatorname{Tr}({}^t \overline{z}_1 x)} e^{\frac{\pi}{2}\operatorname{Tr}({}^t \overline{z}_1 \overline{z}_1)} e^{\frac{\pi i}{2}\operatorname{Tr}({}^t z_1 \overline{T}_{11} z_1)} e^{\pi i \operatorname{Tr}({}^t z_1 \overline{T}_{12} z_2)} e^{-\pi(z_1|z_1)} dz_1 \\
= \; & \alpha\, e^{-\frac{\pi}{2}\operatorname{Tr}({}^t x x)} e^{\frac{\pi i}{2}\operatorname{Tr}({}^t z_2 \overline{T}_{22} z_2)} e^{\pi \operatorname{Tr}({}^t x(-i\overline{T}_{11})(1-i\overline{T}_{11})^{-1}x)} e^{\pi\sqrt{2}\operatorname{Tr}({}^t x(1-i\overline{T}_{11})^{-1}\overline{T}_{12} z_2)} \cdot \\
& e^{-\frac{\pi}{2}\operatorname{Tr}({}^t z_2 {}^t\overline{T}_{12}(1-i\overline{T}_{11})^{-1}\overline{T}_{12} z_2)}, \ \alpha \text{ a constant, by the Lemma} \\
= \; & \alpha\, e^{-\frac{\pi i}{2}\operatorname{Tr}({}^t x \overline{W}_{11} x)} e^{\pi \operatorname{Tr}({}^t x \overline{W}_{12} z_2)} e^{\frac{\pi i}{2}\operatorname{Tr}({}^t z_2 \overline{W}_{22} z_2)}
\end{aligned}$$

where we are using the formulas for $c_b\left(\begin{smallmatrix} T_{11} & T_{12} \\ {}^tT_{12} & T_{22}\end{smallmatrix}\right) = \left(\begin{smallmatrix} W_{11} & W_{12} \\ {}^tW_{12} & W_{22}\end{smallmatrix}\right)$ found in Section 6. ■

Theorem 8.4. *Let $h \in \mathcal{H}(M; \lambda)$. Then for $T \in \mathcal{D}$, $c_b(T) \in \mathcal{S}_b$, the function $f(x, z_2) = q'_{c_b(T)}(x, z_2)h(x, iz_2) \in L^2(\mathcal{M}_b, \mathcal{F}; \lambda)$. We have:*

$$(B_b f)(z) = C\, q_T(z)(J(c_b, T)^* h)(iz), \ C \text{ a constant}$$

Proof. We will need the mean value theorem for harmonic polynomials:

Lemma 8.5. *Let h be a harmonic polynomial on $\mathbb{R}^N$ extended holomorphically to $\mathbb{C}^N$. Then:*

$$\int_{\mathbb{R}^n} e^{-2i\langle x|y\rangle} e^{-\langle x|x\rangle} h(x) dx = \pi^{\frac{n}{2}} e^{-\langle y|y\rangle} h(-iy)$$

This leads to the following (adapted from [14]):

Lemma 8.6. *Let $Z \in \mathrm{Sym}_N(\mathbb{C})$, $\mathrm{Im}(Z) \gg 0$, and h a polynomial on $\mathcal{M}_N$ extended holomorphically to M_N such that $x \mapsto h(gx)$ is harmonic for any $g \in GL(N, \mathbb{R})$. Then:*

$$\int_{\mathcal{M}} e^{2\pi i\langle x|y\rangle} e^{\pi i\langle Zx|x\rangle} h(x) dx = \left(\det \frac{Z}{i}\right)^{-\frac{k}{2}} e^{\pi i\langle -Z^{-1}y|y\rangle} h(-Z^{-1}y).$$

Here, choose the branch of the square root function so that $\left(\det \frac{Z}{i}\right)^{-\frac{1}{2}}$ is positive when $Z = i \cdot 1_N$.

Proof. Both sides are holomorphic in Z, so by analytic continuation, it suffices to prove the result for $Z = i\alpha^2$, where α is a real positive definite symmetric matrix. Then

$$\int_{\mathcal{M}} e^{2\pi i\langle x|y\rangle} e^{-\pi\langle \alpha x|\alpha x\rangle} h(x)dx = (\det\alpha)^{-k} \int_{\mathcal{M}} e^{2\pi\langle x|\alpha^{-1}y\rangle} e^{-\pi\langle x|x\rangle} h(\alpha^{-1}x)dx$$
$$= (\det\alpha)^{-k}\, e^{-\pi\langle \alpha^{-1}y|\alpha^{-1}y\rangle} h(\alpha^{-1}(i\alpha^{-1}y)) = \left(\det\frac{Z}{i}\right)^{-\frac{k}{2}} e^{\pi i\langle -Z^{-1}y|y\rangle} h(-Z^{-1}y)$$

by the mean value theorem for harmonic polynomials. ■

Now we notice:

Lemma 8.7. *Let $h \in \mathcal{H}(M;\lambda)$ and write (x,v) for coordinates on $\mathcal{M}_b \times M_{n-b}$. Then the function $h_v : x \mapsto h(x,v)$ is a polynomial on $\mathcal{M}_b$ such that $x \mapsto h_v(gx)$ is harmonic for any $g \in GL(b,\mathbb{R})$.*

Proof. By Corollary 5.1, $h \in \mathcal{H}(M) \Leftrightarrow \Delta_{i,j}h(z) = 0$, $1 \le i \le j \le n$, where $\Delta_{i,j} = \sum_{\nu=1}^{k} \frac{\partial^2}{\partial z_{i\nu}\partial z_{j\nu}}$. But then $\Delta_{i,j}h(z) = 0$ for $1 \le i \le j \le b$. Thus writing $z = (z_1, z_2) \in M_b \times M_{n-b}$, the polynomial $z_1 \mapsto h(z_1,z_2)$ is in $\mathcal{H}(M_b)$. The Lemma follows. ■

This observation will allow us to use Lemma 8.6 to calculate $B_b f$. Letting $W = c_b(T) \in \mathcal{S}_b$, $f(x,z_2) = q'_W(x,z_2)h(x,iz_2)$, and recalling that $\operatorname{Im} W_{11} \gg 0$ by Lemma 6.1, we obtain

$$\begin{aligned}
&B_b f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\
&= e^{\frac{\pi}{2}\operatorname{Tr}({}^t z_1 z_1)} e^{\frac{\pi i}{2}\operatorname{Tr}({}^t z_2 \overline{W}_{22} z_2)} \int_{\mathcal{M}_b} e^{\frac{\pi i}{2}\operatorname{Tr}{}^t x(i-\overline{W}_{11})x} e^{\pi i \operatorname{Tr}{}^t x(\sqrt{2}z_1 - i\overline{W}_{12}z_2)} h(x, iz_2)dx \\
&= C\det\left(\frac{i-\overline{W}_{11}}{\sqrt{2}i}\right)^{-\frac{k}{2}} q_T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} h\left(\begin{pmatrix} \sqrt{2}i(i-\overline{W}_{11})^{-1} & (i-\overline{W}_{11})^{-1}\overline{W}_{12} \\ 0 & 1_{n-b} \end{pmatrix}\begin{pmatrix} iz_1 \\ iz_2 \end{pmatrix}\right) \\
&= C\, q_T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} (J(c_b^{-1}, W)^{*-1}h)\begin{pmatrix} iz_1 \\ iz_2 \end{pmatrix}
\end{aligned}$$

where C is a constant, using the expression for $J(c_b^{-1}, W)$ from Section 6. Now the relation $J(c_b^{-1}, W)^{-1} = J(c_b, T)$ for $T = c_b^{-1}(W)$ gives the result. ■

Let $R_b : \mathcal{H}(M;\lambda) \to L^2(\mathcal{M}_b, \mathcal{F}; \lambda)$ be the map given by $R_b h(x,v) = h(x,iv)$ for $(x,v) \in \mathcal{M}_b \times M_{n-b}$. By the Theorem, we see that applying the partial inverse Bargmann transform B_b to equation (16) and canceling constants on both sides gives:

$$\begin{aligned}
&\omega_\lambda(g',t')q'_{c_b(T)}R_b(J(c_b,T)^{*-1}h) = q'_{g'^{-1}(c_b(T))}R_b(J(c_b, g(T))^{*-1}J((g,t),T)^{*-1}h) \\
\Leftrightarrow\quad &\omega_\lambda(g',t')q'_{c_b(T)}R_b(h) = q'_{g'^{-1}(c_b(T))}R_b(J(c_b, g(T))^{*-1}J((g,t),T)^{*-1}J(c_b,T)^* h)
\end{aligned}$$

We abuse notation and write $q'_{c_b(T)} \circ R_b$ to denote the map $\mathcal{H}(M;\lambda) \to L^2(\mathcal{M}_b, \mathcal{F}; \lambda)$ given by $h \mapsto q'_{c(T)}R_b(h)$. Let $h_1, h_2 \in \mathcal{H}(M;\lambda)$ and $S,T \in \mathcal{D}$, so

$c_b(S), c_b(T) \in \mathcal{S}_b$. Then by the Theorem:

$$\begin{aligned}
&\langle q'_{c_b(T)} \circ R_b(h_1) | q'_{c_b(S)} \circ R_b(h_2) \rangle_{L^2(\mathcal{M}_b, \mathcal{F}; \lambda)} \\
&= \|\beta\|^2 \langle q_T J(c_b, T)^* h_1 | q_S J(c_b, S)^* h_2 \rangle_{\mathcal{F}(M;\lambda)}, \ \beta \text{ a constant} \\
&= \|\beta\|^2 \langle J(c_b, S) Q(S,T) J(c_b, T)^* h_1 | h_2 \rangle_{\mathcal{H}(M;\lambda)}
\end{aligned}$$

Now define $Q'(c_b(S), c_b(T)) = \|\beta\|^2 J(c_b, S) Q(S,T) J(c_b, T)^*$, so that $Q' : \mathcal{S}_b \times \mathcal{S}_b \to \mathrm{Aut}(V_\tau)$. Q' will be a positive definite kernel function because Q is positive definite, and so by [13], Q' will be the kernel function for a space $H(\mathcal{S}_b, \tau)$ of V_τ-valued functions on $\mathcal{S}_b$. For $(g', t') \in \widetilde{G}^{c_b}$ we obtain:

$$\begin{aligned}
&J((g',t'), c_b(S)) Q'(c_b(S), c_b(T)) J((g',t'), c_b(T))^* \\
&= Q'(g'(c_b(S)), g'(c_b(T))) \qquad (17)
\end{aligned}$$

where we can express $J((g',t'), c_b(S)) = J(c_b, g(S)) J((g,t), S) J(c_b, S)^{-1}$ for $(g,t) \in \widetilde{G}$ such that $(g,t) \stackrel{c_b}{\longmapsto} (g', t')$.

It follows that $\widetilde{G}^{c_b}$ acts on $H(\mathcal{S}_b, \tau)$ by

$$T(g',t') f(W) = J((g',t')^{-1}, W)^{-1} f(g'^{-1}(W)) \qquad (18)$$

for $W \in \mathcal{S}_b$ and that T defines a strongly continuous unitary representation of $\widetilde{G}^{c_b}$. Finally, the map

$$q'_{c_b(T)} R_b(h) \mapsto Q'(\cdot, c_b(T)) h$$

extends to an intertwining map Ξ_λ between $(\omega_\lambda, L^2(\mathcal{M}_b, \mathcal{F}; \lambda))$ and $(T, H(\mathcal{S}_b, V_\tau))$ which can be expressed globally by $\langle \Xi_\lambda f(W) | h \rangle_{\mathcal{H}(M;\lambda)} = \langle f | q'_W R_b(h) \rangle_{L^2(\mathcal{M}_b, \mathcal{F}; \lambda)}$. This map can be written as follows:

Theorem 8.8. *Let $f \in L^2(\mathcal{M}_b, \mathcal{F}; \lambda)$ and let $(x,v) \mapsto I_\lambda(x,v)$ be the polynomial function on $\mathcal{M}_b \times M_{n-b}$ with values in $\mathrm{Hom}(V_\tau, V_\lambda)$ defined by $I_\lambda(x,v)h = h(x, iv)$ for $h \in V_\tau = \mathcal{H}(M;\lambda)$. Then:*

$$\Xi_\lambda f(W) = \int_{\mathcal{M}_b} \int_{M_{n-b}} \overline{q'_W(x,v)} I_\lambda(x,v)^* f(x,v) e^{-\pi(v|v)} dv\, dx$$

The proof is as in Proposition 5.7.

Remark 8.9. When $b = n$, the intertwining map given above is exactly the one found in [14].

9. The Fock Model for $U(p,q)$

For $p \geq q > 0$, define $I_{p,q} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$, and let

$$G = U(p,q) = \{ g \in GL(p+q, \mathbb{C}) : g I_{p,q} g^* = I_{p,q} \}.$$

Then G is a real form of $G_\mathbb{C} = GL(p+q, \mathbb{C})$. We write elements $g \in G_\mathbb{C}$ in block form $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in M_{p,p}(\mathbb{C}), D \in M_{q,q}(\mathbb{C})$. We take the maximal compact subgroup to be $K = G \cap U(p+q) \cong U(p) \times U(q)$, and we obtain the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-$, where

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} : Z \in M_{p,q}(\mathbb{C}) \right\},$$

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} : X_1 \in \mathfrak{gl}(p,\mathbb{C}), X_2 \in \mathfrak{gl}(q,\mathbb{C}) \right\},$$

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} : Y \in M_{q,p}(\mathbb{C}) \right\}.$$

We identify $K_{\mathbb{C}}$ with $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$, and for $(g,Z) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ such that $g \cdot \exp(Z) \in \Omega = P_+ K_{\mathbb{C}} P_-$, we write the automorphy factor

$$j(g,Z) = k(g \cdot \exp(Z)) = (j_1(g,Z), j_2(g,Z))$$

with $j_1(g,Z) \in GL(p,\mathbb{C})$ and $j_2(g,Z) \in GL(q,\mathbb{C})$. The action of g on $\mathfrak{p}_+$ defined by $\exp(g(Z)) = (g \cdot \exp(Z))_+$ is given by

$$g(Z) = (AZ+B)(CZ+D)^{-1}.$$

Here, the image of the Harish-Chandra embedding $G/K \to \mathcal{D} \subset \mathfrak{p}_+$ is given by the generalized unit disk

$$\mathcal{D} = \{q\text{-planes } l_Z : (\cdot,\cdot)|_{l_Z} \gg 0\} = \{Z \in M_{p,q}(\mathbb{C}) : 1_q - Z^* Z \gg 0\}.$$

We now follow the exposition in [1] to establish formulas for the Fock model of the oscillator representation of $G = U(p,q)$.

Let $M_{p+q} = M_{p+q,k}(\mathbb{C})$ and define a hermitian inner product h_0 on M_{p+q} by $h_0(u,u') = \mathrm{Tr}(I_{p,q} u u'^*)$. We get an associated Heisenberg group $H_0 = M_{p+q} \times \mathbb{R}$ with group multiplication

$$(u;t)(u';t') = (u+u'; t+t' - \mathrm{Im}\ h_0(u,u')).$$

Write $u \in M_{p+q}$ as $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with $u_1 \in M_{p,k}(\mathbb{C})$ and $u_2 \in M_{q,k}(\mathbb{C})$. As before write $(u|u') = \mathrm{Tr}(uu'^*)$, $|u|^2 = (u|u)$. Let $\mathcal{F}_{p,q}$ be the Hilbert space of functions f on $M_{p,q}$ such that $u_1 \mapsto f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is holomorphic, $u_2 \mapsto f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is antiholomorphic, and f is square-integrable with respect to the Gaussian measure $d\mu(u) = e^{-\pi(u|u)}du$. Our convention for Lebesgue measure is that $du = d(\mathrm{Re}\, u)\, d(\mathrm{Im}\, u)$. The reproducing kernel K for $\mathcal{F}_{p,q}$ is given by $K(z,w) = e^{\pi(z_1|w_1)} e^{\pi(w_2|z_2)}$, that is,

$$f(z) = \int_{M_{p+q}} K(z,w) f(w) e^{-\pi(w|w)} dw$$

for $f \in \mathcal{F}_{p,q}$ and $z \in M_{p+q}$.

We obtain an irreducible unitary representation ρ_0 of H_0 on $\mathcal{F}_{p,q}$ with central character $\chi(t) = e^{-\pi i t}$ by

$$\rho_0(w;t) f(z) = e^{-\pi i t} K(z,w) e^{-\frac{\pi}{2}(w|w)} f(z-w)$$

for $(w;t) \in H_0$ and $f \in \mathcal{F}_{p,q}$.

Identify $\mathfrak{p}_+$ with $M_{p,q}(\mathbb{C})$ as above. For $T \in \mathfrak{p}_+$ we can define a function on M_{p+q} by

$$q_T(z) = e^{\pi(z_1|Tz_2)} = e^{\pi \operatorname{Tr}(z_1 z_2^* T^*)}.$$

Clearly q_T is holomorphic in z_1 and antiholomorphic in z_2. Moreover, we have from [1, Proposition 1.4]:

Lemma 9.1. *The function q_T lies in $\mathcal{F}_{p,q}$ if and only if $T \in \mathcal{D}$.*

Since the left action of $G = U(p,q)$ on M_{p+q} preserves the form h_0, the action $(w;t) \mapsto (g \cdot w; t)$, where $(w;t) \in H_0$ and $g \in G$, is an automorphism of H_0 which fixes the center of H_0. By the Stone-von Neumann theorem, we obtain a projective unitary representation ω of G defined by $\omega(g)\rho_0(w;t) = \rho_0(g \cdot w;t)\omega(g)$. In fact, we can choose the operators $\omega(g)$ such that ω becomes a representation of G as in [1, Theorem 1.12]:

Theorem 9.2. *For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, define the operator $\omega(g)$ on $\mathcal{F}_{p,q}$ by*

$$\omega(g)f(z) = \int_M K_g(z,w)f(w)e^{-\pi(w|w)}dw, \tag{19}$$

$$K_g(z,w) = (det^{-k}A)q_{BD^{-1}}(z)K\left(\begin{pmatrix} A^{-1}z_1 \\ D^{-1}z_2 \end{pmatrix}, w\right)\overline{q}_{-A^{-1}B}(w).$$

Then $\omega : G \to U(\mathcal{F}_{p,q})$ is a continuous unitary representation.

10. The Dual Pair $(U(p,q), U(k))$

Let $G' = U(k)$ act on $\mathcal{F}_{p,q}$ by right multiplication:

$$c \cdot f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = f\begin{pmatrix} z_1 c \\ z_2 c^{*-1} \end{pmatrix}$$

for $c \in U(k)$ and $f \in \mathcal{F}_{p,q}$. The action of G' commutes with ω, and in fact $(U(p,q), U(k))$ is a dual pair inside $Sp(M_{p,q}) = Sp(2(p+q)k, \mathbb{R})$, where $(M_{p,q}, \operatorname{Im} h_0)$ is regarded as a real symplectic vector space.

As in Section 5 and [2, Chapter 7], we define a map

$$\theta : M_{p+q} \to \mathfrak{p}_+, \qquad \theta(z) = z_1 \cdot z_2^*.$$

Let $\mathcal{P}$ be the dense subspace of polynomial functions in $\mathcal{F} = \mathcal{F}_{p,q}$, $\mathcal{I}$ the ideal in $\mathcal{P}$ generated by the matrix entries of $z \to \theta(z)$, and $\mathcal{H}$ be the orthogonal complement of $\mathcal{I}$. We call $\mathcal{H} = \mathcal{H}_{p,q}$ the space of harmonic polynomials.

Now as in [2, Chapter 7]:

Proposition 10.1. *Let $h \in \mathcal{P}$. Then the following are equivalent:*

(i) $h \in \mathcal{H}_{p,q}$

(ii) $g(\partial)h = 0$ *for all matrix entries g of $z \to \theta(z)$*

(iii) $\Delta_{i,j}h(z) = 0,\ 1 \leq i \leq p, \quad 1 \leq j \leq q$ *where* $\Delta_{i,j} = \sum_{\nu=1}^{k} \frac{\partial^2}{\partial z_{i\nu}\partial \overline{z}_{j\nu}}$.

One can calculate directly the action of $K = U(p) \times U(q)$ on $\mathcal{F}_{p,q}$:

Lemma 10.2. *Let $f \in \mathcal{F}$ and $(g_1, g_2) \in K$. Then*

$$\omega(g_1, g_2) f(z) = (\det g_1)^{-k} f\begin{pmatrix} g_1^{-1} z_1 \\ g_2^* z_2 \end{pmatrix}.$$

Let $\widehat{U(k)}$ denote the unitary dual of $U(k)$, and write λ for a representation in $\widehat{U(k)}$, λ' for its contragredient. As in [14, Section 6], we get:

Theorem 10.3. *The space $\mathcal{H}$ is $K \times G'$ invariant and decomposes as a multiplicity free orthogonal direct sum of unitary $K \times G'$ representations $\tau(\lambda) \otimes \lambda'$.*

For (λ, V_λ) in $\widehat{U(k)}$, let $\mathcal{F}_{p,q}(\lambda)$ be the subspace of functions in $\mathcal{F}_{p,q} \otimes V_\lambda$ satisfying $(c \cdot f)(z) = \lambda(c)^{-1} f(z)$ for all $c \in U(k)$. Let ω_λ denote the representation of G on $\mathcal{F}_{p,q}(\lambda)$. Then we have

$$\mathcal{F}_{p,q} = \oplus_{\lambda \in \widehat{U(k)}} \mathcal{F}_{p,q}(\lambda) \otimes V'_\lambda$$

as a representation of $G \times G'$ and

$$\mathcal{H}_{p,q} = \oplus_{\lambda \in \widehat{U(k)}} \mathcal{H}_{p,q}(\lambda) \otimes V'_\lambda$$

as a representation of $K \times G'$.

Finally, analogous to Proposition 5.4, we have:

Proposition 10.4. *Let $\lambda \in \widehat{U(k)}$ be such that $\mathcal{F}_{p,q}(\lambda) \neq 0$. Then $\mathcal{F}_{p,q}(\lambda)$ is an irreducible G representation.*

For $\lambda \in \widehat{U(k)}$ such that the space $\mathcal{H}_{p,q}(\lambda)$ is non-zero, we construct a G-intertwining map between $(\omega, \mathcal{F}_{p,q}(\lambda))$ and sections of the homogeneous vector bundle over $\mathcal{D}$ associated to the K-module $\mathcal{H}_{p,q}(\lambda)$. The intertwining maps for this example were fully worked out in [1] and are similar to the example in Section 5. Here, we merely give an outline and refer to [1] and our previous work for proofs.

We write $\tau = \tau(\lambda)$ for the K representation given by $\omega_\lambda|_K$ acting on $\mathcal{H}_{p,q}(\lambda)$. We extend τ to a holomorphic representation of $K_{\mathbb{C}} = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ as follows. Let τ_0 be the holomorphic representation of $K_{\mathbb{C}}$ on $\mathcal{H}_{p,q}(\lambda)$ given by

$$(\tau_0(g_1, g_2) h)(z_1, z_2) = h(g_1^{-1} z_1, g_2^* z_2)$$

for $h \in \mathcal{H}_{p,q}(\lambda)$, $(g_1, g_2) \in GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$, and $(z_1, z_2) \in M_{p,k}(\mathbb{C}) \times M_{q,k}(\mathbb{C})$. Then $\tau(g_1, g_2) = (\det g_1)^{-k} \tau_0(g_1, g_2)$ for $(g_1, g_2) \in K_{\mathbb{C}}$.

We write $J(g, Z) = \tau(j(g, Z))$ for $(g, Z) \in G_{\mathbb{C}} \times \mathfrak{p}_+$ such that $j(g, Z)$ is defined. Using the formula (19) for ω and the orthogonality between $\mathcal{H}_{p,q}$ and the ideal $\mathcal{I}$ generated by the non-constant $U(k)$-invariants, we prove just as in Lemma 5.5 [1, Proposition 2.2]:

Lemma 10.5. *Let $h \in \mathcal{H}_{p,q}(\lambda)$. Then $\omega_\lambda(g) h(z) = q_{g(0)}(z) (J(g, 0)^{*-1} h)(z)$ for $g \in G$.*

Corollary 10.6. *Let $T \in \mathcal{D}$ and $h \in \mathcal{H}_{p,q}(\lambda)$. Then:*

$$\omega_\lambda(g) q_T h = q_{g(T)} J(g,T)^{*-1} h.$$

Thus we can use the construction outlined in Section 3. Let $V_\tau = \mathcal{H}_{p,q}(\lambda)$, and take $v, w \in V_\tau$. We obtain a positive definite operator-valued kernel function $Q : \mathcal{D} \times \mathcal{D} \to \mathrm{Aut}(V_\tau)$ by $\langle q_T v | q_S w \rangle = \langle Q(S,T) v | w \rangle$ and calculate that $Q(S,T) = \tau(q(S,T))$. Q gives the reproducing kernel for a space of V_τ-valued holomorphic functions on $\mathcal{D}$, denoted by $H(\mathcal{D}, \tau)$, on which G acts by translation T_τ, where

$$T_\tau(g) F(T) = J(g^{-1}, T)^{-1} F(g^{-1}(T))$$

for $g \in G$, $T \in \mathcal{D}$, and $F \in H(\mathcal{D}, \tau)$.

The intertwining map $\Xi_\lambda : \mathcal{F}_{p,q}(\lambda) \to H(\mathcal{D}, \tau)$ is given on a dense subset of $\mathcal{F}_{p,q}(\lambda)$ by $q_T v \mapsto Q(\cdot, T) v$ and is globally defined by $\Xi_\lambda f(S) = q_S^* f$ for $f \in \mathcal{F}_{p,q}$, regarding q_S as an operator $\mathcal{H}_{p,q}(\lambda) \to \mathcal{F}_{p,q}(\lambda)$ given by multiplication by the function q_S.

Finally, another form of the intertwining map is given by:

Proposition 10.7. *Let $f \in \mathcal{F}_{p,q}(\lambda)$ and let $w \mapsto I_\lambda(w)$ be the $Hom(V_\tau, V_\lambda)$-valued polynomial function on M defined by $I_\lambda(w) h = h(w)$ for $h \in V_\tau = \mathcal{H}_{p,q}(\lambda)$. Then:*

$$\Xi_\lambda f(T) = \int_M \overline{q_T(w)} I_\lambda(w)^* f(w) e^{-\pi(w|w)} dw.$$

11. Cayley Transforms for $U(p,q)$

Retain the notation from Section 6. One can define a maximal set of strongly orthogonal noncompact roots $\Psi = \{\psi_1, \ldots, \psi_q\}$ such that $h_{\psi_i} = E_{i,i} - E_{p+i,p+i} \in i\mathfrak{t}$, $e_{\psi_i} = -iE_{i,p+i} \in \mathfrak{g}_{\mathbb{C}}^{\psi_i}$, $e_{-\psi_i} = iE_{p+i,i} \in \mathfrak{g}_{\mathbb{C}}^{-\psi_i}$ and $[e_{\psi_i}, e_{-\psi_i}] = h_{\psi_i}$ for $i = 1 \ldots q$. Then as before $x_{\psi_i} = e_{\psi_i} + e_{-\psi_i}$ and $y_{\psi_i} = i(e_{\psi_i} - e_{-\psi_i})$ in $\mathfrak{p}$, and for $b = 1 \ldots q$, define $\Gamma_b = \{\psi_1, \ldots, \psi_b\} \subset \Psi$, $x_b = \sum_{\psi \in \Gamma_b} x_\psi$, and $y_b = \sum_{\psi \in \Gamma_b} y_\psi \in \mathfrak{p}$. The matrices for x_b and y_b are exactly the same are those given in Section 6. Finally, the b^{th} partial Cayley transform $c_b = c_{\Gamma_b} = \prod_{\psi \in \Gamma_b} \exp(\frac{\pi i}{4} y_\psi)$ is also identical to the one in Section 6.

For $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{D}$, we have

$$c_b(T) = \begin{pmatrix} (T_{11} + i)(iT_{11} + 1)^{-1} & \sqrt{2}(iT_{11} + 1)^{-1} T_{12} \\ \sqrt{2}\, T_{21}(iT_{11} + 1)^{-1} & T_{22} - i\, T_{21}(iT_{11} + 1)^{-1} T_{12} \end{pmatrix}$$

with the factor of automorphy

$$j(c_b, T) = \left(\begin{pmatrix} \sqrt{2}(iT_{11} + 1)^{-1} & 0 \\ -i\, T_{21}(iT_{11} + 1)^{-1} & 1_{p-b} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}(iT_{11} + 1) & \frac{i}{\sqrt{2}} T_{12} \\ 0 & 1_{q-b} \end{pmatrix} \right).$$

Write $\mathcal{S}_b = c_b(\mathcal{D}) \subset \mathfrak{p}_+$ for the unbounded realizations of G/K obtained as above, $c = c_q$ for the full Cayley transform and $\mathcal{S} = c(\mathcal{D}) \subset \mathfrak{p}_+$. As G^{c_b} preserves the form

$$(u, v)_b = u^* c_b I_{p,q} c_b^* v, \quad u, v \in \mathbb{C}^{p+q}$$

we see that

$$\mathcal{S}_b = \{q\text{-planes } l_W : (\cdot,\cdot)_b|_{l_W} \gg 0\}$$
$$= \left\{ W \in M_{p,q}(\mathbb{C}) : \begin{pmatrix} -i(W_{11}-W_{11}^*) - W_{21}^*W_{21} & -W_{21}^*W_{22} - iW_{12} \\ -W_{22}^*W_{21} + iW_{12}^* & 1 - W_{22}^*W_{22} \end{pmatrix} \gg 0 \right\}. \tag{20}$$

For $W \in \mathcal{S}$, we can write $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ with $W_1 \in M_{q,q}(\mathbb{C}), W_2 \in M_{p-q,q}(\mathbb{C})$. Then $\mathcal{S}$ is the type II Siegel domain [17]:

$$\mathcal{S} = \left\{ \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in \mathfrak{p}_+ : W_1 = X + iY, X = X^*, Y = Y^*, Y - \frac{W_2^*W_2}{2} \gg 0 \right\}.$$

The domains $\mathcal{S}_b, b \neq q$ have the structure of type III Siegel domains [17]. We observe:

Lemma 11.1. *Let* $W = \left(\begin{smallmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{smallmatrix}\right) \in \mathcal{S}_b$. *Write* $W_{11} = X + iY$ *with* $X, Y \in \mathrm{Herm}_b(\mathbb{C})$. *Then* $Y \gg 0$.

Proof. Taking the $b \times b$ principal submatrix of the hermitian matrix in (20) defining $\mathcal{S}_b$ gives

$$-i(W_{11} - W_{11}^*) - W_{21}^*W_{21} \gg 0 \Rightarrow 2Y \gg W_{21}^*W_{21} \geqslant 0. \qquad \blacksquare$$

For the inverse Cayley transforms, taking $W \in \mathcal{S}_b$, we obtain:

$$c_b^{-1}(W) = \begin{pmatrix} (W_{11} - i)(-iW_{11}+1)^{-1} & \sqrt{2}(-iW_{11}+1)^{-1}W_{12} \\ \sqrt{2}\,W_{21}(-iW_{11}+1)^{-1} & W_{22} + i\,W_{21}(-iW_{11}+1)^{-1}W_{12} \end{pmatrix} \tag{21}$$

with factor of automorphy

$$j(c_b^{-1}, W) = \left(\begin{pmatrix} \sqrt{2}(-iW_{11}+1)^{-1} & 0 \\ i\,W_{21}(-iW_{11}+1)^{-1} & 1_{p-b} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}}(-iW_{11}+1) & -\frac{i}{\sqrt{2}}W_{12} \\ 0 & 1_{q-b} \end{pmatrix} \right).$$

12. Generalized Partial Bargmann Transforms

We now define mixed polarization models of the complex Heisenberg group and corresponding models for its irreducible unitary representations.

Fix $b \in \{0 \dots q\}$ and define a hermitian inner product h_b on $M_{p+q} = M_{p+q,k}(\mathbb{C})$ by

$$h_b(u, u') = \mathrm{Tr}(c_b I_{p,q} c_b^* u u'^*) = h_0(c_b^{-1}u, c_b^{-1}u').$$

We get an associated Heisenberg group $H_b = M_{p+q} \times \mathbb{R}$ with group multiplication

$$(u; t)(u'; t') = (u + u'; t + t' - \mathrm{Im}\ h_b(u, u')).$$

The following Lemma is clear from our choice of group law:

Lemma 12.1. *The map $H_0 \to H_b$ defined by $(w;t) \mapsto (c_b(w);t)$ is a group homomorphism.*

We let $L^2(M_b, \mathcal{F}) = L^2(M_b, \mathcal{F}_{p-b,q-b})$ be the space of L^2 functions on M_b with values in $\mathcal{F}_{p-b,q-b}$. Then $L^2(M_b, \mathcal{F})$ is the Hilbert space of classes of measurable functions f on $M_b \times M_{p-b} \times M_{q-b}$ such that

$$\int_{M_b} \int_{M_{p-b+q-b}} f\begin{pmatrix} a \\ z_1 \\ z_2 \end{pmatrix} e^{-\pi|z|^2} dz\, da < \infty$$

and such that $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto f\begin{pmatrix} a \\ z_1 \\ z_2 \end{pmatrix}$ is holomorphic in z_1 and antiholomorphic in z_2.

Write $H_b = M_b \times M_{p-b} \times M_b \times M_{q-b} \times \mathbb{R}$ and $(u;t) \in H_b$ as $u = \left(\begin{pmatrix} a_1 \\ w_1 \\ a_2 \\ w_2 \end{pmatrix}, t\right)$. We obtain an irreducible unitary representation ρ_b of H_b on $L^2(M_b, \mathcal{F})$ with central character $\chi(t) = e^{-\pi i t}$ by

$$\rho_b\left(\begin{pmatrix} a_1 \\ w_1 \\ a_2 \\ w_2 \end{pmatrix}, t\right) f\begin{pmatrix} a \\ z_1 \\ z_2 \end{pmatrix} =$$

$$e^{-\pi i t} e^{-2\pi i \operatorname{Re}(a_1|a)} e^{\pi i \operatorname{Re}(a_1|a_2)} e^{\pi(z_1|w_1)} e^{\pi(w_2|z_2)} e^{\frac{\pi}{2}|w|^2} f\begin{pmatrix} a - a_1 \\ z_1 - w_1 \\ z_2 - w_2 \end{pmatrix}. \quad (22)$$

Let $H_b' = M_b \times M_b \times \mathbb{R}$ be the subgroup of H_b consisting of elements $\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, t\right) \leftrightarrow \left(\begin{pmatrix} a_1 \\ 0 \\ a_2 \\ 0 \end{pmatrix}, t\right)$, and let $\rho_b' = \rho_b|_{H_b'}$. As in Section 7, we define a Fourier-Wigner transform $V_b : L^2(M_b) \times L^2(M_b) \to L^\infty(M_b \times M_b)$ by

$$V(f,g)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \langle \rho_b' \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} f, g \rangle_{L^2(M_b)}.$$

Now we let $\phi_0(a) = e^{-\frac{\pi}{2}(a|a)} \in L^2(M_b)$ and calculate:

$$\begin{aligned} V_b(f,\phi_0)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \langle f | \rho_b' \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix} \phi_0 \rangle \\ &= e^{-\frac{\pi}{2}|z_1|^2} e^{-\frac{\pi}{2}|z_2|^2} \cdot e^{-\pi i (z_1|z_2)} \int_{M_b} f(a) e^{\pi i \sqrt{2}(z_1|a)} e^{\pi\sqrt{2}(a|z_2)} e^{-\pi(a|a)} da \end{aligned}$$

where we have made the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c_b'^{-1} \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}, \qquad c_b' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_b & i \cdot 1_b \\ i \cdot 1_b & 1_b \end{pmatrix}.$$

Define a map $B_b' : L^2(M_b) \to \mathcal{F}_{b,b}$ by

$$B_b' f \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{\frac{\pi}{2}|z|^2} V_b(f,\phi_0)\left(c_b' \begin{pmatrix} -z_1 \\ -z_2 \end{pmatrix}\right)$$

where we write $|z|^2 = |z_1|^2 + |z_2|^2$. Then by calculations similar to those in Lemmas 7.3 and 7.4, we obtain:

Proposition 12.2. *The transform B_b' is an isometry between $L^2(M_b)$ and $\mathcal{F}_{b,b}$ which satisfies*

$$B_b'\,\rho_b'\left(c_b'\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{-\frac{\pi}{2}|w|^2} e^{\pi(z_1|w_1)} e^{\pi(w_2|z_2)} B_b' f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \rho_0'\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} B_b' f\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where $\rho_0'\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \rho_0\begin{pmatrix} w_1 \\ 0 \\ w_2 \\ 0 \end{pmatrix}$.

Definition 12.3. The generalized partial Bargmann transform $B_b f$ of a function $f \in L^2(M_b, \mathcal{F})$ is given by

$$\begin{aligned} B_b f\begin{pmatrix} z_1 \\ v_1 \\ z_2 \\ v_2 \end{pmatrix} &= \left(B_b' f\begin{pmatrix} \cdot \\ v_1 \\ \cdot \\ v_2 \end{pmatrix}\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= e^{-\pi i(z_1|z_2)} \int_{M_b} f\begin{pmatrix} a \\ v_1 \\ v_2 \end{pmatrix} e^{\pi i\sqrt{2}(z_1|a)} e^{\pi\sqrt{2}(a|z_2)} e^{-\pi(a|a)}\, da. \end{aligned}$$

Corollary 12.4. *The transform B_b is an isometry between $L^2(M_b, \mathcal{F}_{p-b,q-b})$ and $\mathcal{F}_{p,q}$ which satisfies*

$$B_b\,\rho_b\left(c_b\begin{pmatrix} w_1 \\ v_1 \\ w_2 \\ v_2 \end{pmatrix}\right) f = \rho_0\begin{pmatrix} w_1 \\ v_1 \\ w_2 \\ v_0 \end{pmatrix} B_b f.$$

By a calculation similar to Lemma 7.5, we obtain the inverse partial Bargmann transforms B_b^{-1}:

Lemma 12.5. *The generalized partial inverse Bargmann transform B_b^{-1} mapping $\mathcal{F}_{p,q}$ to $L^2(M_b, \mathcal{F}_{p-b,q-b})$ is given by*

$$B_b^{-1} g\begin{pmatrix} a \\ v_1 \\ v_2 \end{pmatrix} = e^{-\frac{\pi}{2}(a|a)} \int_{M_b} e^{\pi i(z_2|z_1)} e^{-\sqrt{2}\pi i(a|z_1)} e^{\sqrt{2}\pi(z_2|a)} g\begin{pmatrix} z_1 \\ v_1 \\ z_2 \\ v_2 \end{pmatrix} e^{-\pi|z|^2} dz$$

for $g \in \mathcal{F}_{p,q}$, $dz = dz_1\, dz_2$, provided that this integral converges absolutely.

13. $U(p,q)$-Intertwining Maps for Mixed Models

As in Section 8, we have an action of the group $G^{c_b} = c_b G c_b^{-1}$ on $M_b \times M_{p-b} \times M_b \times M_{q-b}$ by matrix multiplication and hence an action on H_b. We can define a representation $\omega : G^{c_b} \to U(L^2(M_b, \mathcal{F}))$ by $\omega(c_b g c_b^{-1}) = B_b^{-1} \circ \omega(g) \circ B_b$ for $g \in G$. By Corollary 12.4 and the relation $\omega(g)\rho_0(u;t) = \rho_0(g \cdot u; t)\omega(g)$ for $g \in G$ and $(u;t) \in H_0$, we have

$$\omega(g')\rho_b(w;t) = \rho(g' \cdot w; t)\omega(g')$$

for $g' = c_b g c_b^{-1}$ and $(w,t) \in H_b$.

Remark 13.1. This realization of ω acting on $L^2(M_b,\mathcal{F})$ is a mixed model of the oscillator representation of G^{c_b}. For explicit formulas giving the action of ω for a generating set of elements of G^{c_b} in this model, see [15] or [16].

For $\lambda \in \widehat{U(k)}$, let $L^2(M_b,\mathcal{F};\lambda)$ be the subspace of functions f from $M_b\times M_{p-b}\times M_{q-b}$ to V_λ satisfying $\begin{pmatrix} a\\ v_1\\ v_2\end{pmatrix} \mapsto (f\begin{pmatrix} a\\ v_1\\ v_2\end{pmatrix},\phi)\in L^2(M_b,\mathcal{F})$ for every $\phi\in V'_\lambda$ and $f\begin{pmatrix} ac\\ v_1c\\ v_2c\end{pmatrix} = \lambda(c)^{-1}f\begin{pmatrix} a\\ v_1\\ v_2\end{pmatrix}$ for all $c\in U(k)$. The partial Bargmann transform B_b maps $L^2(M_b,\mathcal{F};\lambda)\to \mathcal{F}(M_{p+q};\lambda)$, so we can replace ω with ω_λ above.

We now construct a G^{c_b} intertwining map between $L^2(M_b,\mathcal{F};\lambda)$ and a space of V_τ-valued functions on $\mathcal{S}_b$, where $V_\tau = \mathcal{H}_{p,q}(\lambda)$.

Let $W = \left(\begin{smallmatrix} W_{11} & W_{12}\\ W_{21} & W_{22}\end{smallmatrix}\right)\in\mathcal{S}_b$.

Proposition 13.2. *For $W\in\mathcal{S}_b$, define a function on $M_b\times M_{p-b}\times M_{q-b}$ by*

$$q'_W\begin{pmatrix} a\\ v_1\\ v_2\end{pmatrix} = e^{-\pi i\operatorname{Tr}(aa^*W_{11}^*)}e^{-\pi i\operatorname{Tr}(av_2^*W_{12}^*)}e^{\pi\operatorname{Tr}(v_1a^*W_{21}^*)}e^{\pi\operatorname{Tr}(v_1v_2^*W_{22}^*)}.$$

Then $q'_W\in L^2(M_b,\mathcal{F})$.

Proof. By a calculation similar to the one found in the proof of Proposition 8.2, one can show that $q'_W = \alpha B_b^{-1}q_T$, where $T = c_b^{-1}(W)\in\mathcal{D}$ and α a constant. Here we note that $\left|q_T\left(\begin{smallmatrix} z_1\\ v_1\\ z_2\\ v_2\end{smallmatrix}\right)\right| < e^{C\pi(|z|^2+|v|^2)}$, C a constant, so the integral defining $B_b^{-1}q_T$ converges absolutely and gives a function in $L^2(M_b,\mathcal{F})$. ∎

Theorem 13.3. *Let $h\in\mathcal{H}_{p,q}(\lambda)$. Then for $T\in\mathcal{D}$, $W = c_b(T)\in\mathcal{S}_b$, the function $f_W\begin{pmatrix} a\\ v_1\\ v_2\end{pmatrix} = q'_W\begin{pmatrix} a\\ v_1\\ v_2\end{pmatrix} h\left(\begin{pmatrix} a\\ iv_1\end{pmatrix},\begin{pmatrix} a\\ v_2\end{pmatrix}\right)\in L^2(M_b,\mathcal{F}_{p-b,q-b};\lambda)$. We have:*

$$(B_bf_W)\left(\begin{smallmatrix} z_1\\ v_1\\ z_2\\ v_2\end{smallmatrix}\right) = C\,q_T\left(\begin{smallmatrix} z_1\\ v_1\\ z_2\\ v_2\end{smallmatrix}\right)(J(c_b,T)^*h)\left(\begin{pmatrix} iz_1\\ iv_1\end{pmatrix},\begin{pmatrix} z_2\\ v_2\end{pmatrix}\right),\ C \text{ a constant.}$$

Proof. It will be convenient to work with holomorphic harmonic polynomials, that is, holomorphic V_λ-valued polynomials P on $M_{p+q,k}(\mathbb{C})$ such that

$$\Delta_{i,j}h(z) = 0,\ 1\le i\le p,\quad 1\le j\le q \text{ where } \Delta_{i,j} = \sum_{\nu=1}^{k}\frac{\partial^2}{\partial z_{i\nu}\partial z_{j\nu}}.$$

We will call the space of all such polynomials $\mathcal{H}_{p+q}(\lambda)$. Note that we have a map

$$\mathcal{H}_{p,q}(\lambda)\to\mathcal{H}_{p+q}(\lambda),\qquad h\mapsto P_h$$

given by $P_h(z_1,z_2) = h(z_1,\overline{z}_2), z_1\in M_{p,k}(\mathbb{C}), z_2\in M_{q,k}(\mathbb{C})$. The representation τ_0 of $GL(p,\mathbb{C})\times GL(q,\mathbb{C})$ on $\mathcal{H}_{p,q}(\lambda)$ corresponds to a representation on $\mathcal{H}_{p+q}(\lambda)$, which we also call τ_0, given by $\tau_0(g_1,g_2)P(z_1,z_2) = P(g_1^{-1}z_1,{}^tg_2z_2)$. The following Lemma is adapted from [14, Lemma 4.2]:

Lemma 13.4. *Let* $Z = X + iY \in M_{b,b}(\mathbb{C})$ *with* $X = X^*$, $Y = Y^*$, $Y \gg 0$. *Let* $P \in \mathcal{H}_{p+q}(\lambda)$, *and* $a, a_1, a_2 \in M_{b,k}(\mathbb{C})$. *Then:*

$$\begin{aligned}&\int_{M_b} e^{-\pi i \operatorname{Tr} a^* a_1} e^{-\pi i \operatorname{Tr} a a_2^*} e^{-\pi i \operatorname{Tr} a a^* Z^*} P\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right) da \\ = \ & (i)^{bk} e^{\pi i \operatorname{Tr} a_1 a_2^* Z^{*-1}} \det(-Z^*)^{-k} \cdot \\ & \tau_0\left(\begin{pmatrix} -Z^* & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -Z^{*-1} & 0 \\ 0 & 1 \end{pmatrix}\right) P\left(\begin{pmatrix} a_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a}_2 \\ \overline{v}_2 \end{pmatrix}\right).\end{aligned}$$

Proof. As both sides are holomorphic functions of Z^*, it is enough to verify the equation for $Z = iA^2$, where A is a positive definite hermitian matrix. By the change of variables $a \mapsto A^{-1}a$ and the relation

$$P\left(\begin{pmatrix} Aa \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{Aa} \\ \overline{v}_2 \end{pmatrix}\right) = \tau_0\left(\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} A^* & 0 \\ 0 & 1 \end{pmatrix}\right) P\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right)$$

it is enough to prove the result for $Z = i$. Thus we need to prove:

$$\begin{aligned}&\int_{M_b} e^{-\pi i \operatorname{Tr} a^* a_1} e^{-\pi i \operatorname{Tr} a a_2^*} e^{-\pi \operatorname{Tr} a a^*} P\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right) da \\ = \ & e^{-\pi \operatorname{Tr} a_1 a_2^*} \tau_0\left(\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}\right) P\left(\begin{pmatrix} a_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a}_2 \\ \overline{v}_2 \end{pmatrix}\right).\end{aligned} \tag{23}$$

We think of $M_{b,k}(\mathbb{C})$ as a real vector space embedded in its complexification $M_{b,k}(\mathbb{C}) \times M_{b,k}(\mathbb{C})$ by $a \mapsto (a, \overline{a})$. Then it follows from the mean value theorem for harmonic polynomials (Lemma 8.5) that

$$\begin{aligned}&\int_{M_b} e^{-\pi i \operatorname{Tr} a^* a'} e^{-\pi i \operatorname{Tr} a a'^*} e^{-\pi \operatorname{Tr} a a^*} P\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right) da \\ = \ & e^{-\pi \operatorname{Tr} a' a'^*} P\left(\begin{pmatrix} -ia' \\ iv_1 \end{pmatrix}, \begin{pmatrix} -i\overline{a}' \\ \overline{v}_2 \end{pmatrix}\right).\end{aligned} \tag{24}$$

Here we are using the fact that $a \mapsto P\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right)$ is a harmonic polynomial on $M_{b,k}(\mathbb{C})$ (viewed as a real vector space).

Now we view (24) as an equation between real analytic functions of a'. By analytic continuation, it follows that for any $(a_1, a_2) \in M_{b,k}(\mathbb{C}) \times M_{b,k}(\mathbb{C})$, we have:

$$\begin{aligned}&\int_{M_b} e^{-\pi i \operatorname{Tr} a^* a_1} e^{-\pi i \operatorname{Tr} a a_2^*} e^{-\pi \operatorname{Tr} a a^*} P\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right) da \\ = \ & e^{-\pi \operatorname{Tr} a_1 a_2^*} P\left(\begin{pmatrix} -ia_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} -i\overline{a}_2 \\ \overline{v}_2 \end{pmatrix}\right).\end{aligned} \tag{25}$$

We observe that (25) is equivalent to (23), which proves the Lemma. ∎

Returning to the theorem, we have

$$\begin{aligned}(B_b f_W)\begin{pmatrix} z_1 \\ v_1 \\ z_2 \\ v_2 \end{pmatrix} &= e^{-\pi i \operatorname{Tr} z_1 z_2^*} e^{\pi \operatorname{Tr} v_1 v_2^* W_{22}^*} \int_{M_b} e^{\pi i \sqrt{2} \operatorname{Tr} z_1 a^*} e^{\pi \sqrt{2} \operatorname{Tr} a z_2^*} e^{-\pi \operatorname{Tr} a a^*} \cdot \\ & \quad e^{-\pi i \operatorname{Tr}(a a^* W_{11}^*)} e^{-\pi i \operatorname{Tr}(a v_2^* W_{12}^*)} e^{\pi \operatorname{Tr}(v_1 a^* W_{21}^*)} P_h\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a} \\ \overline{v}_2 \end{pmatrix}\right) da\end{aligned}$$

By Lemma 11.1, we have $W_{11} = X + iY$, with X, $Y \in \mathrm{Herm}_b(\mathbb{C})$ with $Y \gg 0$, and thus the previous Lemma applies. Taking $a_1 = -\sqrt{2}z_1 + iW_{21}^* v_1$, $a_2^* = \sqrt{2}iz_2^* + v_2^* W_{12}^*$, $Z^* = W_{11}^* - i$, the Lemma gives:

$$(B_b f_W)\begin{pmatrix} z_1 \\ v_1 \\ z_2 \\ v_2 \end{pmatrix} = e^{-\pi i \operatorname{Tr} z_1 z_2^*} e^{\pi \operatorname{Tr} v_1 v_2^* W_{22}^*} \cdot (i)^{bk} e^{\pi i \operatorname{Tr} a_1 a_2^* Z^{*-1}} \det(-Z^*)^{-k}$$
$$\tau_0\left(\begin{pmatrix} -Z^* & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -Z^{*-1} & 0 \\ 0 & 1 \end{pmatrix}\right) P_h\left(\begin{pmatrix} a_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a}_2 \\ \overline{v}_2 \end{pmatrix}\right)$$

where we calculate

$$\begin{aligned}
& e^{-\pi i \operatorname{Tr} z_1 z_2^*} e^{\pi \operatorname{Tr} v_1 v_2^* W_{22}^*} \cdot e^{\pi i \operatorname{Tr} a_1 a_2^* Z^{*-1}} \\
= \;& e^{\pi \operatorname{Tr} z_1 z_2^* (-i)(W_{11}^* - i)^{-1}(W_{11}^* + i)} e^{\pi \operatorname{Tr} v_1 z_2^* (-\sqrt{2}i)(W_{11}^* - i)^{-1} W_{21}^*} \cdot \\
& \quad e^{\pi \operatorname{Tr} z_1 v_2^* (-\sqrt{2}i) W_{12}^* (W_{11}^* - i)^{-1}} e^{\pi \operatorname{Tr} v_1 v_2^* (W_{22}^* - W_{12}^* (W_{11}^* - i)^{-1} W_{21}^*)} \\
= \;& e^{\pi \operatorname{Tr}\left(\begin{smallmatrix} z_1 \\ v_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} z_2^* & v_2^* \end{smallmatrix}\right)\left(\begin{smallmatrix} T_{11}^* & T_{21}^* \\ T_{12}^* & T_{22}^* \end{smallmatrix}\right)} = q_T\begin{pmatrix} z_1 \\ v_1 \\ z_2 \\ v_2 \end{pmatrix}
\end{aligned}$$

with $\left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right) = c_b^{-1}\left(\begin{smallmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{smallmatrix}\right)$ given by formula (21) and

$$\begin{aligned}
& (i)^{bk} \det(-Z^*)^{-k} \tau_0\left(\begin{pmatrix} -Z^* & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -Z^{*-1} & 0 \\ 0 & 1 \end{pmatrix}\right) P_h\left(\begin{pmatrix} a_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} \overline{a}_2 \\ \overline{v}_2 \end{pmatrix}\right) \\
= \;& (i)^{bk} \det(-Z^*)^{-k} h\left(\begin{pmatrix} -Z^{*-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} -Z^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ v_2 \end{pmatrix}\right) \\
= \;& (i)^{bk} \det(-W_{11}^* + i)^{-k} \cdot \\
& h\left(\left(\begin{smallmatrix} \sqrt{2}(1+iW_{11}^*)^{-1} & -(W_{11}^* - i)^{-1} W_{21}^* \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} ia_1 \\ iv_1 \end{pmatrix}, \left(\begin{smallmatrix} \sqrt{2}(1-iW_{11})^{-1} W_{12} & -(W_{11} + i)^{-1} W_{12} \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} a_2 \\ v_2 \end{pmatrix}\right) \\
= \;& C\, J(c_b^{-1}, W)^{*-1} h\left(\begin{pmatrix} ia_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ v_2 \end{pmatrix}\right)
\end{aligned}$$

where C is a constant. As $J(c_b^{-1}, W)^{-1} = J(c_b, T)$ for $T = c_b^{-1}(W)$, this proves the theorem. ∎

Now we can proceed exactly as in the $(Sp(n, \mathbb{R}), O(k))$ example in Section 8. First note that the irreducibility of τ and hence of τ_0 implies that $h(iw_1, w_2) = \beta_1 h(w_1, w_2)$, $(w_1, w_2) \in M_{p,k}(\mathbb{C}) \times M_{q,k}(\mathbb{C})$, for some constant β_1. Let I_b denote the map from $\mathcal{H}_{p,q}(\lambda)$ to the space of harmonic polynomials in $L^2(M_b, \mathcal{F}_{p-b,q-b}; \lambda)$ given by

$$I_b h\begin{pmatrix} a \\ v_1 \\ v_2 \end{pmatrix} = h\left(\begin{pmatrix} a \\ iv_1 \end{pmatrix}, \begin{pmatrix} a \\ v_2 \end{pmatrix}\right)$$

for $h \in \mathcal{H}_{p,q}(\lambda)$. By the Theorem, applying the inverse Bargmann transform to (10.6) and canceling constants on both sides leads to:

$$\omega_\lambda(g') q'_{c_b(T)} I_b(h) = q'_{g'^{-1}(c(T))} I_b(J(c_b, g(T))^{*-1} J(g, T)^{*-1} J(c_b, T)^* h)$$

for $T\in\mathcal{D},\ g\in G,\ g'=c_b\,g\,c_b^{-1}\in G^{c_b}$. Then for $h_1,h_2\in\mathcal{H}_{p,q}(\lambda)$ and $S,T\in\mathcal{D}$, we have

$$\begin{aligned}&\langle q'_{c_b(T)}\circ I_b(h_1)|q'_{c_b(S)}\circ I_b(h_2)\rangle_{L^2(M_b,\mathcal{F};\lambda)}\\ =\ &\|\beta\|^2\langle J(c_b,S)Q(S,T)J(c_b,T)^*h_1|h_2\rangle_{\mathcal{H}_{p,q}(\lambda)},\ \beta \text{ a constant}\end{aligned}$$

and we can define a positive definite kernel function $Q'\,:\,\mathcal{S}_b\times\mathcal{S}_b\to\mathrm{Aut}(V_\tau)$ by $Q'(c_b(S),c_b(T))=\|\beta\|^2J(c_b,S)Q(S,T)J(c_b,T)^*$ which satisfies (17). It follows that G^{c_b} acts on $H(\mathcal{S}_b,\tau)$ by the strongly continuous unitary representation T defined by (18). Finally, the map $q'_{c_b(T)}I_b(h)\mapsto Q'(\cdot,c_b(T))h$ extends to an intertwining map Ξ_λ between $(\omega_\lambda,L^2(M_b,\mathcal{F};\lambda))$ and $(T,H(\mathcal{S}_b,V_\tau))$ such that $\langle\Xi_\lambda f(W)|h\rangle_{\mathcal{H}_{p,q}(\lambda)}=\langle f|q'_W I_b(h)\rangle_{L^2(M_b,\mathcal{F};\lambda)}$. This map can be written as follows:

Theorem 13.5. *Let $f\in L^2(M_b,\mathcal{F};\lambda)$ and let $\begin{pmatrix}a\\v_1\\v_2\end{pmatrix}\mapsto I_\lambda\begin{pmatrix}a\\v_1\\v_2\end{pmatrix}$ be the polynomial function on $M_b\times M_{p-b}\times M_{q-b}$ with values in $\mathrm{Hom}(V_\tau,V_\lambda)$ defined by*

$$I_\lambda\begin{pmatrix}a\\v_1\\v_2\end{pmatrix}h=h\left(\begin{pmatrix}a\\iv_1\end{pmatrix},\begin{pmatrix}a\\v_2\end{pmatrix}\right)$$

for $h\in V_\tau=\mathcal{H}_{p,q}(\lambda)$. Then:

$$\Xi_\lambda f(W)=\int_{M_b}\int_{M_{p-b+q-b}}\overline{q'_W}\begin{pmatrix}a\\v_1\\v_2\end{pmatrix}I_\lambda\begin{pmatrix}a\\v_1\\v_2\end{pmatrix}^*f\begin{pmatrix}a\\v_1\\v_2\end{pmatrix}e^{-\pi(v|v)}dv\,dx$$

The proof is as in Proposition 5.7.

Remark 13.6. When $b=q$, the intertwining map given above is exactly the one found in [14].

References

[1] Davidson, M. G., *The Harmonic Representation of $U(p,q)$ and its Connection with the Generalized Unit Disk*, Pacific J. Math. **129** (1987), 33–55.

[2] Davidson, M. G., Enright, T.J. and Stanke, R., *Differential Operators and Highest Weight Representations*, Memoirs of the Amer. Math. Soc. **94** (1991).

[3] Davidson, M. G. and Fabec, R. C., *Geometric Realizations for Highest Weight Representations*, Contemporary Mathematics **191** (1995), 13–31.

[4] Ding, H., *Hankel Transforms and Holomorphic Discrete Series*, Contemporary Mathematics, **191** (1995), 33–40.

[5] Ding, H. and Gross, K. I., *Operator-valued Bessel functions on Jordan Algebras*, J. reine angew. Math. **435** (1993), 157–196.

[6] Folland, G. B., “Harmonic Analysis in Phase Space”, Ann. of Math. Stud. 122, Princeton University Press, 1989.

[7] Gelbart, S., *Holomorphic Discrete Series for the Real Symplectic Group*, Inventiones Math. **19** (1973), 49–58.

[8] Gross, K. I. and Kunze, R. A., *Fourier Bessel Transforms and Holomorphic Discrete Series*, in "Proceedings of the Maryland Conference on Harmonic Analysis," 79–122, Lecture Notes in Mathematics 266, Springer, 1977.

[9] Helgason, S., "Differential Geometry, Lie Groups and Symmetric Spaces", Academic Press, New York, 1978.

[10] Herb, R. and Wolf, J. A., *Wave packets for the relative discreteseries. I. The holomorphic case*, J. Funct Anal. **73** (1987), 1–37.

[11] Howe, R., *Notes on the Oscillator Representation*, Unpublished notes.

[12] —, *L^2 duality for stable reductive dual pairs*, Unpublished notes.

[13] Kunze, R. A., *Positive Definite Operator-valued Kernels and Unitary Representations*, in "Proceedings of the Conference on Functional Analysis at Irvine, California", 235–247,Thomson Book Company, 1966.

[14] Kashiwara, K. and Vergne, M., *On the Segal-Shale-Weil Representations and Harmonic Polynomials*, Invent. Math. **44** (1978), 1–47.

[15] Leslie, C. S., "Geometric Construction of Intertwining Maps for Mixed Models of Compact Dual Pairs", Ph.D. Dissertation, University of California at Berkeley, 1998.

[16] Li, J.-S., *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), 237–255.

[17] Piatetsky-Chapiro (Piatetski-Shapiro), I. I., "Géométrie des domaines classiques et théorie des fonctions automorphes", Dunod, Paris, 1966.

[18] Prasad, D., *Weil Representation, Howe Duality, and the Theta Correspondence*, CRM Proceedings and Lecture Notes **1** (1993), 105–127.

[19] Satake, I., "Algebraic Structures of Symmetric Domains", Iwanami-Shoten and Princeton University Press, Princeton, New Jersey, 1980.

[20] Tirao, J. A., *Square integrable representation of semisimple Lie groups*, Trans. Amer. Math. Soc. **190** (1974), 57–75.

[21] Wallach, N. R., *The Analytic Continuation of the Discrete Series. I*, Trans. Amer. Math. Soc **251** (1979), 1–17.

[22] Wolf, J. A., *Fine Structure of Hermitian Symmetric Spaces*, in "Symmetric Spaces: Short Courses presented at Washington University", 271–357, Marcel Dekker, 1972.

[23] Wolf, J. A. and Korányi, A, *Generalized Cayley Transformations of Bounded Symmetric Domains*, Amer. J. Math. **87** (1965), 899–939.

C. S. Leslie
Department of Computer Science
Columbia University
1214 Amsterdam Avenue, MC: 0401
New York, NY 10027-7003, USA
cleslie@cs.columbia.edu

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