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Polynomial Identities in Smash Products

Yuri Bahturin* **and Victor Petrogradsky**†

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Abstract. Suppose that a group G acts by automorphisms on a (restricted) Lie algebra L over a field K of positive characteristic. This gives rise to smash products $U(L)\#K[G]$ and $u(L)\#K[G]$. We find necessary and sufficient conditions for these smash products to satisfy a nontrivial polynomial identity.

1. Introduction: polynomial identities in group algebras and enveloping algebras

The first observation on the polynomial identities in enveloping algebras was made by V. N. Latyshev [9]. He proved that the universal enveloping algebra of a finite dimensional Lie algebra over a field of characteristic zero satisfies a nontrivial identical relation if and only if this Lie algebra is abelian. Later Yu. Bahturin has noticed in [1] that the condition of finite dimensionality is inessential.

D. S. Passman has obtained the complete description of group algebras satisfying polynomial identities.

Theorem 1.1. ([11]) *The group algebra $K[G]$ of a group G satisfies a nontrivial polynomial identity if and only if the following conditions are satisfied:*

1. *there exists a normal subgroup $A \subset G$ of finite index;*
2. *A is abelian if* $\operatorname{char} K = 0$*, and the commutator subgroup A' is a finite abelian p-group if* $\operatorname{char} K = p > 0$.

Yu. Bahturin has settled the problem of the existence of nontrivial identities for the universal enveloping algebra fields of positive characteristic.

Theorem 1.2. ([2]) *Let L be a Lie algebra over a field K of positive characteristic. Then the universal enveloping algebra $U(L)$ satisfies a nontrivial polynomial identity if and only if the following conditions are satisfied:*

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1. *there exists an abelian ideal $H \subset L$ of finite codimension;*

2. *all inner derivations* $\operatorname{ad} x$, $x \in L$ *are algebraic of bounded degree.*

Petrogradsky [15] and Passman [14] have specified restricted Lie algebras (also called Lie p-algebras) L such that the restricted enveloping algebra $u(L)$ satisfies a nontrivial polynomial identity.

Theorem 1.3. ([14], [15]) *Let L be a Lie p-algebra. Then the restricted enveloping algebra $u(L)$ satisfies a nontrivial polynomial identity if and only if there exist restricted ideals $Q \subset H \subset L$ such that:*

1. $\dim L/H < \infty$, $\dim Q < \infty$*;*

2. *H/Q is abelian;*

3. *Q is abelian with nilpotent p-mapping.*

See also further developments for Lie p-superalgebras [16] and color Lie p-superalgebras [3].

The main body of this paper consists of the proof of our main result which completely describes smash products $u(L)\#K[G]$ that are PI rings (Theorem 3.1). We start with establishing some identical relations that nicely suit our purposes (Section 4). As an important ingredient we begin developing a delta-theory for smash products (Section 5). Next we describe the structure of delta-sets in our case (Section 6).

But the first results in this paper (Section 2) deal with necessary and sufficient conditions under which the smash product $U(L)\#K[G]$ satisfies a nontrivial identity (Theorem 2.3). Actually, this result, as well as Theorem 2.1 could be derived from a result on general smash products by Handelman - Lawrence - Schelter (see [7]) and probably by Passman [13]. But we prefer to keep our proofs here since they are relevant to the techniques of delta-sets used in the next sections for the proof of the main result (Theorem 3.1).

2. Polynomial identities in smash products $U(L)\#K[G]$

We denote the ground field by K. Suppose that a group G acts on an associative algebra A by automorphisms: $\varphi : G \to \operatorname{Aut}(A)$, $\varphi(g) : x \mapsto \varphi(g)(x)$, $g \in G$, $x \in A$. We set $g * x = \varphi(g)(x)$. Now one can form the smash product $R = A\#K[G]$. This is a vector space $R = A \otimes_K K[G]$ endowed with multiplication

$$(a_1, g_1) \cdot (a_2, g_2) = (a_1(g_1 * a_2), g_1 g_2), \qquad a_1, a_2 \in A,\ g_1, g_2 \in G.$$

By linearity also the group ring $K[G]$ acts on A:

$$(\alpha_1 g_1 + \cdots + \alpha_m g_m) * a = \alpha_1(g_1 * a) + \cdots + \alpha_m(g_m * a), \quad g_i \in G, \alpha_i \in K,\ a \in A.$$

Now suppose that G acts on a Lie algebra L by automorphisms. Then this action is naturally extended to the action on the universal enveloping algebra $U(L)$ and we can form the smash product $U(L)\#K[G]$. Such algebras are important because each cocommutative Hopf algebra over an algebraically closed field of characteristic zero can be presented as a smash product $U(L)\#K[G]$ (Kostant, Cartier, et al., see [10]).

The conditions for the existence of nontrivial identities for the smash products $U(L)\#K[G]$ can be derived from [7]. Our next formulation is from [8].

Theorem 2.1. *Let G be a group, L a Lie algebra over a field K of characteristic 0, and G acts on L by automorphisms. Then $U(L)\#K[G]$ satisfies a nontrivial polynomial identity if and only if the following conditions are satisfied:*

1. *L is abelian;*
2. *there exists an abelian normal subgroup $A \subset G$ of finite index;*
3. *A acts trivially on L.*

The proof in [8] as well as the original proof of the theorem about the identical relations in $U(L)$ [2] (see also [1], [3]) is based on the following classical result.

Theorem 2.2. (Posner, [5]) *Let R be a prime algebra with unit over a field satisfying some nontrivial polynomial identity. Let C be the center of R and Q the field of quotients of C. Then the algebra $R^Q = Q \otimes_C R$ of central quotients of R is finite-dimensional central simple over Q and coincides with the left and the right classical rings of quotients of R. Moreover, R and R^Q satisfy the same identities.*

The goal of this section is to prove a result similar to the one just formulated in the case of the fields of positive characteristic. Again we mention a possibility of deriving this result from [7].

Theorem 2.3. *Let G be a group, L a Lie algebra over a field K of characteristic $p > 0$ and G act on L by automorphisms. Then $U(L)\#K[G]$ satisfies a nontrivial polynomial identity if and only if the following conditions are satisfied:*

1. *there exists an abelian G-invariant ideal $H \subset L$ of finite codimension and all derivatives $\operatorname{ad} x$, $x \in L$ are algebraic of bounded degree;*
2. *there exists a normal subgroup $A \subset G$ of finite index with the commutator subgroup A' being a finite abelian p-group.*
3. *A acts trivially on L.*

Let us comment on this result. We observe that $K[G]$ and $U(L)$ are the subrings of $U(L)\#K[G]$ and thus Theorems 1.1 and 1.2 apply. This gives us the structure for G and L described in the first two claims except for the fact that H is G-invariant. So, the most essential here is the third claim.

We start with recalling the notion of delta-sets. They provide us with the key instrument to the study of identities of enveloping algebras. One defines the sets of elements of "finite width"

$$\begin{aligned} \delta_n(L) &= \{x \in L \mid \dim[x, L] \le n\}, \quad n \in \mathbb{N}; \\ \delta(L) &= \bigcup_{n=1}^{\infty} \delta_n(L). \end{aligned}$$

These sets have appeared in [2] as Lie algebra analogues of delta-sets for the groups. Those delta-sets were crucial in the study identical relations for group rings [11], [12]. Namely, if G is a group then one can define the sets of elements having finitely many conjugates as follows

$$\begin{aligned}\delta_n(G) &= \{g \in G \,|\, |g^G| \le n\}, \quad n \in \mathbb{N};\\ \delta(G) &= \bigcup_{n=1}^{\infty} \delta_n(G).\end{aligned}$$

Lemma 2.4. ([3]) *Let L be a Lie algebra. Then the delta-sets have the following properties.*

1. *if $x \in \delta_i(L)$, $y \in \delta_j(L)$ then $\alpha x + \beta y \in \delta_{i+j}(L)$, $\alpha, \beta \in K$;*
2. *if $x \in \delta_i(L)$, $y \in L$ then $[x, y] \in \delta_{2i}(L)$;*
3. *let $x \in \delta_i(L)$ and suppose that L is a restricted Lie algebra. Then $x^{[p]} \in \delta_i(L)$;*
4. *$\delta_i(L)$ is invariant under the automorphisms of L, $i \in \mathbb{N}$;*
5. *$\delta(L)$ is a (restricted) invariant ideal of L.*

Recall that a subalgebra is called *invariant* if it is stable under all automorphisms and *restricted* if closed under the p-map.

We use this lemma to prove Theorem 2.3

Proof. First, let us check that our conditions are sufficient. Let $f_1, \dots, f_k$ form a basis of L modulo H. By the hypothesis, each $\operatorname{ad} f_i$ annihilates some nonzero polynomial $q_i(t)$, $i = 1, \dots, k$. Recall that a polynomial of the form $q(t) = \sum_{i=0}^{s} \alpha_i t^{p^i}$ is called a *p-polynomial* [6]. Such polynomials have the following property. Let x be an element of an associative algebra over the field of characteristic p, viewed as a Lie algebra under the bracket operation $[a, b] = ab - ba$. Then

$$\operatorname{ad}(q(x)) = \operatorname{ad}(\sum_{i=0}^{s} \alpha_i x^{p^i}) = \sum_{i=0}^{s} \alpha_i (\operatorname{ad} x)^{p^i} = q(\operatorname{ad} x). \qquad (1)$$

Any polynomial is a divisor of some nonzero p-polynomial [6]. So, we may assume that $q_i(t)$ are some p-polynomials. By (1) $z_i = q_i(f_i)$, $i = 1, \dots, k$ are central elements in $U(L)$. Let d_i, $i = 1, \dots, k$, be the degrees of polynomials q_i. We denote by B the ring generated by $U(H)$ along with $z_1, \dots, z_k$. Then ${}_BU(L)$ is a free B-module with a finite basis $\{f_1^{i_1} \cdots f_k^{i_k} | 0 \le i_j < d_j, 1 \le j \le k\}$ [6]. Let $g_1, \dots, g_s$ be the right coset representatives of A in G. Since A acts trivially on the whole of L we obtain that $\widetilde{B} = B \otimes K[A]$ is a commutative subring of $R = U(L) \# K[G]$. Also, ${}_{\widetilde{B}}R$ is a free module with a basis $\{f_1^{\alpha_1} \cdots f_k^{\alpha_k} g_j | 0 \le \alpha_i < d_i, 1 \le j \le s\}$ and rank $t = d_1 \cdots d_k s$.

We identify any $x \in R$ with an operator of the right multiplication: $R \to R$, $a \mapsto ax$, $a \in R$. This yields an embedding of R into a matrix ring over the commutative ring $\widetilde{B}$:

$$R \subset \operatorname{End}_{\widetilde{B}} R \cong \mathrm{M}_t(\widetilde{B}) \cong \mathrm{M}_t(K) \otimes_K \widetilde{B}.$$

By Regev's Theorem about tensor products of PI-rings [17] we conclude that $R = U(L)\#K[G]$ is a PI-ring.

Now suppose that $U(L)\#K[G]$ satisfies a nontrivial polynomial identity.

First, let us prove that there exists a G-invariant abelian ideal $H \subset L$ of finite codimension. We need to recall the steps of the proof of Theorem 1.2 in [2] (see also this construction in [15], [16], and [3]).

1) The existence of a nontrivial identity in $U(L)$ implies that for some number m we have $\delta(L) = \delta_m(L)$ and $\dim L/\delta(L) < \infty$. We set $D = \delta(L)$.

2) We apply P.M.Neumann's Theorem on bilinear maps (see Theorem 6.7 below) and conclude that the commutator subalgebra $D^2 = [D, D]$ is finite-dimensional. We set $C = C_D(D^2) = \{x \in D | [x, D^2] = 0\}$. Then $\dim D/C < \infty$, and $C^3 = 0$.

3) We again use the identity in the enveloping algebra for C and prove that $\dim C/H < \infty$, where $H = Z(C)$ is the center of C.

One can trace these steps and see by Lemma 2.4 that all these subalgebras are invariant ideals. Hence, we obtain the G-invariant abelian ideal $H \subset L$ of finite codimension.

Next, we apply Theorem 1.1 and obtain a subgroup of finite index $B \subset G$ such that the commutator subgroup B' is a finite abelian p-group. Now our task is reduced to the following. We consider the smash product $U(L)\#K[B]$ and find a subgroup of finite index $A \subset B$ acting trivially on the whole of L. Let $\{U_n(L) | n = 0, 1, 2 \dots\}$ be the standard filtration on the universal enveloping algebra. Then it induces a filtration on the smash product, the associated graded algebra is also PI and

$$\mathrm{gr}\{U_n(L)\#K[B] | n = 0, 1, 2, \dots\} \cong (\mathrm{gr}\{U_n(L) | n = 0, 1, 2, \dots\})\#K[B].$$

But $\mathrm{gr}\{U_n(L) | n = 0, 1, 2, \dots\} \cong U(\bar{L})$, where $\bar{L}$ is an abelian Lie algebra with the same vector space L. So, we may assume that L is abelian so that $U(L)$ is a polynomial ring.

Let $g \in B$ be an element of infinite order. We claim that there exists $m > 0$ such that g^m acts trivially on L. Let $\langle g \rangle$ be the cyclic subgroup generated by g. By way of contradiction suppose that $\langle g \rangle$ acts faithfully on L. One easily verifies that the ring $R = U(L)\#\langle g \rangle$ has no zero divisors, hence is prime. Let C be the center of R and consider some central element $c = \sum_i c_i g^i \in C$, $c_i \in U(L)$. Suppose that there exists $c_j \neq 0$, $j \neq 0$. Since $\langle g \rangle$ acts faithfully on L we find $a \in L$ with $g^j * a \neq a$. Remark that $(g^j * a - a)c_j \neq 0$ because $U(L)$ has no zero divisors. Then

$$[c, a] = \sum_i c_i g^i a - a \sum_i c_i g^i = \sum_i (g^i * a - a) c_i g^i \neq 0. \qquad (2)$$

This is a contradiction with the fact that c is central. Therefore $c_i = 0$ for $i \neq 0$, so $C \subset U(L)$. Let Q be the field of fractions for C. Then the elements g^i, $i \in \mathbb{N}$ are linearly independent over Q in the ring of fractions for R. This contradicts Posner's theorem. Hence for any $g \in B$ there exists m such that g^m acts trivially on L.

Let us consider an arbitrary finitely generated subgroup $W \subset B$. Suppose that the action of W on L is faithful. Then our argument implies that the generating elements of W are of finite order. In view of the structure of B we conclude that W is finite. Now we consider the ring $R_0 = U(L)\#K[W]$ and apply one result on the smash products of type $A\#H$, where H is a finite dimensional Hopf algebra [10, p. 55]. Namely, $A\#H$ is a prime ring if and only if A is a faithful left and right $A\#H$-module and the invariant subring A^H is prime. Of course, in our case $U(L)^{K[W]}$ is prime. Let us check the conditions of faithful action. If $H = K[W]$ then the left and right actions are defined by

$$\begin{aligned}(a\#g)\cdot b &= a(g * b), \qquad a,b \in A,\ g \in W;\\ a\cdot(b\#g) &= g^{-1} * (ab), \qquad a,b \in A,\ g \in W.\end{aligned}$$

Suppose that a nonzero element

$$a_1 g_1 + \cdots + a_m g_m, \qquad 0 \neq a_i \in U(L),\ g_i \in W, \tag{3}$$

acts trivially on the left on $U(L)$. Let m be taken minimal among nonzero elements (3) that act trivially on $U(L)$. Then

$$a_1(g_1 * x) + \cdots + a_m(g_m * x) = 0, \quad x \in U(L). \tag{4}$$

We replace x by xy and multiply (4) by $g_1 * y$ on the right. Here we also use the commutativity of $U(L)$.

$$\begin{aligned}&a_1(g_1 * x)(g_1 * y) + \cdots + a_m(g_m * x)(g_m * y) = 0, \quad x,y \in U(L);\\ &a_1(g_1 * x)(g_1 * y) + \cdots + a_m(g_m * x)(g_1 * y) = 0, \quad x,y \in U(L);\\ &a_2((g_2 - g_1) * y)(g_2 * x) + \cdots + a_m((g_m - g_1) * y)(g_m * x) = 0,\ x,y \in U(L).\end{aligned}$$

Since $U(L)$ has no zero divisors and by the choice of m in (3) we conclude that $(g_2 - g_1) * y = 0$ for all $y \in U(L)$, contradicting to the fact that W acts faithfully on L. We can check that the right action is faithful in the same way.

Now we can apply Posner's theorem. The same computation (2) shows that the center C of R is contained in $U(L)$.

Let Q be the field of fractions for C. Again all elements $g \in W$ are linearly independent over Q in the ring of quotients R_Q for R. Let d be the degree of a nontrivial polynomial identity satisfied by the smash product $U(L)\#K[G]$. By Posner's theorem $|W| \le \dim_Q R_Q \le [d/2]$.

Let now W be an arbitrary finitely generated subgroup of B. We set $\mathrm{St}_W L = \{w \in W | w * x = x, x \in L\}$. Then by the above arguments $|W : \mathrm{St}_W L| \le s = [d/2]$. Consider $A_1 = \mathrm{St}_B L = \{b \in B | b * x = x, x \in L\}$. We claim that $|B : A_1| \le s$. By way of contradiction suppose that $|B : A_1| > s$, then we can take elements $g_1, \ldots, g_{s+1} \in B$ lying in different left classes modulo A_1. We consider the subgroup W generated by $g_1, \ldots, g_{s+1}$. Then these elements belong to different cosets of $\mathrm{St}_W L \subset A_1$, proving $|W : \mathrm{St}_W L| > s$, a contradiction with the above. Thus we should have $|B : A_1| \le s$ and by construction A_1 acts trivially on L.

To finish the proof it is enough to choose a normal subgroup of finite index $A \subset A_1$. ■

3. Polynomial identities in the smash products $u(L)\#K[G]$

The main goal of this paper is to prove the following result.

Theorem 3.1. *Suppose that a group G acts by automorphisms on a Lie p-algebra L. Then $u(L)\#K[G]$ is a PI-algebra if and only if*

1. *there exist G-invariant restricted subalgebras $Q \subset H \subset L$ with*
   (a) $\dim L/H < \infty$, $\dim Q < \infty$;
   (b) $[H,H] \subset Q$;
   (c) *Q is abelian with a nilpotent p-mapping.*
2. *there exists a subgroup $A \subset G$ with*
   (a) $|G:A| < \infty$;
   (b) *the commutator subgroup A' is a finite abelian p-group;*
3. *A acts trivially on H/Q.*

We remark that $K[G]$ and $u(L)$ are the subrings of $u(L)\#K[G]$ and we can apply Theorems 1.1 and 1.3. This gives us the structure of G and L described in the first two claims except for the fact that H, Q are G-invariant. But the most difficult here is the third claim about the action of G on L.

While studying polynomial identities for $u(L)$, a crucial example is the infinite-dimensional Heisenberg algebra. By δ_{ij} we mean the Kronecker symbol.

Example 3.2. ([15, 16, 3]) We consider the infinite-dimensional Heisenberg Lie algebra

$$L = \langle x_1, x_2, \dots, y_1, y_2, \dots, z | [x_i, y_j] = \delta_{ij} z, [x_i, z] = [y_j, z] = 0, i, j \in \mathbb{N} \rangle_K.$$

Then the existence of a nontrivial identity for $u(L)$ depends on the value of the p-map on the central element z

1. if $z^{[p]} = 0$, then $u(L)$ satisfies a nontrivial identity $(XY - YX)^p \equiv 0$;
2. if $z^{[p]} = z$, then $u(L)$ does not satisfy any nontrivial identity.

Let us illustrate our main result by examples. These examples are similar to the Heisenberg algebra, we only need to remember that G acts by automorphisms on L.

Example 3.3. Let $L = \langle y, x_j, x_j^{[p]}, \dots, x_j^{[p^k]}, \dots | j = 1, 2, \dots; y^{[p]} = 0 \rangle_K$ be an abelian restricted Lie algebra and the group $G = (\mathbb{Z}_p)^{\mathbb{N}}$ acts on L as

$$g_i * x_j = x_j + \delta_{ij} y; \quad g_i * x_j^{[p^k]} = x_j^{[p^k]}, \ k \geq 1; \quad g_i * y = y;$$

where $g_i = (0, \dots, 0, 1, 0, \dots)$, with 1 on i-th place, $i \in \mathbb{N}$. We consider the smash product $R = u(L)\#K[G]$. Then

1. R satisfies the identity $(XY - YX)^p \equiv 0$;
2. G acts faithfully on L.

Proof. We have $[g, x_j] = gx_j - x_j g = (g * x_j - x_j)g = \lambda yg$, $\lambda \in K$, $g \in G$. Remark that all other pairs of generating elements for R commute. Therefore, any commutator $[a, b]$, $a, b \in R$ contains as a factor the central element y. Recall that $y^p = 0$, therefore R satisfies the claimed identity.

Let us check the second claim. For $g = (n_1, n_2, \dots, n_i, \dots) \in G$ one has $g * x_j = x_j + n_j y$, so that G acts faithfully on L. ■

This example fits into the wording of the theorem by setting $A = G$, $H = L$, $Q = \langle y \rangle_K$. This example shows that the action of A on Q may be nontrivial. Moreover, one can check that we cannot avoid this by taking a somewhat smaller subgroup $A_1 \subset A$ of finite index and a subalgebra $H_1 \subset H$ of finite codimension.

Let us change the p-mapping on y in the previous example.

Example 3.4. Let $L = \langle y, x_j, x_j^{[p]}, \dots, x_j^{[p^k]}, \dots | j = 1, 2, \dots, y^{[p]} = y \rangle_K$ be an abelian restricted Lie algebra and the group $G = (\mathbb{Z}_p)^{\mathbb{N}}$ act on L by

$$g_i * x_j^{[p^k]} = x_j^{[p^k]} + \delta_{ij} y, \ k \geq 0; \quad g_i * y = y.$$

Then $R = u(L)\#K[G]$ does not satisfy any nontrivial identity.

Proof. If R is PI then it must satisfy the identity given below in Lemma 4.2 and so we have

$$F_2(x_1, \dots, x_n, g_1, \dots, g_n) = \sum_{\pi \in S_n} \alpha_\pi (g_1 - 1) * x_{\pi(1)} \cdots (g_n - 1) * x_{\pi(n)} = y^n \neq 0,$$

because only the summand for the identity permutation is nontrivial. This contradiction proves that R is not PI. ■

The same argument applies also for the following example.

Example 3.5. Let $L = \langle y, e_1, e_2, \dots | e_j^{[p]} = e_j, j \in \mathbb{N}; y^{[p]} = y \rangle_K$ be an abelian restricted Lie algebra and the group $G = (\mathbb{Z}_p)^{\mathbb{N}}$ act on L by

$$g_i * e_j = e_j + \delta_{ij} y, \quad g_i * y = y.$$

Then $R = u(L)\#K[G]$ does not satisfy any identity.

By $\omega K[G]$ we denote the augmentation ideal of the group ring $\omega K[G] = \{\sum_i \alpha_i g_i | \sum_i \alpha_i = 0; \alpha_i \in K, g_i \in G\}$. If L is a Lie p-algebra then by $\omega u(L)$ we denote also the augmentation ideal of the restricted enveloping algebra $\omega u(L) = u(L)L = Lu(L)$.

Next we prove the sufficiency in Theorem 3.1.

Proof. We set $R = u(L)\#K[G]$, $R_1 = u(H)\#K[A]$, $R_0 = u(Q)\#K[A']$.

Let I be the subring of R_0 generated by Q and $\{h - 1 | h \in A'\}$; this is an ideal of codimension 1 in R_0. First, let us prove that I is nilpotent. We have $(\omega u(Q))^q = 0$ for some q since Q is abelian finite dimensional with a nilpotent p-mapping. Also $(\omega K[A'])^t = 0$ for some number t because A' is an abelian finite

p-group. Thus $Q = Q_0 \supset Q_1 \supset \cdots \supset Q_t = 0$, where $Q_i = (\omega K[A'])^i Q$. Now let us look at the commutators of the nilpotent elements of $\omega u(Q)$ and $\omega K[A']$.

$$(h-1)z = hzh^{-1}h - z = (h*z)h - z = ((h-1)*z+z)h - z = z(h-1) + ((h-1)*z)h,$$

where $h \in A'$ and $z \in Q_i$, in this case $(h-1)*z \in Q_{i+1}$. This relation yields that these two commutative nilpotent subrings generate a nilpotent subring. Indeed, consider a product consisting of $z_i \in Q$ and $(h_j - 1)$ where $h_j \in A'$. By the above relation the number of z's is bounded by $s_1 = q-1$. Also, the number of factors $(h_j - 1)$ is bounded by $s_2 = (t-1) + (t-1)(q-1)$. Hence, $I^s = 0$ for $s = 1 + s_1 + s_2 = qt$.

Second, we claim that R_1 is a PI-algebra. We consider the left ideal $J = R_1 I = R_1(Q + \omega K[A'])$. The following commutator relations hold for arbitrary $x \in H$, $z \in Q$, $g \in A$, and $h \in A'$

$$\begin{aligned}
zx &= xz + [z,x], & [z,x] \in Q;\\
zg &= gg^{-1}zg = g(g^{-1} * z), & g^{-1} * z \in Q;\\
(h-1)g &= g(g^{-1}hg - 1), & g^{-1}hg \in A';\\
(h-1)x &= h(x - h^{-1}xh) + x(h-1) & \\
&= h((1-h^{-1}) * x) + x(h-1), & (1-h^{-1}) * x \in Q,
\end{aligned}$$

the latter relation being true because A acts trivially on H/Q. These relations describe the commutators of all possible products of the form $u \cdot v$ where u, v are the generating elements of subrings I and R_1, respectively. It follows that $IR_1 \subset R_1 I$. By symmetry we have a two-sided ideal $J = R_1 I = IR_1 \triangleleft R_1$ and $J^s = (R_1 I)^s \subset R_1 I^s = 0$. Next we use the fact that $R_1/J \cong u(H/Q)\#K[A/A']$. By assumption of the theorem this is a commutative algebra. Therefore, R_1 satisfies the identity $(XY - YX)^s \equiv 0$.

Let $f_1, \ldots, f_k$ form a basis of L modulo H. Let $g_1, \ldots, g_{k_0}$ be the right coset representatives of A in G. Now ${}_{R_1}R$ is a free R_1-module with the finite basis $\{f_1^{\alpha_1} \cdots f_k^{\alpha_k} g_j | 0 \le \alpha_t < p, 1 \le j \le k_0\}$, the rank being $r = p^k k_0$. Indeed, suppose that we have a relation

$$\sum_{\alpha j} r_{\alpha j} f_1^{\alpha_1} \cdots f_k^{\alpha_k} g_j = 0, \quad r_{\alpha j} \in R_1. \tag{5}$$

Let $A = \{a_\lambda | \lambda \in \Lambda\}$, then

$$\sum_{\alpha j} \left(\sum_{\lambda} u_{\alpha j \lambda} a_\lambda \right) f_1^{\alpha_1} \cdots f_k^{\alpha_k} g_j = 0, \quad u_{\alpha j \lambda} \in u(H);$$

$$\sum_{j \lambda} \left(\sum_{\alpha} u_{\alpha j \lambda} (a_\lambda * f_1)^{\alpha_1} \cdots (a_\lambda * f_k)^{\alpha_k} \right) a_\lambda g_j = 0;$$

$$\sum_{\alpha} u_{\alpha j \lambda} (a_\lambda * f_1)^{\alpha_1} \cdots (a_\lambda * f_k)^{\alpha_k} = 0. \tag{6}$$

Since H is G-invariant the elements $\{a_\lambda * f_1, \ldots, a_\lambda * f_k\}$ form a basis of L modulo H. We apply PBW-Theorem to (6) and obtain that $u_{\alpha j \lambda} = 0$. Hence, (5) is trivial.

If we identify any $x \in R$ with the right multiplication by x we obtain an embedding of R into a matrix ring over the PI-ring R_1:

$$R \subset \mathrm{End}({}_{R_1}R) \cong \mathrm{M}_r(R_1) \cong \mathrm{M}_r(K) \otimes_K R_1.$$

By Regev's Theorem on the tensor product of PI-rings [17] we conclude that $R = U(L)\#K[G]$ is a PI-ring. ■

4. Useful identities

We start with constructing a special identity. It is similar to the identities used in the study of identical relations of restricted enveloping algebras [3].

Lemma 4.1. *Suppose that R is a PI-algebra over an arbitrary field. Then it satisfies a nontrivial identity of the form*

$$F(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = \sum_{\pi \in S_n} \alpha_\pi Y_1 X_{\pi(1)} \cdots Y_n X_{\pi(n)} \equiv 0, \quad \alpha_\pi \in K,\ \alpha_e = 1.$$

Proof. Let $A = A(X_1, \ldots, X_m, \ldots, Y_1, \ldots, Y_m, \ldots,)$ be the free associative algebra. For any permutation $\pi \in S_n$ we define a monomial

$$f_\pi = Y_1 X_{\pi(1)} \cdots Y_n X_{\pi(n)} \in A. \tag{7}$$

We denote by $P_m(Z_1, \ldots, Z_m)$ the subspace of all multilinear polynomials in m variables $Z_1, \ldots, Z_m$ in the free associative algebra $\bar{A} = \bar{A}(Z_1, \ldots, Z_m, \ldots)$ in a countable set of variables $Z_1, \ldots, Z_m, \ldots$. By $P'_m(Z_1, \ldots, Z_m)$ we denote the subspace of elements in $P_m(Z_1, \ldots, Z_m)$ that are the left hand sides of identities for R. Let R satisfy a nontrivial identity of degree d. The following estimate is well-known [1]

$$\dim P_m(Z_1, \ldots, Z_m)/P'_m(Z_1, \ldots, Z_m) < d^{2m}, \quad m \in \mathbb{N}. \tag{8}$$

Let us apply this estimate to A. We consider the subspace $P_{2n}(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \subset A$ of multilinear polynomials of degree $2n$ depending on the variables $X_1, \ldots, X_n,\ Y_1, \ldots, Y_n$. This subspace contains $n!$ monomials of the form (7) which are linearly independent. We apply (8)

$$\dim P_{2n}(X_1, \ldots, X_n, Y_1, \ldots, Y_n)/P'_{2n}(X_1, \ldots, X_n, Y_1, \ldots, Y_n) < d^{4n}, \quad n \in \mathbb{N}.$$

If $n! > d^{4n}$ then f_π, $\pi \in S_n$ are linearly dependent modulo $P'_{2n}(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$, thus yielding the desired identity. Since $n! > (n/e)^n > (n/3)^n$, the number $n = 3d^4$ is sufficiently large, and the result follows. ■

Next we construct some special weak identities for the smash products. A relation is called a *weak identity* of an algebra R if it vanishes whenever the selected indeterminates are replaced by the elements from the selected subsets of R. Let $\widetilde{A}$ be the free associative algebra generated by the set of symbols $\{z_i * X_j | i, j \in \mathbb{N}\}$. We consider weak identities as the elements from $\widetilde{A}$. We say that a weak identity is nontrivial if it is a nonzero element of $\widetilde{A}$.

Lemma 4.2. *Let $R = u(L)\#K[G]$ be a PI-algebra. Then it satisfies the following weak identities*

1.
$$\begin{aligned} & F_1(X_1,\dots,X_n,z_1,\dots,z_n) \\ = & \sum_{\pi\in S_n} \alpha_\pi (z_1 * X_{\pi(1)})\cdots(z_n * X_{\pi(n)}) \\ \equiv & \ 0; X_1,\dots,X_n \in u(L);\ z_1,\dots,z_n \in K[G]. \end{aligned}$$

2.
$$\begin{aligned} & F_2(X_1,\dots,X_n,g_1,\dots,g_n) \\ = & \sum_{\pi\in S_n} \alpha_\pi (g_1 * X_{\pi(1)} - X_{\pi(1)})\cdots(g_n * X_{\pi(n)} - X_{\pi(n)}) \\ \equiv & \ 0;\ X_1,\dots,X_n \in u(L);\ g_1,\dots,g_n \in G. \end{aligned}$$

where $\alpha_\pi \in K$, and $\alpha_e = 1$.

Proof. We take the identity of the previous lemma, set $Y_1 = g_1$, $Y_2 = g_1^{-1}g_2, \dots,$ $Y_n = g_{n-1}^{-1}g_n$, $g_i \in G$, and multiply on the right by g_n^{-1}. The for all $X_1,\dots,X_n \in u(L)$ and $g_1,\dots,g_n \in G$ we get

$$F_1(X_1,\dots,X_n,g_1,\dots,g_n) = \sum_{\pi\in S_n} \alpha_\pi (g_1 * X_{\pi(1)})\cdots(g_n * X_{\pi(n)}) \equiv 0. \qquad \blacksquare$$

By linearity we can substitute elements of the group ring for $g_1,\dots,g_n$. Thus we derive the first identity.

We decompose $F_2(X_1,\dots,X_n,g_1,\dots,g_n)$, $g_1,\dots,g_n \in G$, into 2^n summands, each being the result of substitution of the identity element $e \in G$ into $F_1(X_1,\dots,X_n,g_1,\dots,g_n)$ on some places $1 \le i_1 < i_2 < \cdots < i_s \le n$:

$$\begin{aligned} & F_2(X_1,\dots,X_n,g_1,\dots,g_n) = \\ & = \sum_{s=0}^{n} \sum_{1\le i_1<i_2<\cdots<i_s\le n} (-1)^s F_1(X_1,\dots,X_n,g_1,\dots,g_n)|_{g_{i_1}=\cdots=g_{i_s}=e} \equiv 0. \end{aligned}$$

We remark that the following decomposition holds

$$\begin{aligned} & F_1(X_1,\dots,X_n,z_1,\dots,z_n) \\ = & \sum_{i=1}^{n} (z_1 * X_i) H_i(X_1,\dots,\hat{X}_i,\dots,X_n,z_2,\dots,z_n); \qquad (9) \\ & H_i(X_1,\dots,\hat{X}_i,\dots,X_n,z_2,\dots,z_n) \\ = & \sum_{\pi(1)=i} \alpha_\pi (z_2 * X_{\pi(2)})\cdots(z_n * X_{\pi(n)}), \end{aligned}$$

where H_i are the polynomials of the same type as F_1. Moreover, if F_1 is nontrivial then some H_i is also nontrivial. Similar decompositions also hold true for polynomials of type F and F_2.

Suppose that L is a restricted Lie algebra. Fix some basis in L. Then we have the standard PBW-basis for the restricted enveloping algebra $u(L)$ [6]. Let

$u_n(L)$ denote the span of all basis monomials for $u(L)$ of length not greater than n. Now we have the standard filtration $u_0(L) \subset u_1(L) \subset \cdots \subset u_n(L) \subset \cdots$. It induces the filtration for $R = u(L)\#K[G]$

$$R_0 \subset R_1 \subset \cdots \subset R_n \subset \cdots; \qquad R_n = u_n(L)K[G],\ n = 0,1,2,\ldots$$
$$\operatorname{gr} R = \bigoplus_{i=0}^{\infty} \bar{R}_i, \quad \bar{R}_i = R_i/R_{i-1},\ i = 0,1,\ldots$$

Observe that $\operatorname{gr} R \cong \operatorname{gr}\{u_n(L)|n = 0,1,2\ldots\}\#K[G]$ where the action of G on the vector space $\bar{L} \cong u_1(L)/u_0(L)$ is the same as the action of G on L. Recall also that $\operatorname{gr}\{u_n(L)|n = 0,1,2\ldots\}$ is isomorphic to the ring of truncated polynomials.

Lemma 4.3. *Let $R = u(L)\#K[G]$ be a PI-algebra. Then $\operatorname{gr} R$ satisfies a weak identity*

$$F_3(X_1,\ldots,X_n,z_1,\ldots,z_n) = \sum_{\pi\in S_n} \alpha_\pi (z_{\pi(n)} * X_n)\cdots(z_{\pi(1)} * X_1) \equiv 0;$$
$$X_1,\ldots,X_n \in u(L), \quad z_1,\ldots,z_n \in K[G];$$

where $\alpha_\pi \in K$, and $\alpha_e = 1$.

Proof. We rewrite $F_1(X_1,\ldots,X_n,z_1,\ldots,z_n)$ using the commutativity of $\operatorname{gr} u(L)$. ∎

The nontrivial elements of the form F_3 can be also decomposed as

$$\begin{aligned} &F_3(X_1,\ldots,X_n,z_1,\ldots,z_n)\\ =\ &\sum_{i=1}^{n}(z_i * X_n)H_i(X_1,\ldots,X_{n-1},z_1,\ldots\hat{z}_i,\ldots,z_n); \qquad (10)\\ &H_i(X_1,\ldots,X_{n-1},z_1,\ldots\hat{z}_i,\ldots,z_n)\\ =\ &\sum_{\pi(n)=i} \alpha_\pi(z_{\pi(n-1)} * X_{n-1})\cdots(z_{\pi(1)} * X_1), \end{aligned}$$

where H_i are the polynomials of the same type as F_3 and some H_i is also nontrivial.

5. Delta-theory for smash products

Recall that there are some delta-sets inside G and L that have been effectively used in the study of the inner structure of G and L; the notation δ, Δ was used for these sets in [3], [11], [14]. A definition of delta-sets for Hopf algebras in terms of their inner actions can be found in [4].

Suppose that a group G acts on a Lie algebra L by automorphisms. In this section we introduce four more families of delta-sets defined with respect to this action. We specially reserve the symbol $\mathcal{D}$ for the pairing between $K[G]$ and L.

First, we define a series of delta-sets inside L:

$$\begin{aligned} \mathcal{D}_{m,G}(L) &= \{x \in L \mid \dim K[G] * x \le m\}, \quad m \in \mathbb{N};\\ \mathcal{D}_G(L) &= \bigcup_{m=1}^{\infty} \mathcal{D}_{m,G}(L) \subset L. \end{aligned}$$

Another series of delta-sets lives inside G:

$$\begin{aligned}\mathcal{D}_{m,L}(G) &= \{g \in G \mid \dim(g-1)*L \le m\}, \quad m = 0,1,2\ldots;\\ \mathcal{D}_L(G) &= \bigcup_{m=0}^{\infty} \mathcal{D}_{m,L}(G) \subset G.\end{aligned}$$

Finally, we define families of delta-sets inside the group ring $K[G]$ and the restricted enveloping algebra $u(L)$:

$$\begin{aligned}\mathcal{D}_{m,L}(K[G]) &= \{a \in K[G] \mid \dim(a*L) \le m\}, \quad m = 0,1,2,\ldots;\\ \mathcal{D}_L(K[G]) &= \bigcup_{m=0}^{\infty} \mathcal{D}_{m,L}(K[G]) \subset K[G].\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{m,G}(u(L)) &= \{v \in u(L) \mid \dim K[G]*v \le m\}, \quad m \in \mathbb{N};\\ \mathcal{D}_G(u(L)) &= \bigcup_{m=1}^{\infty} \mathcal{D}_{m,G}(u(L)) \subset u(L).\end{aligned}$$

In our study of identities in smash products we shall use essentially three families except those inside $u(L)$. For convenience we often omit subscripts L, G and simply write, for example, $\mathcal{D}(L)$ instead of $\mathcal{D}_G(L)$.

One can easily check the following properties of these sets.

Lemma 5.1. *The sets $\mathcal{D}_i(L)$ satisfy*

1. *if $x \in \mathcal{D}_i(L)$, $y \in \mathcal{D}_j(L)$ then $\alpha x + \beta y \in \mathcal{D}_{i+j}(L)$, $\alpha, \beta \in K$;*
2. *if $x \in \mathcal{D}_i(L)$, $y \in \mathcal{D}_j(L)$ then $[x,y] \in \mathcal{D}_{i\cdot j}(L)$; and $x^{[p]} \in \mathcal{D}_{i^p}(L)$;*
3. *$\mathcal{D}_i(L)$ are G-invariant;*
4. *$\mathcal{D}(L)$ is a restricted G-invariant subalgebra in L.*

Proof. For example, let us prove the second claim. In this case $K[G]*[x,y] = \langle g*[x,y] | g \in G\rangle_K = \langle [g*x, g*y] | g \in G\rangle_K \subset [K[G]*x, K[G]*y]$, so $\dim K[G]*[x,y] \le \dim([K[G]*x) \cdot \dim([K[G]*y) \le ij$. ■

Lemma 5.2. *The sets $\mathcal{D}_i(G)$ satisfy*

1. *if $g \in \mathcal{D}_i(G)$, $h \in \mathcal{D}_j(G)$ then $gh \in \mathcal{D}_{i+j}(G)$;*
2. *if $g \in \mathcal{D}_i(G)$, then $g^{-1} \in \mathcal{D}_i(G)$;*
3. *if $g \in \mathcal{D}_i(G)$, $h \in G$ then $h^{-1}gh \in \mathcal{D}_i(G)$;*
4. *$1 \in \mathcal{D}_i(G)$ for all $i \ge 0$;*
5. *$\mathcal{D}(G)$ is a normal subgroup in G.*

Proof. In order to prove claims 1), 2), and 3) we observe that for arbitrary $g, h \in G$ one has

$$(gh-1)*L = ((g-1)h + h - 1)*L \subset (g-1)*L + (h-1)*L;$$
$$(g^{-1}-1)*L = g^{-1}*((1-g)*L);$$
$$(h^{-1}gh-1)*L = (h^{-1}(g-1)h)*L \subset h^{-1}*((g-1)*L).$$

Other claims are obvious. ■

Lemma 5.3. *The sets* $\mathcal{D}_i(K[G])$ *satisfy*

1. *if* $a \in \mathcal{D}_i(K[G])$, $b \in \mathcal{D}_j(K[G])$ *then* $\alpha a + \beta b \in \mathcal{D}_{i+j}(K[G])$, $\alpha, \beta \in K$;
2. $K[G] \cdot \mathcal{D}_i(K[G]) \cdot K[G] \subset \mathcal{D}_i(K[G])$;
3. $\mathcal{D}(K[G])$ *is a two-sided ideal in* $K[G]$.

Proof. Let us check the second claim. Suppose that $a \in \mathcal{D}_i(K[G])$, so $\dim(a*L) \le i$. Then for arbitrary $x, y \in K[G]$ we have $(xay)*L \subset x*(a*L)$ and $\dim((xay)*L) \le i$. Hence, $xay \in \mathcal{D}_i(K[G])$. ■

Lemma 5.4. *The sets* $\mathcal{D}_i(u(L))$ *satisfy*

1. *if* $v \in \mathcal{D}_i(u(L))$, $w \in \mathcal{D}_j(u(L))$ *then* $\alpha v + \beta w \in \mathcal{D}_{i+j}(u(L))$, $\alpha, \beta \in K$;
2. *if* $v \in \mathcal{D}_i(u(L))$, $w \in \mathcal{D}_j(u(L))$ *then* $vw \in \mathcal{D}_{i+j}(u(L))$;
3. $\mathcal{D}(u(L))$ *is a subalgebra in* $u(L)$.

Proof. is similar to that of Lemma 5.1. ■

Let us establish the relationship between $\mathcal{D}(G)$ and $\mathcal{D}(K[G])$.

Lemma 5.5. *1.* $1 + (\mathcal{D}(K[G]) \cap (G-1)) = \mathcal{D}(G)$;

2. $K[G] \cdot \omega K[\mathcal{D}(G)] = \omega K[\mathcal{D}(G)] \cdot K[G] \subset \mathcal{D}(K[G])$.

Proof. The first claim follows directly from definitions. Let us prove the second one. We consider $v = \alpha_1 g_1 + \cdots + \alpha_m g_m \in \omega K[\mathcal{D}(G)]$, $\alpha_i \in K$, $g_i \in \mathcal{D}(G)$. Then $\alpha_1 + \cdots + \alpha_m = 0$ and $v = \alpha_1(g_1 - 1) + \cdots + \alpha_m(g_m - 1)$. There exists a number s such that $\{g_1, \ldots, g_m\} \subset \mathcal{D}_s(G)$. Let $w, u \in K[G]$ then

$$(wvu)*L \subset w*((g_1-1)*L + \cdots + (g_m-1)*L),$$

and $wvu \in \mathcal{D}_{ms}(K[G])$. ■

But we lack any bounds in this lemma. We only suggest the following conjecture that would make Theorem 6.5 below unnecessary.

Conjecture *Let* $G = \mathcal{D}_s(G)$ *then* $\omega K[G] = \mathcal{D}_t(K[G])$ *for some number* $t = f(s)$.

On the other hand, the next example shows that the inclusion in Lemma 5.5 is strict: $K[G] \cdot \omega K[\mathcal{D}(G)] \ne \mathcal{D}(K[G])$.

Example 5.6. Let $L = \langle x_1, x_2, \ldots, y_1, y_2, \ldots | x_i^{[p]} = y_i^{[p]} = 0, i \in \mathbb{N}\rangle_K$ be an abelian restricted Lie algebra. We consider the group $G = (\mathbb{Z}_p)^{\mathbb{N}}$ and set $g_i = (0, \ldots, 0, 1, 0, \ldots)$, with 1 on i-th place, $i \in \mathbb{N}$. Suppose that G acts on L by

$$g_i * x_j = x_j + \xi_i y_j, \quad g_i * y_j = y_j, \quad i, j \in \mathbb{N},$$

where $\{\xi_i \in K | i \in \mathbb{N}\}$ are the scalars linearly independent over $\mathbb{Z}_p$. We consider the smash product $R = u(L)\#K[G]$. Then

1. $\mathcal{D}(G) = \{e\}$;
2. $\mathcal{D}_L(K[G]) = \mathcal{D}_{0,L}(K[G])$ and $\dim K[G]/\mathcal{D}(K[G]) = 2$;
3. R is not PI;
4. R satisfies some weak identities of the type

$$\begin{aligned} & F_2(X_1, \ldots, X_n, h_1, \ldots, h_n) \\ = & \sum_{\pi \in S_n} \alpha_\pi (h_1 * X_{\pi(1)} - X_{\pi(1)}) \cdots (h_n * X_{\pi(n)} - X_{\pi(n)}) \\ \equiv & \ 0; X_1, \ldots, X_n \in L, \ h_1, \ldots, h_n \in G. \end{aligned}$$

Proof. Let $e \neq g = (n_1, n_2, \ldots) \in G$, then $(g-1) * x_j = \lambda y_j$, where $\lambda = \sum_i n_i \xi_i \neq 0$ by assumption. Now the first claim follows by the definition of the delta-sets $\mathcal{D}_{i,L}(G)$.

To prove the second claim we take $z = \sum_i \alpha_i g_i \in K[G]$ and consider

$$z * x_j = \Big(\sum_i \alpha_i\Big) x_j + \Big(\sum_i \alpha_i \xi_i\Big) y_j, \quad j \in \mathbb{N}.$$

Then $z \in \mathcal{D}_{m,L}(K[G])$ for some $m \geq 0$ if and only if $\sum_i \alpha_i = 0$ and $\sum_i \alpha_i \xi_i = 0$. Hence, $\mathcal{D}_L(K[G]) = \mathcal{D}_{0,L}(K[G])$ and $\dim K[G]/\mathcal{D}_L(K[G]) = 2$.

Suppose that R is PI. We apply the first identity of Lemma 4.2 and substitute the values $a_1 = x_1$, $a_2 = x_2 x_3$, $\ldots$, $a_n = x_{(n-1)n/2+1} \cdots x_{n(n+1)/2}$, and $g_{i_1}, \ldots, g_{i_n} \in G$:

$$\begin{aligned} F_1(a_1, a_2, \ldots, a_n, g_{i_1}, \ldots, g_{i_n}) &= \sum_{\pi \in S_n} \alpha_\pi (g_{i_1} * a_{\pi(1)}) \cdots (g_{i_n} * a_{\pi(n)}) \\ &= \Big(\sum_{\pi \in S_n} \alpha_\pi \xi_{i_1}^{\pi(1)} \xi_{i_2}^{\pi(2)} \cdots \xi_{i_n}^{\pi(n)}\Big) y_1 \cdots y_{n(n+1)/2} + \text{ terms with } x\text{-s.} \end{aligned}$$

Therefore, $0 \neq f(X_1, \ldots, X_n) = \sum_{\pi \in S_n} \alpha_\pi X_1^{\pi(1)} X_2^{\pi(2)} \cdots X_n^{\pi(n)} \in K[X_1, \ldots, X_n]$ is annihilated by any substitution from the countable set of scalars $X_i \in \{\xi_j | j \in \mathbb{N}\}$, $i = 1, \ldots, n$. This contradiction proves that R is not PI.

By linearity, the last claim can be checked for the action on x-s only. We remark that any element $h \in G$ acts as the generators g_i, namely $h * x_i = x_i + \mu_h y_i$, $\mu_h \in K$, $i \in \mathbb{N}$. Then

$$\begin{aligned} F_2(x_1, \ldots, x_n, h_1, \ldots, h_n) &= \sum_{\pi \in S_n} \alpha_\pi (h_1 * x_{\pi(1)} - x_{\pi(1)}) \cdots (h_n * x_{\pi(n)} - x_{\pi(n)}) \\ &= \Big(\sum_{\pi \in S_n} \alpha_\pi\Big) \mu_1 \mu_2 \cdots \mu_n y_1 y_2 \cdots y_n. \end{aligned}$$

If $\sum_{\pi\in S_n}\alpha_\pi = 0$, then we obtain a weak identity.[3] ■

Also, this example shows that it is not enough to study the action of G on L, but we also need to take into account the action of G on $u(L)$ as well (see the proof of Theorem 6.2 below).

Let us also establish the relationship between $\mathcal{D}(L)$ and $\mathcal{D}(u(L))$.

Lemma 5.7. *1.* $\mathcal{D}(u(L))\cap L = \mathcal{D}(L)$*;*

2. $\omega u(\mathcal{D}(L)) \subset \mathcal{D}(u(L))$.

Proof. The first claim follows from definitions. The subalgebra $\omega u(\mathcal{D}(L))$ is generated by the elements from $\mathcal{D}(L)$. These elements have finite width by Lemma 5.4. ■

We suggest to study whether there exists some bounds similar to Conjecture above. Another interesting question is to investigate if the inclusion in Lemma 5.7 is strict.

6. Structure of delta-sets

In this section we establish crucial facts about the structure of three families of delta-sets, provided that the smash product $R = u(L)\#K[G]$ satisfies a nontrivial polynomial identity. ¿From now on we assume that the number n is fixed in Lemma 4.2 (remark that all lemmas of Section 4 fix the same number n).

Let us consider the associated graded algebra $\operatorname{gr} R \cong (\operatorname{gr} u(L))\#K[G]$. It satisfies the same weak identities of Lemmas 4.2, 4.3. We observe that the action of G on the space $u_1(L)/u_0(L) \cong L$ inside $\operatorname{gr} u(L)$ remains the same. Therefore, we may assume that L is abelian with the trivial p-mapping, while studying the action of G on L in this section.

Let L be a restricted Lie algebra and some basis $L = \{e_i | i \in I\}$ be fixed. Suppose that the decomposition of $v \in u(L)$ via the standard basis for the restricted enveloping algebra depends on the elements $e_{i_1},\dots,e_{i_s}$. Then we denote the *support* of v by $\operatorname{supp} v = \{e_{i_1},\dots,e_{i_s}\}$.

Let G be a group and T a subset of G. We say that T has *finite index* in G if there exist $g_1, g_2, \dots, g_m \in G$ with

$$G = g_1T \cup g_2T \cup \dots \cup g_mT.$$

We then define the *index* $|G : T|$ to be the minimum possible integer m with the above property[11], [12]. If T is a subgroup of G, then this agrees with the usual definition of index.

Lemma 6.1. ([11]) *Let T be a subset of G with $|G : T| \le m$. We set $T^* = T \cup \{1\} \cup T^{-1}$. Then*

$$(T^*)^{4^m} = \underbrace{T^*\cdot T^*\cdot\dots\cdot T^*}_{4^m\ times}$$

is a subgroup of G.

[3]This example was produced in collaboration with M. Kotchetov

Theorem 6.2. *Let $R = u(L)\#K[G]$ be a PI-algebra. Then there exists a subgroup $G_0 \subset G$ with $|G : G_0| < n$ and $G_0 \subset \mathcal{D}_{n^3 4^n, L}(G)$.*

Proof. Let n be the number fixed in Lemma 4.3. Let us prove that $|G : \mathcal{D}_{n^3}(G)| < n$. We fix arbitrary $g_1, \dots, g_n \in G$. It suffices to prove that there exist $i \neq j$ such that $g_i^{-1} g_j \in \mathcal{D}_{n^3}(G)$. We apply the weak identity of Lemma 4.3

$$F_3(X_1, \dots, X_n, g_1, \dots, g_n) = \sum_{\pi \in S_n} \alpha_\pi (g_{\pi(n)} * X_n) \cdots (g_{\pi(1)} * X_1) \equiv 0;$$
$$X_1, \dots, X_n \in u(L).$$

In the identical relations that follow we denote by $X's$, and $Y's$ the variables that range over some sets of elements inside $u(L)$.

Let us prove by induction on m the following. Suppose that $g_1, \dots, g_m \in G$ are fixed and satisfy

$$F_3(X_1, \dots, X_m, g_1, \dots, g_m) \equiv 0; \qquad X_i \in u_i(L),\ i = 1, \dots, m,$$

where F_3 is nontrivial. Then we claim that there exist $i \neq j$ such that $g_i^{-1} g_j \in \mathcal{D}_{m^3}(G)$.

We consider $m = 1$, we have $g_1 * X_1 \equiv 0$, $X_1 \in L$. Then $L = 0$ and the assertion is trivial.

Suppose that the statement is valid for $m-1$, $m > 1$. We apply (10):

$$\begin{aligned} & F_3(X_1, \dots, X_m, g_1, \dots, g_m) \\ = & \sum_{i=1}^{m} (g_i * X_m) H_i(X_1, \dots, X_{m-1}, g_1, \dots \hat{g}_i, \dots, g_m) \\ \equiv & \ 0; X_i \in u_i(L),\ 1 \leq i \leq m. \end{aligned} \tag{11}$$

Without loss of generality we may assume that H_m is a non-trivial polynomial. Now we consider two cases. First, suppose that

$$H_m(X_1, \dots, X_{m-1}, g_1, \dots, g_{m-1}) \equiv 0; \quad X_i \in u_i(L),\ i = 1, \dots, m-1.$$

Then by the inductive assumption there exist $i \neq j$, $1 \leq i, j \leq m-1$ such that $g_i^{-1} g_j \in \mathcal{D}_{(m-1)^3}(G) \subset \mathcal{D}_{m^3}(G)$.

Second, there exist $a_1 \in u_1(L)$, $a_2 \in u_2(L)$, $\dots$, $a_{m-1} \in u_{m-1}(L)$ such that $h_m = H_m(a_1, \dots, a_{m-1}, g_1, \dots, g_{m-1}) \neq 0$. Let $\{e_i | i \in I\}$ be an ordered basis for L. Since H_m is linear in $X's$, we can consider $a_1, \dots, a_{m-1}$ to be monomials in $\{e_{\alpha_1}, \dots, e_{\alpha_t}\}$, where $t \leq 1 + 2 + \dots + m - 1 < m(m-1)$. We set

$$V_0 = \langle g_i * e_{\alpha_j} | 1 \leq i \leq m-1, 1 \leq j \leq t \rangle_K.$$

Then $\dim V_0 \leq t(m-1) \leq m(m-1)^2$. Now we substitute $a_1, \dots, a_{m-1}$ for $X_1, \dots, X_{m-1}$ in (11), set $X = X_m$, and obtain the relation

$$\sum_{i=1}^{m} (g_i * X) h_i \equiv 0, \quad X \in u_m(L); \tag{12}$$

$$h_m \neq 0, \quad \operatorname{supp} h_m \subset V_0. \tag{13}$$

By substituting XY for X we get

$$\sum_{i=1}^{m}(g_i * X)(g_i * Y)h_i \equiv 0, \quad X \in u_{m-1}(L),\ Y \in L. \tag{14}$$

Multiplying (12) by $g_1 * Y$ and subtracting from (14) we obtain

$$\sum_{i=2}^{m}(g_i * X)(g_i * Y - g_1 * Y)h_i \equiv 0, \quad X \in u_{m-1}(L),\ Y \in L. \tag{15}$$

Here we have two possibilities. First, $(g_m - g_1) * y \in V_0$ for all $y \in L$. Then $(g_1^{-1}g_m - 1) * L \subset g_1^{-1} * V_0$, therefore $g_1^{-1}g_m \in \mathcal{D}_b(G)$, where $b = \dim V_0 \leq m(m-1)^2 < m^3$. Second, there exists $y_0 \in L$ such that $(g_m - g_1) * y_0 = v_1 \notin V_0$. We set $V_1 = V_0 + \langle v_1 \rangle_K$ and change the basis of L outside V_0 so that v_1 coincides with some basis element. We fix $Y = y_0$ in (15), and by the construction of v_1 and (13) we obtain

$$\begin{aligned} &\sum_{i=2}^{m}(g_i * X)h_i^{(1)} \equiv 0, \quad X \in u_{m-1}(L); \\ &h_i^{(1)} = (g_i * y_0 - g_1 * y_0)\, h_i, \quad i = 2, \dots, m; \\ &h_m^{(1)} = v_1 h_m \neq 0, \qquad \operatorname{supp} h_m^{(1)} \subset V_1. \end{aligned} \tag{16}$$

We continue this process further by deleting in (16) the term for $i = 2$.

$$\sum_{i=3}^{m}(g_i * X)(g_i * Y - g_2 * Y)h_i^{(1)} \equiv 0, \quad X \in u_{m-2}(L),\ Y \in L.$$

Similarly, either $(g_m - g_2) * L \subset V_1$ and we are done (see below), or there exists $y_1 \in L$ such that $(g_m - g_2) * y_1 = v_2 \notin V_1$. In the latter case we change the basis of L outside V_1 so that v_2 is one of the basis elements, set $V_2 = V_1 + \langle v_2 \rangle_K$, $h_i^{(2)} = (g_i * y_1 - g_2 * y_1)h_i^{(1)}$, $i = 3, \dots, m$ and obtain the relation

$$\begin{aligned} &\sum_{i=3}^{m}(g_i * X)h_i^{(2)} \equiv 0, \quad X \in u_{m-2}(L); \\ &h_m^{(2)} = v_2 h_m^{(1)} \neq 0, \qquad \operatorname{supp} h_m^{(2)} \subset V_2. \end{aligned}$$

This process terminates with the relation

$$\begin{aligned} &(g_m * X)h_m^{(m-1)} \equiv 0, \quad X \in u_1(L); \\ &h_m^{(m-1)} \neq 0; \\ &\operatorname{supp} h_m^{(m-1)} \subset V_{m-1} = V_0 + \langle v_1, \dots, v_{m-1} \rangle_K. \end{aligned} \tag{17}$$

If we substitute $X = 1$ in (17) then we obtain a contradiction. This contradiction proves that this process had to stop even before, and the desired relation holds: $g_i^{-1}g_m \in \mathcal{D}_c(G)$, where $c = \dim V_{i-1} \leq \dim V_{m-1} \leq m(m-1)^2 + m - 1 \leq m^3$.

If we set $T = \mathcal{D}_{n^3}(G)$, then we have proved that $|G : T| \leq n$. Remark that, by Lemma 5.2, $1 \in T$ and $T = T^{-1}$. We set $G_0 = T^{4^n}$ and conclude that G_0 is a subgroup by Lemma 6.1. Of course, $|G : G_0| \leq n$ and $G_0 \subset \mathcal{D}_{n^3 4^n}(G)$ by Lemma 5.2. ∎

Suppose that W is a subset in a vector space V. We say that W has finite codimension in V if there exist $v_1, \dots, v_m \in V$ with $V = W + \langle v_1, \dots, v_m \rangle_K$. If m is the minimum possible integer with such property then we set $\dim V/W = m$. We also introduce the notation $m \cdot W = \{w_1 + \dots + w_m | w_i \in W\}$, $m \in \mathbb{N}$.

Lemma 6.3. *Let L be a vector space. Suppose that a subset $T \subset L$ is stable under multiplication by scalars and such that* $\dim L/T \le m$*. Then $\langle T \rangle_K = 4^m \cdot T$.*

Proof. We prove this statement by induction on m. If $m = 0$ then the assertion is trivial. Suppose that $\dim L/T = m$. Then there exist $h_1, \dots, h_m$ such that

$$L = T + \langle h_1, \dots, h_m \rangle_K. \tag{18}$$

If $2 \cdot T = T$ then T is a subspace. Otherwise there exist $t_1, t_2 \in T$ with $t_1 + t_2 \notin T$. By (18) $t_1 + t_2 = t_3 + \alpha_1 h_1 + \dots + \alpha_m h_m$, $t_3 \in T$, $\alpha_i \in K$, where one of scalars is nonzero. Let $\alpha_m \ne 0$, then $h_m \in 3 \cdot T + \langle h_1, \dots, h_{m-1} \rangle_K$. We substitute in (18) and obtain $L = 4 \cdot T + \langle h_1, \dots, h_{m-1} \rangle_K$. By the inductive assumption $\langle 4 \cdot T \rangle_K = 4^{m-1} \cdot (4 \cdot T) \subset 4^m \cdot T$. Lemma is proved. ∎

Theorem 6.4. *Let $R = u(L)\#K[G]$ be a PI-algebra. Then there exists a G-invariant restricted subalgebra $L_0 \subset L$ with* $\dim L/L_0 < n$ *and $L_0 \subset \mathcal{D}_{p^n 4^n n^5, G}(L)$.*

Proof. Let n be the number fixed above. While studying the action of G on L, we temporarily assumethat L is abelian with the trivial p-mapping. First, let us prove that $\dim L/\mathcal{D}_{n^2}(L) < n$.

We fix arbitrary $a_1, \dots, a_n \in L$. By Lemma 4.2 we have

$$F_1(a_1, \dots, a_n, g_1, \dots, g_n) = \sum_{\pi \in S_n} \alpha_\pi (g_1 * a_{\pi(1)}) \cdots (g_n * a_{\pi(n)}) \equiv 0; \quad g_1, \dots, g_n \in G.$$

In this theorem g_i's, g are the variables that range over G. Let us prove by induction on m the following. Suppose that $a_1, \dots, a_m \in L$ are fixed and satisfy the condition

$$F_1(a_1, \dots, a_m, g_1, \dots, g_m) = \sum_{\pi \in S_m} \alpha_\pi (g_1 * a_{\pi(1)}) \cdots (g_m * a_{\pi(m)}) \equiv 0, \qquad g_1, \dots, g_m \in G; \tag{19}$$

where $F_1(X_1, \dots, X_m, g_1, \dots, g_m)$ is some nontrivial polynomial. Then $a_1, \dots, a_m$ are linearly dependent modulo $\mathcal{D}_{m^2}(L)$.

In the case $m = 1$ we have $g_1 * a_1 \equiv 0$, $g_1 \in G$. Since G acts by automorphisms we have $a_1 = 0 \in \mathcal{D}_1(L)$.

Suppose that the statement is valid for $m-1$, $m > 1$. We apply (9) to (19)

$$\begin{aligned} & F_1(a_1, \dots, a_m, g_1, \dots, g_m) \\ = & \sum_{i=1}^{m} (g_1 * a_i) H_i(a_1, \dots, \hat{a}_i, \dots, a_m, g_2, \dots, g_m) \\ \equiv & \ 0; \ g_1, \dots, g_m \in G. \end{aligned} \tag{20}$$

Without loss of generality we assume that $H_1(X_2, \dots, X_m, g_2, \dots, g_m)$ is a nontrivial polynomial. Now two cases are possible. In the first case $H_1(a_2, \dots, a_m, g_2, \dots, g_m) \equiv 0$ for all $g_2, \dots, g_m \in G$. Then by the inductive hypothesis $a_2, \dots, a_m$ are linearly dependent modulo $\mathcal{D}_{(m-1)^2}(L)$ and we are done. In the second case there exist $h_2, \dots, h_m \in G$ such that $f_1 = H_1(a_2, \dots, a_m, h_2, \dots, h_m) \neq 0$. We substitute these values into (20) and set $f_i = H_i(a_1, \dots, \hat{a}_i, \dots, a_m, h_2, \dots, h_m)$, $i = 1, \dots, m$:

$$\sum_{i=1}^{m} (g_1 * a_i) f_i \equiv 0, \qquad g_1 \in G.$$

We discard the summands with $f_i = 0$, set $g = g_1$, and rewrite our relation as follows

$$\sum_{i=1}^{r} (g * a_i) f_i \equiv 0, \quad g \in G; \quad f_i \neq 0,\ i = 1, \dots, r. \tag{21}$$

We set $V = \langle g_i * a_j | 2 \leq i \leq m, 1 \leq j \leq m \rangle_K$, then $\dim V < m^2$ and $f_1, \dots, f_r$ belong to the subalgebra of $u(L)$ generated by V. Next we prove by induction on r that (21) implies linear dependence of $a_1, \dots, a_r$ modulo $\mathcal{D}_{m^2}(L)$.

If $r = 1$ then

$$(g * a_1) f_1 \equiv 0, \quad g \in G. \tag{22}$$

Let us prove that $G * a_1 \subset V$, so $a_1 \in \mathcal{D}_{m^2}(L)$. By way of contradiction suppose that there exists $d \in G$ with $d * a_1 = e_0 \notin V$. Choose an ordered basis for L whose first elements is e_0, followed by a basis of $V = \langle v_1, \dots, v_t \rangle_K$. Now f_1 is the sum of products, each product consists of $m - 1$ factors of the type $g_i * a_j \in V$. Using the standard basis of the restricted enveloping algebra we have

$$f_1 = \sum_j \alpha_j v_{j_1} \cdots v_{j_{m-1}}, \quad \alpha_j \in K. \tag{23}$$

Multiplying f_1 by e_0 we obtain a nonzero element. Thus, seting $g = d$ we arrive at a contradiction with (22).

We consider $r > 1$. Suppose that $G * a_r \subset V$ in (21). Then $a_r \in \mathcal{D}_{m^2}(L)$ and the result follows. So, we assume that $e_0 = d * a_r \notin V$ for some $d \in G$. By analogy with the preceding argument we choose an ordered basis for L. We set $d * a_j = \alpha_j e_0 + w_j$, $j = 1, \dots, r - 1$, $\alpha_j \in K$ and each w_j being a linear combination of the basis elements of L except e_0. By setting $g = d$ in (21) we obtain

$$e_0(\alpha_1 f_1 + \cdots + \alpha_{r-1} f_{r-1} + f_r) + w_1 f_1 + \cdots + w_{r-1} f_{r-1} = 0. \tag{24}$$

We set $f = \alpha_1 f_1 + \cdots + \alpha_{r-1} f_{r-1} + f_r$. Suppose that $f \neq 0$. By analogy with the preceding argument, f is of the form (23), which means that the first summand in (24) has degree m and may be written as

$$e_0 f = \sum_j \alpha_j e_0 v_{j_1} \cdots v_{j_{m-1}}, \quad \alpha_j \in K. \tag{25}$$

Other nonzero summands in (24), being written in the form (25), do not contain e_0 as their factor. This is a contradiction. So, $f = \alpha_1 f_1 + \cdots + \alpha_{r-1} f_{r-1} + f_r = 0$. We express f_r from this relation and substitute into (21):

$$\sum_{i=1}^{r-1} (g * (a_i - \alpha_i a_r)) f_i \equiv 0, \quad g \in G;$$
$$f_i \neq 0,\ i = 1, \dots, r-1.$$

By the inductive assumption $a_1 - \alpha_1 a_r, \dots, a_{r-1} - \alpha_{r-1} a_r$ are linearly dependent modulo $\mathcal{D}_{m^2}(L)$, therefore $a_1, \dots, a_r$ are also linearly dependent modulo this set.

Thus we have proved that $\dim L/\mathcal{D}_{n^2}(L) < n$. Now we return to the original structure of a Lie p-algebra on L. By Lemmas 6.3, 5.1 we construct a subspace $W_1 = \langle \mathcal{D}_{n^2}(L) \rangle_K = 4^n \cdot \mathcal{D}_{n^2}(L) \subset \mathcal{D}_{4^n n^2}(L)$. We consider the chain $W_i = W_{i-1} + [W_{i-1}, W_1]$, $i = 2, 3 \dots$. Due to the finiteness of codimension we have stabilization $\cup_{i=1}^{\infty} W_i = W_n$ and by Lemma 5.1 $W_n \subset \mathcal{D}_{4^n n^4}(L)$. By construction, W_n is a G-invariant subalgebra. To obtain a restricted subalgebra we consider its p-hull $L_0 = (W_n)_p = \langle w^{[p^s]} | w \in W_n, s \geq 0 \rangle_K$. Again by the codimension argument, we can assume here that $s \leq n$. By Lemma 5.1 we get $L_0 \subset \mathcal{D}_{p^n 4^n n^5}(L)$, also $\dim L/L_0 < n$. The theorem follows. ■

Theorem 6.5. *Let $R = u(L) \# K[G]$ be a PI-algebra. Then there exists a two-sided ideal $J \subset K[G]$ with $\dim K[G]/J < n$ and $J \subset \mathcal{D}_{4^n n^2, L}(K[G])$.*

Proof. Similar to that for the previous theorem. Let n be the number fixed above. We may assume that L is abelian with the trivial p-mapping. First, let us prove that $\dim K[G]/\mathcal{D}_{n^2}(K[G]) < n$.

We fix arbitrary $z_1, \dots, z_n \in K[G]$. By Lemma 4.3 we have

$$F_3(a_1, \dots, a_n, z_1, \dots, z_n) = \sum_{\pi \in S_n} \alpha_\pi (z_{\pi(n)} * a_n) \cdots (z_{\pi(1)} * a_1) \equiv 0; \quad a_1, \dots, a_n \in L.$$

In this theorem a_i's, a denote the variables that range over L. We prove by induction on m the following. Let $z_1, \dots, z_m \in K[G]$ be fixed and satisfy the condition

$$F_3(a_1, \dots, a_m, z_1, \dots, z_m) = \sum_{\pi \in S_m} \alpha_\pi (z_{\pi(m)} * a_m) \cdots (z_{\pi(1)} * a_1) \equiv 0, \quad (26)$$
$$a_1, \dots, a_m \in L;$$

where $F_3(X_1, \dots, X_m, z_1, \dots, z_m)$ is some nontrivial polynomial. Then $z_1, \dots, z_m$ are linearly dependent modulo $\mathcal{D}_{m^2}(K[G])$.

In the case $m = 1$ we have $z_1 * a_1 \equiv 0$, $a_1 \in L$. Then $z_1 \in \mathcal{D}_0(K[G])$ and the result follows.

Suppose that the statement is valid for $m-1$, $m > 1$. We apply decomposition (10) to the relation (26)

$$\begin{aligned} & F_3(a_1, \dots, a_m, z_1, \dots, z_m) \\ = & \sum_{i=1}^{m} (z_i * a_m) H_i(a_1, \dots, a_{m-1}, z_1, \dots, \hat{z}_i, \dots, z_m) \\ \equiv & \ 0; \ a_1, \dots, a_m \in L. \end{aligned} \quad (27)$$

Without loss of generality we may assume that $H_1(X_1,\dots,X_{m-1},z_2,\dots,z_m)$ is a nontrivial polynomial. We have two cases. 1) $H_1(a_1,\dots,a_{m-1},z_2,\dots,z_m)\equiv 0$ for all $a_1,\dots,a_{m-1}\in L$. Then by the inductive hypothesis $z_2,\dots,z_m$ are linearly dependent modulo $\mathcal{D}_{(m-1)^2}(K[G])$ and we are done. 2) There exist $b_1,\dots,b_{m-1}\in L$ such that $H_1(b_1,\dots,b_{m-1},z_2,\dots,z_m)\neq 0$. We substitute these values into (27) and set $f_i=H_i(b_1,\dots,b_{m-1},z_1,\dots,\hat{z}_i,\dots,z_m)$, $i=1,\dots,m$:

$$\sum_{i=1}^{m}(z_i * a_m)f_i\equiv 0,\qquad a_m\in L.$$

We omit the summands with $f_i=0$, denote $a=a_m$, and obtain

$$\sum_{i=1}^{r}(z_i * a)f_i\equiv 0,\quad a\in L;\quad f_i\neq 0,\ i=1,\dots,r. \tag{28}$$

We set $V=\langle z_i * b_j \mid 1\le i\le m, 1\le j< m-1\rangle_K$, then $\dim V<m^2$ and $f_1,\dots,f_r$ belong to the subalgebra of $u(L)$ generated by V. Next we prove by induction on r that (28) implies linear dependence of $z_1,\dots,z_r$ modulo $\mathcal{D}_{m^2}(K[G])$.

If $r=1$ then

$$(z_1 * a)f_1\equiv 0,\quad a\in L. \tag{29}$$

Let us prove that $z_1 * L\subset V$, so $z_1\in\mathcal{D}_{m^2}(K[G])$. By way of contradiction suppose that there exists $y\in L$ with $z_1 * y=e_0\notin V$. We choose an ordered basis for L whose first element is e_0, followed by a basis of $V=\langle v_1,\dots,v_t\rangle_K$. In the standard basis of the restricted enveloping algebra we have

$$f_1=\sum_j \alpha_j v_{j_1}\cdots v_{j_{m-1}}\neq 0,\quad \alpha_j\in K.$$

Multiplying f_1 by e_0 we obtain a nonzero element. Thus, if we set $a=y$ then (29) leads to contradiction.

We consider $r>1$. Suppose that $z_r * L\subset V$ in (28). Then $z_r\in\mathcal{D}_{m^2}(K[G])$ and the result follows. So, we assume that $e_0=z_r * y\notin V$ for some $y\in L$. By analogy with the preceding argument we choose an ordered basis for L. We set $z_j * y=\alpha_j e_0+w_j$, $j=1,\dots,r-1$, $\alpha_j\in K$ and each w_j being a linear combination of the basis elements of L except e_0. By setting $a=y$ in (28) we obtain

$$e_0(\alpha_1 f_1+\dots+\alpha_{r-1}f_{r-1}+f_r)+w_1 f_1+\dots+w_{r-1}f_{r-1}=0. \tag{30}$$

Let us set $f=\alpha_1 f_1+\dots+\alpha_{r-1}f_{r-1}+f_r$. Suppose that $f\neq 0$. By analogy with the preceding theorem the first summand in (30) has degree m and is written in the standard basis as

$$e_0 f=\sum_j \alpha_j e_0 v_{j_1}\cdots v_{j_{m-1}},\quad \alpha_j\in K.$$

Other nonzero summands in (30), being written in the standard basis, have no e_0 as their factor. This is a contradiction. Therefore, $f=\alpha_1 f_1+\dots+\alpha_{r-1}f_{r-1}+f_r=0$. We express f_r from this relation and substitute into (28):

$$\sum_{i=1}^{r-1}((z_i-\alpha_i z_r) * a)f_i\equiv 0,\quad a\in L;$$
$$f_i\neq 0,\ i=1,\dots,r-1.$$

By the inductive assumption $z_1 - \alpha_1 z_r, \ldots, z_{r-1} - \alpha_{r-1} z_r$ are linearly dependent modulo $\mathcal{D}_{m^2}(K[G])$, therefore $z_1, \ldots, z_r$ are also linearly dependent modulo this set.

Thus we have proved that $\dim K[G]/\mathcal{D}_{n^2}(K[G]) < n$. By Lemma 6.3, we form a subspace by $J = \langle \mathcal{D}_{n^2}(K[G])\rangle_K = 4^n \cdot \mathcal{D}_{n^2}(K[G])$. Due to Lemma 5.3 J is an ideal and $J \subset \mathcal{D}_{4^n n^2}(K[G])$. Of course, also $\dim K[G]/J < n$. The theorem is proved. ∎

We can also describe the properties of the delta-sets as follows, but in this case it is not possible to evaluate the numbers n_1, n_2, n_3 below.

Corollary 6.6. *Let $R = u(L)\#K[G]$ be a PI-algebra and n be the number fixed above. Then there exist numbers n_1, n_2, n_3 such that*

$$\begin{aligned} &\mathcal{D}_L(G) = \mathcal{D}_{n_1,L}(G), && |G : \mathcal{D}_L(G)| < n;\\ &\mathcal{D}_G(L) = \mathcal{D}_{n_2,G}(L), && \dim L/\mathcal{D}_G(L) < n;\\ &\mathcal{D}_L(K[G]) = \mathcal{D}_{n_3,L}(K[G]), && \dim K[G]/\mathcal{D}_L(K[G]) < n. \end{aligned}$$

Proof. Let us check the first equality. For $T = \mathcal{D}_{n^3}(G)$, we have $|G : T| < n$ (Theorem 6.2). We consider $T_i = (\mathcal{D}_i(G))^{4^n}$, $i \geq n^3$. By Lemma 6.1, we obtain the chain of subgroups and $T_i \subset \mathcal{D}_{4^n i}(G)$ by Lemma 5.2. Since $|G : T_i| < n$, this chain must stabilize. In other two cases similar chains also stabilize by the codimension argument. ∎

Next we are going to use the next result on bilinear maps.

Theorem 6.7. (P. M. Neumann, [1]) *Let $\varphi : U \times V \to W$ be a bilinear map, where U, V, W are vector spaces over a field K. Suppose that $\dim \varphi(u, V) \leq m$ for each $u \in U$ and $\dim \varphi(U, v) \leq l$ for all $v \in V$. Then $\dim\langle\varphi(U, V)\rangle_K \leq ml$.*

The goal of three previous theorems has been to establish the following result.

Theorem 6.8. *Let $R = u(L)\#K[G]$ be a PI-algebra. Then there exist a G-invariant restricted subalgebra of finite codimension $L_0 \subset L$ and a subgroup of finite index $G_0 \subset G$ such that $\dim(\omega K[G_0] * L_0) < \infty$.*

Proof. We apply Theorems 6.2, 6.4, and 6.5. These theorems yield us the subgroup of finite index $G_0 \subset G$, the G-invariant restricted subalgebra L_0 with $\dim L/L_0 < n$, and the ideal $J \triangleleft K[G]$ with $\dim K[G]/J < n$. Also, their elements have finite width with respect to the action of G on L. Namely,

$$\begin{aligned} &\dim((g-1) * L) \leq 4^n n^3, && g \in G_0;\\ &\dim(K[G] * x) \leq p^n 4^n n^5, && x \in L_0;\\ &\dim(z * L) \leq 4^n n^2, && z \in J. \end{aligned} \tag{31}$$

We have $\dim \omega K[G_0]/(J \cap \omega K[G_0]) < n$. Hence, there exist $g_1, \ldots, g_n \in G_0$ such that

$$\omega K[G_0] = \langle (g_1 - 1), \ldots, (g_n - 1)\rangle_K + J \cap \omega K[G_0].$$

Then

$$\omega K[G_0] * L_0 \subset (g_1 - 1) * L_0 + \cdots + (g_n - 1) * L_0 + J * L_0.$$

We use the conditions of finite width (31) and apply Theorem 6.7 to the bilinear mapping $\varphi : J \times L_0 \to L_0$, $\varphi(z,x) = z * x$, $z \in J$, $x \in L_0$. Finally, we have

$$\begin{aligned}\dim(\omega K[G_0] * L_0) &\le \dim(g_1 - 1) * L_0 + \cdots + \dim(g_n - 1) * L_0 + \dim(J * L_0)\\ &\le n \cdot 4^n n^3 + 4^n n^2 \cdot p^n 4^n n^5 < \infty.\end{aligned}$$

■

7. Proof of the main result

If H is a subalgebra in a restricted Lie algebra L then by H_p we denote the p-hull. This is a minimal restricted subalgebra containing H, and in fact $H_p = \langle h^{[p^i]} \mid h \in H, i = 0,1,2,\dots\rangle_K$.

Suppose that Z is a finite-dimensional abelian p-algebra. Then its structure is determined by the p-mapping, which satisfies the relation

$$(\lambda_1 x_1 + \lambda_2 x_2)^{[p]} = \lambda_1^p x_1^{[p]} + \lambda_2^p x_2^{[p]}, \quad x_1, x_2 \in Z;\ \lambda_1, \lambda_2 \in K.$$

For shortness we often write x^p instead of $x^{[p]}$. By $\mathcal{N}(Z)$ we denote the set of p-nilpotent elements of Z. Over an algebraically closed field there exists the following decomposition

$$Z = \langle e_1, \dots, e_q | e_i^p = e_i \rangle_K \oplus \mathcal{N}(Z). \tag{32}$$

Suppose that a group G acts on a space V and W is a G-invariant subspace. We set

$$\begin{aligned}V^G &= \{v \in V | g * v = v,\ g \in G\};\\ V^G(\operatorname{mod} W) &= \{v \in V | g * v = v(\operatorname{mod} W),\ g \in G\}.\end{aligned}$$

Proposition 7.1. *Let $R = u(L)\#K[G]$ be a PI-algebra, L has a finite - dimensional central restricted ideal Z with $[L,L] \subset \mathcal{N}(Z)$, and $\omega K[G] * L \subset Z$. Then there exists a subgroup of finite index $B \subset G$ such that*
$$\dim L/(L^B(\operatorname{mod}\mathcal{N}(Z))) < \infty.$$

Proof. We suppose that the ground field K is algebraically closed. Let us factor out $\mathcal{N}(Z)$ and for simplicity keep the same notations. By (32) we have

$$Z = \langle e_1, \dots, e_q | e_i^p = e_i \rangle_K. \tag{33}$$

Recall that x is called a *p-element* if $x^p = x$. An easy check shows that there are only finitely many p-elements in (33), namely $\{n_1 e_1 + \cdots + n_q e_q | n_i \in \{0,1,\dots,p-1\}\}$. Then $\operatorname{Aut}(Z) \subset S_{p^q}$. Hence, there exists a subgroup of finite index $B \subset G$ acting trivially on Z.

Let us prove by induction on $q = \dim Z$ that $\dim L/L^B < \infty$. Let $q = 1$, then $Z = \langle e_0 \rangle_K$ where $e_0^p = e_0$. We have

$$g * x = x + \beta(g,x) e_0, \quad \beta(g,x) \in K,\ g \in B, x \in L.$$

We observe that $\beta : B \times L \to (K,+)$ is the mapping into the additive group of the field, which is K-linear by the second argument. We set $L_1 = L$. If

$\beta(B,L_1)\neq 0$ then there exist $g_1\in B$, $x_1\in L_1$ such that $\beta(g_1,x_1)=1$. We consider $L_2=\{x\in L_1|\beta(g_1,x)=0\}$, then $\dim L_1/L_2=1$. If again $\beta(B,L_2)\neq 0$ then there exist $g_2\in B$, $x_2\in L_2$ such that $\beta(g_2,x_2)=1$. Then we define $L_3=\{x\in L_2|\beta(g_2,x)=0\}$ and $\dim L_2/L_3=1$, etc. Suppose that we can make n steps. Then we have the elements $g_1,\dots,g_n\in B$; $x_1\in L_1,\ \dots,\ x_n\in L_n$ such that

$$(g_i-1)*x_i=e_0,\ i=1,\dots,n;\quad (g_i-1)*x_j=0,\ 1\le i<j\le n. \tag{34}$$

We apply identity of Lemma 4.2

$$F_2(x_1,\dots,x_n,g_1,\dots,g_n)=\sum_{\pi\in S_n}\alpha_\pi(g_1-1)*x_{\pi(1)}\cdots(g_n-1)*x_{\pi(n)}=e_0^n\equiv 0, \tag{35}$$

because by (34) the only nontrivial term is given by the identity permutation.

Of course, (35) is a contradiction. This contradiction proves that we cannot make n steps. Therefore, for some $i\in\{1,\dots,n\}$ we have $\beta(B,L_i)=0$. We remark that $L_i=L^B$ and $\dim L/L^B<n$. Since B acts trivially on Z, we have $Z\subset L^B$; also we observe that L^B is a restricted subalgebra.

Now suppose that $\dim Z=q>1$. Set $D=\langle e_2,\dots,e_q\rangle_K$. We consider $\widetilde{L}=L/D$, $\widetilde{Z}=Z/D$. We set $L_1=L^B(\operatorname{mod} D)$, by the inductive assumption for $q=1$, we have $\dim L/L_1<\infty$. Hence, $\omega K[B]*L_1\subset D$. We apply inductive assumption for $D\subset L_1$, where $\dim D=q-1$, and obtain that $\dim L_1/L_1^B<\infty$. It remains to remark that $L_1^B=L^B$. Now we have proved that if K is algebraically closed then there exist the subgroup of finite index $B\subset G$ and the number l such that $\dim L/(L^B(\operatorname{mod}\mathcal{N}(Z)))<l$.

Now we consider the case of an arbitrary field K. Note that we can extend K by adjoining finitely many roots of p-polynomials and obtain the decomposition (32) (see [6]). Let $\bar K\supset K$ be such an extension and $\dim_K\bar K=m$. Given a K-space V we set $\bar V=\bar K\otimes_K V$. Then $\dim_K\bar L/(\bar L^B(\operatorname{mod}\mathcal{N}(\bar Z)))<t=lm$. Pick arbitrary $x_1,\dots,x_t\in L$. Then there exist $\alpha_1,\dots,\alpha_t\in K$ such that $\alpha_1x_1+\cdots+\alpha_tx_t\in\bar L^B(\operatorname{mod}\mathcal{N}(\bar Z))$. This means that for any $g\in B$ we have

$$(g-1)*(\alpha_1x_1+\cdots+\alpha_tx_t)\in\mathcal{N}(\bar Z)\cap L\subset\mathcal{N}(Z).$$

This relation reads also as $\dim_K L/(L^B(\operatorname{mod}\mathcal{N}(Z)))<t$. This concludes the proof. ∎

Now we come back to the proof Theorem 3.1.

Proof. The sufficiency was proved above. Now suppose that $u(L)\#K[G]$ satisfies a nontrivial polynomial identity.

First, let us prove that there exist G-invariant restricted subalgebras $Q_*\subset H_*\subset L$ satisfying conditions 1) of Theorem. We recall the steps of the proof of Theorem 1.3 in [15] (see also this construction in [16] and [3]).

1) The existence of a nontrivial identity in $u(L)$ implies that for some number m we have $\delta(L)=\delta_m(L)$ and $\dim L/\delta(L)<\infty$. We set $D=\delta(L)$.

2) We apply Theorem 6.7 on bilinear maps and conclude that the commutator subalgebra $D^2=[D,D]$ is finite dimensional. We set

$$C = C_D(D^2) = \{x \in D | [x, D^2] = 0\}.$$

Then $\dim D/C < \infty$, and $C^3 = 0$.

3) We have $\dim C^2 < \infty$ and $(C^2)_p$ is a finitely generated abelian p-algebra. By the structure of such algebras $Q_* = \mathcal{N}((C^2)_p)$ is finite dimensional. Then the proof shows that $H_* = C(C, Q_*) = \{x \in C | [x, C] \subset Q_*\}$ has finite codimension in C.

We trace these steps and see by Lemma 2.4 that D, C, Q_*, and H_* are restricted invariant subalgebras. Hence, we obtain the required G-invariant restricted subalgebras $Q_* \subset H_* \subset L$, where H_* is nilpotent of step 2.

Second, we apply results of Section 6.. By Theorem 6.8 there exist the G-invariant restricted subalgebra of finite codimension $L_0 \subset L$ and the subgroup of finite index $G_0 \subset G$ such that $\dim(\omega K[G_0] * L_0) < \infty$. Without loss of generality we may assume that G_0 to be normal in G. We set $Q_0 = Q_* \cap L_0$, $H_0 = H_* \cap L_0$.

Next we consider $Q_1 = Q_0 + \omega K[G_0] * H_0 \subset H_0$, by this construction $\dim Q_1 < \infty$. We observe that Q_1 is a subalgebra, since $[H_0, H_0] \subset Q_0$. Let us check that Q_1 is restricted. By the axioms of the p-map [6]

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y), \quad x, y \in H_0, \tag{36}$$

where each $s_i(x, y)$ is the linear span of commutators in x, y of length p. Since H_0 is nilpotent of step 2, all $s_i(x, y)$ are equal to zero in the case $p > 2$ or belong to Q_0 for $p = 2$. We compute

$$\begin{aligned} ((g-1) * x)^{[p]} &= (g * x - x)^{[p]} = (g * x)^{[p]} - x^{[p]} \\ &= g * (x^{[p]}) - x^{[p]} = (g-1) * (x^{[p]}), \quad g \in G_0, \quad x \in H_0, \end{aligned}$$

where in the case $p = 2$ we have some additional summands from Q_0. For arbitrary $x \in Q_1$ we treat x^p again by (36) and conclude that Q_1 is restricted. We also easily observe that Q_1 is G-invariant because H_0, Q_0 are G-invariant and G_0 is a normal subgroup of G in which case $G \cdot \omega K[G_0] = \omega K[G_0] \cdot G$.

We set $H_1 = C_{H_0}(Q_1) = \{x \in H_0 | [x, Q_1] = 0\}$. Then H_1 is G-invariant, $\dim H_0/H_1 < \infty$, Q_1 is central in H_1, $[H_1, H_1] \subset Q_0 \subset \mathcal{N}(Q_1)$, and $\omega K[G_0] * H_1 \subset Q_1$. Now we can apply Proposition 7.1 to the smash product $K[G_0] \# H_1$ where $Z = Q_1$. We obtain the subgroup of finite index $G_1 \subset G_0$ and the subalgebras

$$Q = \mathcal{N}(Q_1), \quad H = H_1^{G_1} (\operatorname{mod} Q), \tag{37}$$

where $\dim H_1/H < \infty$. The proof of Proposition 7.1 allows us to assume that G_1 is normal in G. The construction (37) shows that Q, H are G-invariant and that G_1 acts trivially on H/Q.

Finally, we use Theorem 1.1 and find the subgroup $A \subset G_1$ such that the conditions 2) of Theorem are satisfied. ∎

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Yu. Bahturin
Department of Mathematics
and Statistics
Memorial University of Newfoundland
St. John's, NF
A1C 5S7 Canada
bahturin@mun.ca

V. M. Petrogradsky
Faculty of Mathematics
Ulyanovsk State University
Lev Tolstoy 42
432700 Ulyanovsk
Russia
vmp@mmf.ulsu.ru

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