Journal of Lie Theory
Volume **12** (2002) 539–549

Moment Polytopes of Projective G-Varieties and Tensor Products of Symmetric Group Representations

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Communicated by J. Hilgert

Abstract. We present a new description of the moment polytope associated with a complex projective variety acted on by a reductive group. We apply this to give a short proof of certain inequalities due to Manivel and Strassen concerning the decomposition of (inner) tensor products of irreducible representations of the symmetric group, and to exhibit, in a concrete example, a complete system of inequalities.

1. Introduction

Let G be a connected reductive linear algebraic group over the complex numbers, V a finite-dimensional rational G-module, and X an irreducible closed G-stable subvariety of the projective space $\mathbf{P}V$. We are interested in the G-module structure of the homogeneous coordinate ring $\mathbf{C}[X]$, as encoded in the moment polytope of X.

Choose a Borel subgroup B of G and a maximal torus $T \subseteq B$, and denote by $\mathfrak{X}(T)$ its character group. Let D be the (rational) positive Weyl chamber in $E = \mathfrak{X}(T) \otimes_{\mathbf{Z}} \mathbf{Q}$ with respect to B. For each $\chi \in \mathfrak{X}(T) \cap D$ we choose a simple G-module $V_{(\chi)}$ with highest weight χ. For rational G-modules V_1, V_2 we write V_1^* for the dual of V_1, and $V_1 < V_2$ to mean that some G-module isomorphic to V_1 is a submodule of V_2.

The *moment polytope* of the projective G-variety X is defined by

$$P_G(X) = \left\{ \frac{\chi}{n} : n \geq 1, V_{(\chi)}^* < \mathbf{C}[X]_n \right\} \subseteq D.$$

It is indeed a polytope (see Mumford [9], Brion [3], or the theorem below). Note that $P_G(X)$, like $\mathbf{C}[X]$, depends on the embedding of X in projective space. The description of moment polytopes by linear inequalities has been a subject of recent research, for example by Sjamaar [11], Brion [4], and Berenstein–Sjamaar [1].

Let $\Gamma(V)$ denote the set of weights of V with respect to T. We write $v = \bigoplus_{\gamma \in \Gamma(V)} v_\gamma$ for the decomposition of $v \in V$ into weight vectors. The *sup-*

ISSN 0949–5932 / $2.50

port $\operatorname{supp} x$ of a point $x = [v] \in X$ is the set of all $\gamma \in \Gamma(V)$ such that $v_\gamma \neq 0$, and $\operatorname{conv}(\operatorname{supp} x)$ is its rational convex hull in E. Set

$$P(x) = \bigcap_{u \in U^-} \operatorname{conv}(\operatorname{supp} ux),$$

where U^- denotes the unipotent radical of the Borel subgroup B^- opposite to B. Note that $P(x)$ is a polytope and that there are only finitely many of them as x ranges over X, simply because $\Gamma(V)$ and hence also the number of possible supports is finite.

Theorem 1.1. *For all* $x \in X$*,*

$$P_G(X) \supseteq P(x) \cap D,$$

with equality for all x *in a non-empty Zariski open subset of* X*.*

The proof of the theorem appears in the next section. In the subsequent sections we demonstrate its usefulness by applying it to the decomposition of tensor products of irreducible representations of symmetric groups. Manivel [8] and Strassen (unpublished) have proven inequalities imposing restrictions on the possible decompositions. Explicitly, they studied for $m \in \mathbf{N}^k$ linear inequalities satisfied by the set

$$P(m) = \left\{ \left(\frac{\pi_1}{n}, \dots, \frac{\pi_k}{n} \right) : n \geq 1, \pi_j \vdash n, l(\pi_j) \leq m_j, [\pi_1] < [\pi_2] \otimes \cdots \otimes [\pi_k] \right\},$$

where $\pi \vdash n$ denotes a partition of n with length $l(\pi)$ and corresponding isomorphism class $[\pi]$ of simple S_n-modules. $P(m)$ is in fact a moment polytope, which enables us to employ the above theorem. That does not only lead to a simple and transparent proof of Manivel's and Strassen's inequalities, but also opens up the way to further ones. We show this by giving a complete system of inequalities defining $P(3,3,3)$.

Acknowledgements. The author thanks Volker Strassen for introducing him to this problem, for access to his unpublished notes, and for helpful discussions, furthermore him, Annette A'Campo-Neuen, Jürgen Hausen, and the referee for their valuable comments on earlier versions of this paper. Finally, use of the beautiful LiE program [5] is gratefully acknowledged.

2. The moment polytope

Recall that a vector $0 \neq v \in V$ or a point $[v] \in \mathbf{P}V$ is called *unstable* (with respect to G) if 0 lies in the closure of the orbit Gv, and *semistable* otherwise. The unstable vectors in V together with 0 are precisely the elements of the *null cone* of V, *i.e.*, the subvariety defined by all non-constant homogeneous G-invariant polynomials on V.

The following lemma appears more or less explicit in [10], Lemma 3.8, and in [12], § 2.

Lemma 2.1. *A point* $x \in X$ *is semistable if and only if* $0 \in P(x)$*.*

Proof. We denote the set of one-parameter subgroups of an algebraic group H by $\mathfrak{Y}(H)$. According to the Hilbert–Mumford criterion, x is unstable with respect to G if and only if it is unstable with respect to some $\lambda \in \mathfrak{Y}(G)$. We may even assume $\lambda = g^{-1}\mu g$ for some $\mu \in \mathfrak{Y}(T)$ and some $g \in U^-$. Let us recall the argument: The intersection of B^- with the parabolic subgroup

$$P(\lambda) = \{\, g \in G : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G\}$$

contains a maximal torus T' of both groups (*cf.* [2], Cor. 14.13 or [13], Cor. 8.3.10). Since any two maximal tori of a linear algebraic group are conjugated, it follows that λ is conjugated in $P(\lambda)$ to some $\nu \in \mathfrak{Y}(T')$, and ν to some $\mu \in \mathfrak{Y}(T)$ in $B^- = U^- \rtimes T$, hence even by some $g \in U^-$. This proves the claim because x is also unstable with respect to ν, as one readily verifies.

The lemma now follows from the fact that gx is semistable with respect to all $\lambda \in \mathfrak{Y}(T)$ if and only if $0 \in \operatorname{conv}(\operatorname{supp} gx)$. ■

Proof of Theorem 1.1. Let $n \geq 1$ and $\chi \in \mathfrak{X}(T) \cap D$. Denote by x^- the point in $\mathbf{P}V^*_{(\chi)}$ with $\operatorname{supp} x^- = \{-\chi\}$, and by C_χ its (closed) G-orbit. As observed by Mumford [9], $\gamma = \chi/n \in P_G(X)$ if and only if $Y = \varphi_n(X) \times C_\chi \subseteq \mathbf{P}(\operatorname{Sym}^n V \otimes V^*_{(\chi)})$ has semistable points, where $\varphi_n \colon \mathbf{P}V \hookrightarrow \mathbf{P}\operatorname{Sym}^n V$ is the Veronese embedding.

Let S_γ be the set of all $x \in X$ such that $(\varphi(x), x^-) \in Y$ is semistable. The null cone of a G-module is Zariski closed, so S_γ is Zariski open, and only empty if Y has no semistable points. Hence, if we look for semistable points of Y, it suffices to test those of the form $(\varphi_n(x), x^-)$. Any $u \in U^-$ fixes x^-, and

$$\begin{aligned}\operatorname{conv}\bigl(\operatorname{supp} u(\varphi_n(x), x^-)\bigr) &= \operatorname{conv}\bigl(\operatorname{supp}\,(u\varphi_n(x), x^-)\bigr)\\ &= \operatorname{conv}\bigl(\operatorname{supp} \varphi_n(ux)\bigr) - \chi = n \operatorname{conv}(\operatorname{supp} ux) - \chi.\end{aligned}$$

Therefore, the preceding lemma implies that $(\varphi_n(x), x^-)$ is semistable if and only if $\gamma \in P(x)$, whence

$$P_G(X) = \bigcup_{x\in X} P(x) \cap D.$$

If $\gamma_1, \dots, \gamma_k$ are the vertices of the finite number of polytopes appearing in this union, then for all $x \in S_{\gamma_1} \cap \cdots \cap S_{\gamma_k}$ they are all contained in the polytope $P(x) \cap D$, and finally $P_G(X)$ as well. ■

Though in principle the theorem gives a complete description of the moment polytope, we will use it only as an outer approximation because it is very difficult to determine the support of ux for all $u \in U^-$. Typically, the argument goes as follows: One shows that for all x in a dense subset of X there is a $u \in U^-$ such that $\operatorname{supp} ux \subseteq S$ for a fixed S; then the theorem implies $P_G(X) \subseteq \operatorname{conv}(S) \cap D$.

We finally formulate an inner approximation of the moment polytope which also appears in [11], Lemma 7.1; it will be used in Section 5. Call a subset $S \subseteq E$ *free* if no two elements of S differ by a root of G (with respect to T).

Proposition 2.2. *If $S \subseteq \Gamma(V)$ is free, then* $\operatorname{conv}(S) \cap D \subseteq P_G(\mathbf{P}V)$.

Proof. We begin with a general remark about the moment polytope of a projective G-variety $X \subseteq \mathbf{P}V$: Let K be a compact form of G such that $T \cap K$ is a maximal torus of K, and denote the Lie algebras of G, T, and K by $\mathfrak{g}$, $\mathfrak{t}$, and $\mathfrak{k}$, respectively. Equip V with a K-invariant Hermitian scalar product. Then, as shown in [9], the moment polytope $P_G(X)$ is the intersection of D with the image of the moment map[1]

$$\mu\colon X \to i\mathfrak{k}^*, \qquad \mu([v])(\xi) = \frac{\langle \xi v, v\rangle}{\langle v, v\rangle}$$

for $[v] \in X$ and $\xi \in \mathfrak{g}$. Here $i\mathfrak{k}^* \subseteq \mathfrak{g}^*$ denotes those $\mathbf{C}$-linear forms on $\mathfrak{g}$ which assume purely imaginary values on $\mathfrak{k}$, and E is considered as a rational subspace of $\mathfrak{t}^* \cap i\mathfrak{k}^*$ by identifying characters with their differentials.

Now let $\chi = \sum_{\gamma\in S} t_\gamma \gamma \in \operatorname{conv}(S)$ be some convex combination. Choose $v \in V$ such that $\|v_\gamma\|^2 = t_\gamma$ for all $\gamma \in S$, and 0 otherwise. We claim that $[v]$ is a preimage of χ under the moment map of $\mathbf{P}V$: For $\xi \in \mathfrak{t}$,

$$\mu([v])(\xi) = \sum_{\gamma,\gamma'} \langle \xi v_\gamma, v_{\gamma'}\rangle = \sum_{\gamma} \gamma(\xi)\|v_\gamma\|^2 = \sum_{\gamma} \gamma(\xi) t_\gamma = \chi(\xi),$$

and for any root α of $\mathfrak{g}$ we have $S \cap (S+\alpha) = \emptyset$ by hypothesis, hence for all $\xi \in \mathfrak{g}_\alpha$

$$\mu([v])(\xi) = \sum_{\gamma,\gamma'} \langle \xi v_\gamma, v_{\gamma'}\rangle = 0,$$

which proves the claim. ∎

3. The symmetric group

Let π be a partition, *i.e.*, a non-increasing finite sequence of natural numbers. We write $\pi \vdash n$ if π partitions the number $n \in \mathbf{N}$, and $l(\pi)$ for the number of non-zero elements of π. We will consider π as a vector in $\mathbf{Q}^m$ for all $m \geq l(\pi)$. If $\pi \vdash n$, we denote by $[\pi]$ the module of the symmetric group S_n determined by π, and by V^π the $GL(V)$-module determined by π if V is a complex vector space of finite dimension not less than $l(\pi)$. (More precisely, $[\pi]$ and V^π denote isomorphism classes of such modules.) For example, if $\pi = (n)$, then $V^\pi = \operatorname{Sym}^n V$ (for $V \neq 0$), and $[\pi]$ is the one-dimensional trivial S_n-module.

For the general linear group GL_m we follow the usual conventions: We choose the group T_m of diagonal matrices as maximal torus and the upper triangular matrices as Borel subgroup. U_m^- is the group of lower diagonal matrices with ones on the diagonal. The weights of the standard action of T_m on $\mathbf{C}^m$ are denoted by $\varepsilon_1, \ldots, \varepsilon_m$. We identify the space E with $\mathbf{Q}^m$, hence the Weyl chamber consists exactly of those vectors whose coordinates form a non-increasing sequence. In addition, we denote by $E(i', i, c) \in U_m^-$ the elementary matrix which differs from the identity only by the entry $c \in \mathbf{C}$ at position (i', i), $i' > i$.

[1]For a true moment map for the K-action on $\mathbf{P}V$ in the sense of symplectic geometry, μ must be scaled as to map to $\mathfrak{k}$, the exact factor depending on the symplectic form chosen for $\mathbf{P}V$.

We want to describe the decomposition of tensor products $[\pi_1]\otimes\cdots\otimes[\pi_k]$ of S_n-modules into simple modules in a way which makes the dependence on the factors $[\pi_j]$ as clear as possible. To make life easier, we will neglect multiplicities and only ask which simple modules appear in such tensor products. Of course, it suffices to describe whether the trivial module appears or not.

In order to bring the representation theory of the general linear group into play, and in particular our theorem, we content ourselves with the study of the following set, which gives an "asymptotic" description of the possible decompositions: For $m \in \mathbf{N}^k$ define the subset

$$P(m) = \left\{ \left(\frac{\pi_1}{n}, \dots, \frac{\pi_k}{n}\right) : n \geq 1, \pi_j \vdash n, l(\pi_j) \leq m_j, [n] < [\pi_1]\otimes\cdots\otimes[\pi_k] \right\}$$

of $E(m) = \mathbf{Q}^{m_1}\oplus\cdots\oplus\mathbf{Q}^{m_k}$. (This is equivalent to the definition given in the introduction since S_n-modules are isomorphic to their duals.) Note the symmetry of $P(m)$, which is due to the commutativity of tensor products: If, say, $m_1 = m_2$, then $P(m)$ is stable under the transposition of the first two "components" of elements of $E(m)$. If all elements of m are equal, then $P(m)$ is stable with respect to the canonical action of S_k on the ambient space.

Proposition 3.1. *Let $m \in \mathbf{N}^k$. Then*

$$P(m) = P_{GL_{m_1}\times\cdots\times GL_{m_k}}\big(\mathbf{P}(\mathbf{C}^{m_1}\otimes\cdots\otimes\mathbf{C}^{m_k})\big).$$

In particular, $P(m)$ is a rational polytope.

Proof. This is a trivial corollary of the following well-known fact, which is itself an easy consequence of the Schur correspondence between representations of the symmetric and those of the general linear groups (a good reference for this is [7], in particular Lecture 6):

Let $V_1, \dots, V_k$ be finite dimensional complex vector spaces and $\pi_1, \dots, \pi_k$ partitions of $n \in \mathbf{N}$ with $l(\pi_i) \leq \dim V_i$ for all i. Then the multiplicity of the isomorphism class $V_1^{\pi_1}\otimes\cdots\otimes V_k^{\pi_k}$ in $\mathrm{Sym}^n(V_1\otimes\cdots\otimes V_k)$, considered as a $GL(V_1)\times\cdots\times GL(V_k)$-module, equals that of the trivial representation of S_n in $[\pi_1]\otimes\cdots\otimes[\pi_k]$. Furthermore, all simple summands of the symmetric power are of this type. ∎

Hence, we are led to study the moment polytope of the G-variety $\mathbf{P}V$, where $V = \mathbf{C}^{m_1}\otimes\cdots\otimes\mathbf{C}^{m_k}$ and $G = GL_{m_1}\times\cdots\times GL_{m_k}$. The canonical basis of V yields a decomposition into weight spaces

$$V_{(\varepsilon_{i_1},\dots,\varepsilon_{i_k})} = \mathbf{C}\, e_{i_1}\otimes\cdots\otimes e_{i_k};$$

we will think of these basis vectors as the integral points in a hypercube (with different edge lengths), and of $v \in V$ as a $\mathbf{C}$-valued function on these points. By the i-th *slice* in direction j we mean the collection of the points with j-th coordinate equal to i, and by the *column* over a point in direction j those points whose j-th coordinate is greater than that of the given point, all other coordinates being equal. The action of an element $E_j(i', i, c) \in U^-_{m_j} \subseteq U^-$ on v consists in adding the c-fold values of the i-th slice in direction j to the corresponding values of the i'-th slice, $i' > i$, leaving all others constant.

Figure 1: $P(4,4)$: Support of a generic vector (a) after 6 moves, (b) at the end.

We now want to describe a method to determine sets to which the support of a generic element of V can be reduced. Let $v \in V$. Generically (*i.e.*, for all v in a Zariski open subset), all coordinates v_I of v are non-zero. If I and I' are two positions with I' in the column over I, say, in direction j, we can therefore clear the entry at I' by applying the element $E_j(I'_j, I_j, -v_{I'}/v_I) \in U^-_{m_j} \subseteq U^-$ to v. Generically, all other coordinates are still non-zero, and we can repeat this for new I and I', unless we are certain that we have already cleared the I-th coordinate. Note that, if the slice through I' in direction j contains cleared coordinates, another elimination step as described above will generically turn them non-zero again, unless the corresponding coordinates in the slice through I have been cleared, too. Dropping the word "generically", we arrive at the following "algorithm":

Method 3.2. We consider a function on the integral points $\{1,\ldots,m_1\}\times\cdots\times\{1,\ldots,m_k\}$ in a hypercube; each entry may assume the values "zero" or "non-zero". Initially, all entries are non-zero. We are allowed to make an arbitrary, but finite number of "moves" to eliminate entries, *i.e.*, to make them zero, at the cost of turning other, already eliminated entries back to the non-zero state. More precisely, if the entries at positions I and I' are non-zero, where I' is in the column over I in direction j, one may change the I'-the entry to zero, but all zero entries of the slice through I' in direction j return to non-zero, unless the corresponding entries in the slice through I are zero as well.

By Theorem 1.1 and the motivation given above, we conclude that after each move the convex hull of the support of the filled boxes is an outer approximation of the moment polytope, as is the intersection of the convex hulls of all (with respect to inclusion) minimal supports which can be obtained this way.

Example 3.3. Let us see how this method works for $P(m_1, m_2)$, $m_1 \leq m_2$: Use the first slice in the first direction (*i.e.*, the first column) to clear all other entries in the first row. Since the entry $(1,1)$ is still non-zero, we may now take the first slice in the second direction (*i.e.*, the first row) to clear all other entries in the first column. *Cf.* Figure 1 (a) for the remaining support. Next we take the second column resp. row and clear the same way all boxes in them but the first two. Previously cleared boxes will never turn back to the non-zero state, because the first box in the second row resp. column is cleared. If we go on like this, the support will be reduced to the "diagonal", as shown in Figure 1 (b).

From this we conclude (with the canonical identification $\mathbf{Q}^{m_1} \subseteq \mathbf{Q}^{m_2}$)

$$\begin{aligned} P(m_1, m_2) &\subseteq \operatorname{conv}\{(\varepsilon_i; \varepsilon_i) : 1 \le i \le m_1\} \cap D \\ &= \{(\chi; \chi) : \chi \in \mathbf{Q}^{m_1},\ \chi_1 \ge \cdots \ge \chi_{m_1} \ge 0,\ \chi_1 + \cdots + \chi_{m_1} = 1\}. \end{aligned}$$

In fact, we have equality, as follows directly from the definition of $P(m_1, m_2)$ (and the selfduality of representations of symmetric groups).

4. Inequalities

As an application of the method described in the previous section, we derive some inequalities proven by Strassen using a different approach (unpublished).

Proposition 4.1. *Let $m \in \mathbf{N}^k$, and choose vectors $a^j \in \mathbf{Q}^{m_j}$, $2 \le j \le k$. Let $b \in \mathbf{Q}^{m_1}$ be the vector of the m_1 largest elements, in descending order, of the set $\{\sum_j a^j_{i_j} : i_1, \dots, i_k\}$, counted with multiplicities. Then, for every $(\chi^1; \dots; \chi^k)$ in the polytope $P(m)$,*

$$\langle b, \chi^1 \rangle \ge \sum_{j=2}^{k} \langle a^k, \chi^k \rangle . \qquad (*)$$

In fact, one may always assume that the components of the vectors a^j are non-increasing, as are those of b, because these inequalities imply all others. The preference of the first component of m is of course arbitrary; this is just to avoid a too clumsy notation.

Proof. Without loss of generality, we may assume that all sums $\sum_j a^j_{i_j}$ are distinct. This assumption simplifies the exposition of our proof.

Let $I(i) \in \mathbf{N}^k$ denote the unique index such that $b_i = \sum_j a^j_{I(i)_j}$. Obviously, all weights but those at positions $(i_1, I(i))$ with $i_1 > i$ satisfy inequality $(*)$. Hence, in terms of our method, we have to show that we can clear all entries with these coordinates. But this is simple: Start by eliminating the column over $(1, I(1))$ in direction 1 with the slice $i_1 = 1$. If you do the same for the column over $(2, I(2))$ with the slice $i_1 = 2$, the previously cleared entries remain untouched. Now continue until all undue boxes are cleared. ■

Note that, while proving a single inequality, we have always eliminated entries along a fixed direction (as the proposition is stated, the first). That is, we have only used some subgroup $U^-_{m_j}$ of the whole unipotent radical of G. This hints to the fact that Strassen's inequalities are far from being complete; we will give an example of this in the following section.

As a special case, one may choose an $r \in \mathbf{N}^k$ and set

$$a^j_i = \begin{cases} 1 & i \le r_j, \\ 0 & i > r_j. \end{cases} \qquad (**)$$

The corresponding inequalities are due to Manivel [8], Proposition 3.5. They are strictly weaker than Strassen's: The point $(1/3, 1/3, 1/6, 1/6; 2/3, 1/3; 2/3, 1/3, 0)$ for instance satisfies all inequalities of the form $(**)$ for $P(4, 2, 3)$, but violates $(*)$ for $a^2 = (1, 0)$ and $a^3 = (2, 1, 0)$.

5. An example

In this section we take a closer look at $P(3,3,3)$. Similar to the previous sections, we will write an element of $E(3,3,3) = \mathbf{Q}^3 \oplus \mathbf{Q}^3 \oplus \mathbf{Q}^3$ as a triple $\chi = (\chi^1;\chi^2;\chi^3)$. Recall from Section 3 that $P(3,3,3)$ is invariant under the action of S_3 on $E(3,3,3)$ which permutes χ^1, χ^2, and χ^3.

Proposition 5.1. *The polytope $P(3,3,3)$ is the convex hull of the points*

$$\begin{aligned}
\gamma_1 &= (1/3,1/3,1/3;\ 1/3,1/3,1/3;\ 1/3,1/3,1/3),\\
\gamma_2 &= (1/3,1/3,1/3;\ 1/3,1/3,1/3;\ 1/2,1/2,\ 0),\\
\gamma_3 &= (1/3,1/3,1/3;\ 1/3,1/3,1/3;\ 1,\ 0,\ 0),\\
\gamma_4 &= (1/3,1/3,1/3;\ 1/2,1/2,\ 0;\ 1/2,1/2,\ 0),\\
\gamma_5 &= (1/3,1/3,1/3;\ 1/2,1/2,\ 0;\ 2/3,1/6,1/6),\\
\gamma_6 &= (1/3,1/3,1/3;\ 2/3,1/3,\ 0;\ 2/3,1/3,\ 0),\\
\gamma_7 &= (1/2,1/4,1/4;\ 1/2,1/2,\ 0;\ 3/4,1/4,\ 0),\\
\gamma_8 &= (1/2,1/2,\ 0;\ 1/2,1/2,\ 0;\ 1/2,1/2,\ 0),\\
\gamma_9 &= (1/2,1/2,\ 0;\ 1/2,1/2,\ 0;\ 1,\ 0,\ 0),\\
\gamma_{10} &= (1/2,1/2,\ 0;\ 2/3,1/6,1/6;\ 2/3,1/6,1/6),\\
\gamma_{11} &= (\ 1,\ 0,\ 0;\ 1,\ 0,\ 0;\ 1,\ 0,\ 0),
\end{aligned}$$

and their images under the S_3-action. Dually, $P(3,3,3)$ is the intersection of the Weyl chamber of $GL(3) \times GL(3) \times GL(3)$ with the subspace determined by the equalities $\chi^j_1 + \chi^j_2 + \chi^j_3 = 1$, $j \in \{1,2,3\}$, the halfspaces determined by $\chi^j_3 \geq 0$, $j \in \{1,2,3\}$, and the halfspaces determined by the inequalities $\langle h, \chi\rangle \leq 0$, where h runs through the vectors

$$\begin{aligned}
h_1 &= (1,0,0;\ 1,0,0;\ -2,-1,-1),\\
h_2 &= (1,1,0;\ 1,0,0;\ -2,-2,-1),\\
h_3 &= (0,1,0;\ 1,0,0;\ -1,-2,-1),\\
h_4 &= (2,0,1;\ 2,0,1;\ -4,-2,-3),\\
h_5 &= (2,0,1;\ 2,1,0;\ -4,-3,-2),\\
h_6 &= (1,2,0;\ 2,0,1;\ -3,-4,-2),\\
h_7 &= (1,2,0;\ 2,1,0;\ -4,-3,-2),
\end{aligned}$$

and their images under S_3. Furthermore, both descriptions are minimal. In all, $P(3,3,3)$ is a 6-dimensional polytope with 33 vertices and 45 facets.

Note that only the first two inequalities are covered by the results of the preceding section.

Proof. That both descriptions are minimal and dual to each other can be checked with any program for convex hull computations, *e.g.*, the "convex" package [6]. Granting this, it suffices to show that $P(3,3,3)$ contains the given points and is itself contained in the second set of halfspaces (the equalities and first inequalities being obvious).

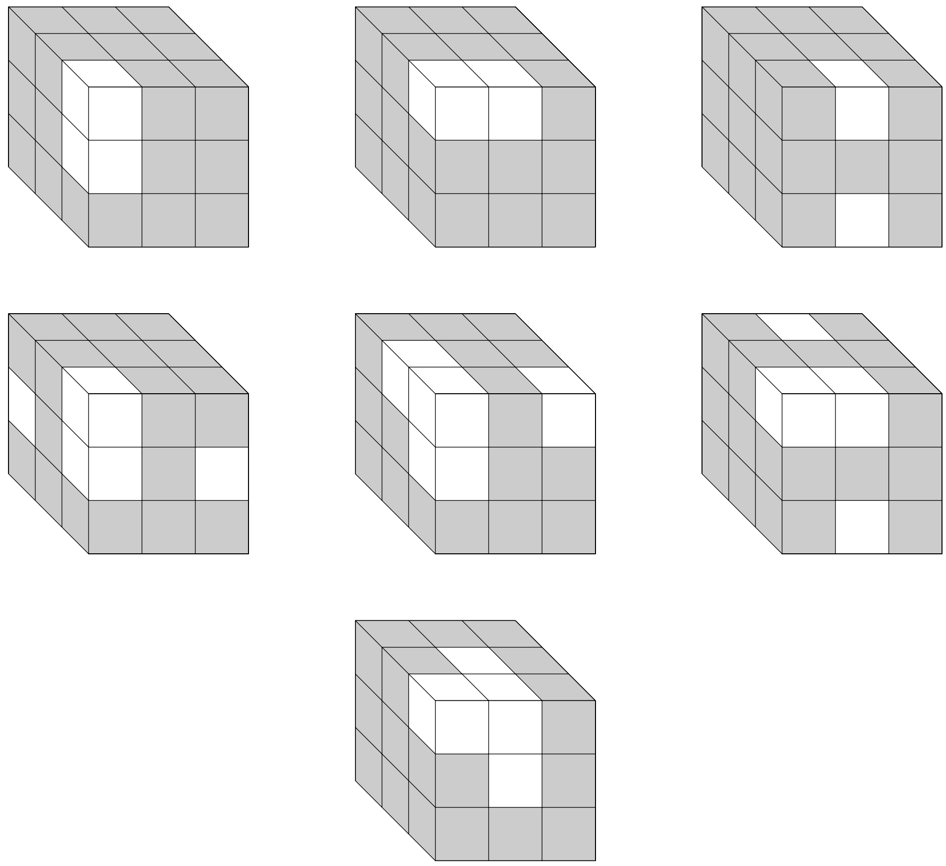

Figure 2: Weights satisfying the inequalities determined by $h_1, \ldots, h_7$

We prove the inequalities by the method described in Section 3. Figure 2 shows, for each inequality, to which weights the support of a generic element of $\mathbf{C}^3 \otimes \mathbf{C}^3 \otimes \mathbf{C}^3$ must be reduced. We describe explicitely how this may be done for the inequality determined by h_7, the others being even simpler:

$$(2,1,1) \to (2,1,2), \qquad (2,1,1) \to (2,1,3), \qquad (1,1,2) \to (1,1,3)$$
$$(1,2,3) \to (2,2,3), \qquad (2,1,1) \to (2,1,2).$$

(Here we have written $I \to I'$ if one clears the box I' with the slice through I in the direction connecting I and I'.)

As for the points, one possibility is to use the definition of $P(3,3,3)$ and to show that some tensor product contains a suitable submodule. This may become quite difficult because the minimal positive integer n for which $n\gamma \in \mathfrak{X}(T)$ does not always do the trick: For example, we have $\gamma_2 \in P(3,3,3)$, though $[3,3]$ is not a submodule of $[2,2,2] \otimes [2,2,2]$, so one has to calculate at least in S_{12}.

Alternatively, one may resort to Proposition 2.2, which reduces the problem to finding suitable free sets of weights. We content ourselves with displaying γ_2 as a convex combination of such a set:

$$\begin{aligned} 6\gamma_2 &= \varepsilon_1 \oplus \varepsilon_1 \oplus \varepsilon_1 + \varepsilon_2 \oplus \varepsilon_2 \oplus \varepsilon_1 + \varepsilon_3 \oplus \varepsilon_3 \oplus \varepsilon_1 \\ &+ \varepsilon_1 \oplus \varepsilon_2 \oplus \varepsilon_2 + \varepsilon_2 \oplus \varepsilon_3 \oplus \varepsilon_2 + \varepsilon_3 \oplus \varepsilon_1 \oplus \varepsilon_2. \end{aligned}$$ ■

Remark 5.2. While proving inequality h_7 by our method we have cleared the entry $(2,1,2)$ in the first step, refilled it in step 4 and cleared it again at the end. Though one can proceed differently, there is no sequence of moves such that the support of the generic vector is non-increasing. This makes searching for all minimal positions algorithmically much more difficult.

Remark 5.3. Note that the vertex $(1,0,0;\,1,0,0;\,1,0,0)$ lies in the interior of the halfspace determined by h_7, though the corresponding facet is not induced by a wall of the Weyl chamber. This shows that the moment polytope $P_G(\mathbf{P}V_{(\chi)})$, $V_{(\chi)}$ a simple G-module, need not be the intersection of the Weyl chamber and the cone with apex χ generated by the moment polytope.

Remark 5.4. We have seen the efficiency of Proposition 2.2 in proving that some point belongs to the moment polytope. Unfortunately, this does not always work: The point $\gamma = (1/2,1/2,0;\,1/2,1/2,0;\,1/2,1/3,1/6)$ lies in $P(3,3,3)$, but not in the convex hull of any free set of weights. The second assertion can be seen as follows: Two weights of $\mathbf{C}^3 \otimes \mathbf{C}^3 \otimes \mathbf{C}^3$ differ by a root if and only if two of their three components are equal. By looking at the first two components of γ we see that any minimal free set S with $\gamma \in \operatorname{conv}(S)$ has at most 4 elements, hence we must have

$$(1/2,1/2,0;\,1/2,1/2,0) = a_{11}\,\varepsilon_1 \oplus \varepsilon_1 + a_{12}\,\varepsilon_1 \oplus \varepsilon_2 + a_{21}\,\varepsilon_2 \oplus \varepsilon_1 + a_{22}\,\varepsilon_2 \oplus \varepsilon_2$$

for some a_{11}, a_{12}, a_{21}, $a_{22} \in [0,1]$. This implies $a_{11} = a_{22}$ and $a_{12} = a_{21}$. But now there is no way to choose the third component of these weights such that the above convex combination adds up to γ while the weights remain free.

Hence, free sets of weights in general do not suffice to determine the whole moment polytope. There are probably other examples where even some vertex fails to be accessible this way.

Further results on moment polytopes of projective spaces of representations can be found in [11], §7.1, and the references given therein.

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Received July 24, 2001
and in final form September 27, 2001