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Complete Filtered Lie Algebras over a Vector Space of Dimension Two

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Abstract. There may exist many non-isomorphic complete filtered Lie algebras with the same graded algebra. In [6], we found elements in the Spencer cohomology that determined all complete filtered Lie algebras having certain graded algebra provided that obstructions do not exist in the cohomology at higher levels. In this paper we use the Spencer cohomology to classify all graded and filtered algebras over a real vector space of dimension two.

1. Introduction

Closed transitive Lie algebras are subalgebras of the Lie algebra $D(\mathbb{K}^n)$ of formal vector fields. If $\mathbb{K}$ a field of characteristic zero, X is a formal vector field in $D(\mathbb{K}^n)$ if

$$X = \sum_i X_i(x_1, \dots , x_n)\frac{\partial}{\partial x_i},$$

where X_i in $\mathbb{K}[[x_1, \dots , x_n]]$. The vector space $D(\mathbb{K}^n)$ is a Lie algebra under the usual bracket operation

$$[X, Y] = \sum_{i,j} \left\{ X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right\} \frac{\partial}{\partial x_j}.$$

If $D^k(\mathbb{K}^n)$ is the set of $X \in D(\mathbb{K}^n)$ such that each X^i has no terms of degree k or less, then $D(\mathbb{K}^n)$ has a natural filtration

$$D(\mathbb{K}^n) \supset D^0(\mathbb{K}^n) \supset D^1(\mathbb{K}^n) \supset D^2(\mathbb{K}^n) \supset \cdots .$$

Guillemin and Sternberg studied local geometries by examining Lie algebras of formal vector fields [3]. More specifically, if we choose a coordinate system and replace each infinitesimal automorphism (which is a vector field) with its Taylor series expansion about the origin, we obtain a subalgebra L of $D(\mathbb{K}^n)$. Letting $L_k = D^k(\mathbb{K}^n) \cap L$, we have

$$L \supset L_0 \supset L_1 \supset L_2 \supset \cdots$$

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with $[L_i, L_j] \subset L_{i+j}$. Guillemin and Sternberg limited their study to transitive geometries. That is, for any two points there exists a local transformation that takes one point to other. In infinitesimal terms, there exists an $X \in L$ such that $X(0) = v$ for each $v \in \mathbb{K}^n$. We also demand that L be closed. If $X \in D(\mathbb{K}^n)$ and there exists an $X_i \in L$ such that X and X_i agree on terms of up to order i for $i = 1, 2, \dots$, then $X \in L$. A subalgebra $L \subset D(\mathbb{K}^n)$ satisfying these properties is a *closed transitive Lie algebra.* Two such algebras are isomorphic when they are equivalent by a formal change of coordinates.

A *complete filtered Lie algebra* over a field $\mathbb{K}$ of characteristic zero is a Lie algebra with a decreasing sequence of subalgebras $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ satisfying the following conditions.

1. $\bigcap_i L_i = 0$.
2. $[L_i, L_j] \subset L_{i+j}$ (by convention $L_{-2} = L$).
3. $\dim L_i/L_{i+1} < \infty$.
4. If $x \in L_i$ for $i \geq 0$ and $[L, x] \subset L_i$, then $x \in L_{i+1}$.
5. Whenever $\{x_i\}$ is a sequence in L such that $x_i - x_{i+1} \in L_i$ for $i \geq 0$, then there exists an $x \in L$ such that $x - x_i \in L_i$.

Every complete filtered Lie algebra is isomorphic to a closed transitive subalgebra of $D(\mathbb{K}^n)$ [3].

A *graded Lie algebra* is a Lie algebra $\prod_{p=-1}^{\infty} G_p$ that satisfies the following conditions.

1. $[G_i, G_j] \subset G_{i+j}$ (by convention $G_{-2} = 0$).
2. $\dim G_i < \infty$.
3. If $x \in G_i$ for $i \geq 0$ and $[G_{-1}, x] = 0$, then $x = 0$.

Any graded Lie algebra is a complete filtered Lie algebra if we let $L_i = G_i \times G_{i+1} \times \cdots$. Conversely, if L is a complete filtered Lie algebra, then the bracket operation on L induces a bracket operation on

$$G_L = \prod_{p=-1}^{\infty} L_p/L_{p+1}.$$

We refer to G_L as the *associated graded algebra* of L. An isomorphism of two complete filtered Lie algebras is a Lie algebra isomorphism preserving the filtration. Similarly, an isomorphism of two graded Lie algebras is a Lie algebra isomorphism preserving the gradation.

There may exist many non-isomorphic complete filtered Lie algebras with the same graded algebra. Given a graded Lie algebra $\prod G_p$, it is an interesting problem to try to reconstruct all complete filtered Lie algebras L whose associated graded algebras are isomorphic to $\prod G_p$. One of the primary tools for analyzing this problem has been the Spencer cohomology. A complete filtered Lie algebra is isomorphic to its graded algebra provided certain cohomology groups vanish [3, 7, 9, 12]. It is more difficult to determine the complete filtered Lie algebras that are not isomorphic to their graded algebras. Many of the known results have hypothesis that are difficult to verify. In [6] we outlined a theory, where certain elements in the Spencer cohomology determine all the complete filtered Lie algebras having a certain graded algebra provided that obstructions do not exist in the cohomology at a higher level. In this paper we use the theory to classify all graded and filtered algebras over a real vector space of dimension two. Cartan first classified these algebras as pseudogroups on $\mathbb{R}^2$ [1].

2. Graded Algebras with $\dim G_{-1} = 2$

If $\prod G_p$ is a graded Lie algebra, then $V = G_0$ is a linear Lie algebra acting faithfully on G_{-1} by $[G_0, G_{-1}] \subset G_{-1}$. For $p \geq 0$, we may consider G_p to be a subspace of $V \otimes S^{p+1}(V^*)$. If $X \in G_p$ and $v_0, \ldots, v_p \in V$, define $\overline{X} \in V \otimes S^{p+1}(V^*)$ by

$$\overline{X}(v_0, \ldots, v_p) = [\cdots[[X, v_0], v_1], \cdots v_p].$$

Since $[G_{-1}, G_{-1}] = 0$, the Jacobi identity implies that $\overline{X}(v_0, \ldots, v_p)$ is symmetric in $v_0, \ldots, v_p$. The bracket operation on $\prod G_p$ then becomes

$$\begin{aligned} [\overline{X}, \overline{Y}](v_0, \ldots, v_{p+q}) &= \frac{1}{p!(q+1)!} \sum \overline{X}(\overline{Y}(v_{j_0}, \ldots, v_{j_q}), v_{j_{q+1}}, \ldots, v_{j_{p+q}}) \\ &- \frac{1}{(p+1)!q!} \sum \overline{Y}(\overline{X}(v_{k_0}, \ldots, v_{k_p}), v_{k_{p+1}}, \ldots, v_{k_{p+q}}). \end{aligned}$$

In particular, if $X \in G_p$ with $p > 0$ and $v \in G_{-1}$, then

$$[\overline{X}, v](v_1, \ldots, v_p) = \overline{X}(v, v_1, \ldots, v_p).$$

Conversely, given a sequence $V = G_{-1}, G_0, G_1, \ldots$ in $V \otimes S^{p+1}(V^*)$, we know that $\prod G_p$ is a graded algebra under the bracket operation described above if $[G_p, G_q] \subset G_{p+q}$.

Given a finite sequence $V = G_{-1}, G_0, G_1, \ldots, G_{n-1}$ with $G_p \subset V \otimes S^{p+1}(V^*)$ and $[G_p, G_q] \subset G_{p+q}$ with p, q, and $p+q$ all less than n, we wish to impose conditions on subspaces $G_i \subset V \otimes S^{i+1}(V^*)$ with $i \geq n$ that will allow $\prod G_p$ to be a graded algebra. Define the *first prolongation* $\Lambda^1 P$ of a subspace $P \subset V \otimes S^{p+1}(V^*)$ to be the subspace of maps $T \in V \otimes S^{p+2}(V^*)$ such that for all fixed $v \in V$, $T(v, v_1, \ldots, v_p) \in P$. The k-th *prolongation* is defined inductively by $\Lambda^1 \Lambda^{k-1} P$. Thus, $G_n \subset \Lambda^1 G_{n-1}$ and $[G_n, G_0] \subset G_n$. Hence, G_n must be an invariant subspace under this representation. Since $[G_p, G_q] \subset G_n$ whenever $p < n$, $q < n$, and $p + q = n$, we must not choose G_n to be too small. If such a G_n can be selected, then we are guaranteed a graded algebra containing G_n.

For a given Lie algebra $G_0 \subset \mathfrak{gl}(V)$ acting on a vector space $V = G_{-1}$, it is often possible to compute all graded algebras arising from G_{-1} and G_0. Suppose that $\dim V = 2$ and G_0 is a subalgebra of $\mathfrak{gl}(V)$. The prolongation $\Lambda^1 G_0$ of G_0 consists of $T \in V \otimes S^2(V^*)$ such that for $v \in G_{-1}$, $T(v) \in G_0$. We can represent elements $T \in V \otimes S^2(V^*)$ using matrices

$$\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{22} \\ a^2_{11} & a^2_{12} & a^2_{22} \end{pmatrix},$$

where

$$T(e_i, e_j) = a^1_{ij} e_1 + a^2_{ij} e_2,$$

if $\{e_1, e_2\}$ is a fixed basis for V. Hence, T is in $\Lambda^1 G_0$ if and only if the matrix is in G_0 whenever the first or the last column of the matrix is deleted. In general, we shall write

$$\begin{pmatrix} a^1_{11\cdots 1} & a^1_{1\cdots 12} & \cdots & a^1_{12\cdots 2} & a^1_{22\cdots 2} \\ a^2_{11\cdots 1} & a^2_{1\cdots 12} & \cdots & a^2_{12\cdots 2} & a^2_{22\cdots 2} \end{pmatrix}$$

for an element in $\Lambda^n G_0$.

Proposition 2.1. *Let V be a real vector space with $\dim V = 2$. The following subalgebras G_0 are the only subalgebras of $\mathfrak{gl}(V)$ up to conjugation.*

1. $\dim G_0 = 1$ *and* $\lambda \in \mathbb{R}$,
$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & \lambda a \end{pmatrix}, \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}, \begin{pmatrix} \lambda a & -a \\ a & \lambda a \end{pmatrix}.$$
2. $\dim G_0 = 2$ *and* $\lambda \in \mathbb{R}$,
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1) a \end{pmatrix}.$$
3. $\dim G_0 = 3$,
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \mathfrak{sl}(V).$$
4. $\dim G_0 = 4$, $\mathfrak{gl}(V)$.

A complete determination of Lie algebras of dimension less than or equal to three can be found in Jacobson [5]. To construct the faithful representations of these algebras in $\mathfrak{gl}(V)$ up to conjugation, see [4, 5].

Proposition 2.2. *Let V be a real vector space of dimension two. The prolongations of $G_0 \subset \mathfrak{gl}(V)$ are the algebras* (1), (3), (4), (6)–(8), (11), (14), (16), (21), (25), (35), *and* (37) *in Table* 1.

As an example, we will compute the prolongations in (7) and (21). Let e_1, e_2 be a basis for V and recall that we can represent elements $T \in V \otimes S^2(V^*)$ using matrices

$$\begin{pmatrix} a^1_{11} & a^1_{12} & a^1_{22} \\ a^2_{11} & a^2_{12} & a^2_{22} \end{pmatrix},$$

where

$$T(e_i, e_j) = a^1_{ij} e_1 + a^2_{ij} e_2.$$

Since T is in $\Lambda^1 G_0$ if and only if the matrix is in G_0 whenever the first or the last column of the matrix is deleted, the first prolongation of (7) must be zero. On the other hand, the first prolongation of (21) is

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Continuing, we see that the nth prolongation is

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n+1} & a_{n+2} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For a more in depth treatment of prolongation, see [3, 12].

Theorem 2.3. *Table 1 is a complete list of all graded algebras up to isomorphism with V, a real vector space of dimension two, and $G_0 \subset \mathfrak{gl}(V)$.*

Table 1e. Graded Algebras $\prod G_p$ with $G_0 \subset \mathfrak{gl}(2,\mathbb{R})$

1. $$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

2. $G_k = 0$ for $k > n$
$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & a \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

3. $\lambda \neq 0$
$$\begin{pmatrix} a & 0 \\ 0 & \lambda a \end{pmatrix} - (0) - \cdots$$

4. $$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

5. $G_k = 0$ for $k > n$
$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - (0) - \cdots$$

6. $$\begin{pmatrix} a & a \\ 0 & a \end{pmatrix} - (0) - \cdots$$

7. $\lambda \in \mathbb{R}$
$$\begin{pmatrix} \lambda a & -a \\ a & \lambda a \end{pmatrix} - (0) - \cdots$$

8. $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} - \cdots$$

9. $$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$
or
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} - (0) - \cdots$$

10. $G_k = 0$ for $k > n$
$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} - (0) - \cdots$$

11. $$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} - \cdots$$

12. $$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$
or
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \end{pmatrix} - (0) - \cdots$$

13. $G_k = 0$ for $k > n$

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

14.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} - \begin{pmatrix} a & -b & -a \\ b & a & -b \end{pmatrix} - \begin{pmatrix} a & -b & -a & b \\ b & a & -b & -a \end{pmatrix} - \cdots$$

15. $G_k = 0$ for $k > n$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} - \begin{pmatrix} a & -b & -a \\ b & a & -b \end{pmatrix} - \cdots - \begin{pmatrix} a & -b & -a & \cdots \\ b & a & -b & \cdots \end{pmatrix} - (0) - \cdots$$

16. $\lambda \neq -1$

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & (\lambda+1)a \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & \lambda a & b \\ 0 & \cdots & 0 & 0 & (\lambda+1)a \end{pmatrix} - \cdots$$

17. $\lambda \neq -1$ and $G_k = 0$ for $k > n$

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

or

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - \cdots$$

18. $\lambda \neq -1$

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

19. $\lambda = 1$

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & (\lambda+1)a \end{pmatrix} - (0) - \cdots$$

20. $\lambda = -(n+1)/(n-1), n = 2, 3, \ldots$ and $G_k = 0$ for $k > n$

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

21.

$$\begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

22.

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

23. $\dim G_k = 1$ for $k > n$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - \cdots$$

24. $G_k = 0$ for $k > n$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

25.

$$\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & b \end{pmatrix} - \cdots$$

26. $G_k = 0$ for $k > n$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

27. $\dim G_k = 1$ for $k > n$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - \cdots$$

28.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

29.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \end{pmatrix} - \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \end{pmatrix} - \cdots$$

30.

$$\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

31.

$$\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

or

$$\begin{pmatrix} a_1 & a_2 \\ 0 & b \end{pmatrix} - \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & b \end{pmatrix} - \cdots$$

32. $\lambda \neq 0$
$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & \lambda a & b \\ 0 & 0 & 0 & a \end{pmatrix} - \cdots$
or
$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$

33. $\lambda = 1/2$
$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & 0 \\ 0 & 0 & a \end{pmatrix} - (0) - \cdots$

34. $\lambda = (n-1)/2, n = 3, 4, \dots$, and $G_k = 0$ for $k > n$
$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & \lambda a & b \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$

35. $\mathfrak{sl}(2,\mathbb{R}) - \Lambda^1\mathfrak{sl}(2,\mathbb{R}) - \Lambda^2\mathfrak{sl}(2,\mathbb{R}) - \cdots$

36. $\mathfrak{sl}(2,\mathbb{R}) - (0) - \cdots$

37. $\mathfrak{gl}(2,\mathbb{R}) - \Lambda^1\mathfrak{gl}(2,\mathbb{R}) - \Lambda^2\mathfrak{gl}(2,\mathbb{R}) - \cdots$

38. $\mathfrak{gl}(2,\mathbb{R}) - (0) - \cdots$
or
$\mathfrak{gl}(2,\mathbb{R}) - \Lambda^1\mathfrak{sl}(2,\mathbb{R}) - \Lambda^2\mathfrak{sl}(2,\mathbb{R}) - \cdots$
or
$\mathfrak{gl}(2,\mathbb{R}) - \begin{pmatrix} 2a & b & 0 \\ 0 & a & 2b \end{pmatrix} - (0) - \cdots$

For the proof of (35)–(38) refer to Singer and Sternberg [12]. Koch proved (25)–(34) in [10]. It remains to show that (1)–(24) are the only possible graded algebras with $\dim G_0 = 1$ or 2. We will calculate the Lie brackets on a basis for each graded algebra obtained from the prolongation of G_0 with $\dim G_0 \leq 2$ in the following lemmas. The proof of the theorem follows directly from the following lemmas and the fact that $[G_p, G_q] \subset G_{p+q}$. For the remainder of the paper we shall let $\{e_1, e_2\}$ be a canonical basis for $V = G_{-1}$.

Lemma 2.4. *Let $\{e_1, e_2, A_0, A_1, \dots\}$ be a basis for* (1)*, where*

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \dots .$$

Then the only nonzero bracket relations are $[A_0, e_2] = e_1$ and $[A_i, e_2] = A_{i-1}$, where $i \geq 1$.

Proof. Clearly, these relations hold as well as the relations $[e_1, e_2] = 0$ and $[A_i, e_1] = 0$ for $i \geq 0$. It remains to show that $[A_i, A_j] = 0$. For $j > 0$ and $k = 1$ or 2, we have

$$[[A_0, A_j], e_k] = [A_0, [A_j, e_k]] + [A_j, [A_0, e_k]].$$

If $k=1$, the righthand expression is zero. If $k=2$, then

$$[[A_0, A_j], e_2] = [A_0, A_{j-1}] = 0$$

by induction on j; hence, $[A_0, A_j] = 0$. Similarly, if we fix j and induct on i, then $[A_i, A_j] = 0$. ∎

The proofs of the following lemmas are similar.

Lemma 2.5. *Let* $\{e_1, e_2, A_0, A_1, \ldots\}$ *be a basis for* (4)*, where*

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots .$$

The only nonzero bracket operations are $[A_0, e_1] = e_1$ *and* $[A_i, e_1] = A_{i-1}$ *for* $i \geq 1$.

Lemma 2.6. *Let* $\{e_1, e_2, A_0, A_1, \ldots, B_0, B_1, \ldots\}$ *be a basis for* (8)*, where*

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots ,$$

and

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots .$$

The nonzero bracket operations are

$$\begin{array}{rclrcl} [A_0, e_1] & = & e_1, & [A_i, e_1] & = & A_{i-1}, \\ [B_0, e_2] & = & e_2, & [B_i, e_2] & = & B_{i-1}, \end{array}$$

for $i \geq 1$, *and*

$$\begin{array}{rcl} [A_i, A_j] & = & \dfrac{(i-j)(i+j+1)!}{(i+1)!(j+1)!} A_{i+j}, \\ [B_i, B_j] & = & \dfrac{(i-j)(i+j+1)!}{(i+1)!(j+1)!} B_{i+j}. \end{array}$$

Lemma 2.7. *Let* $\{e_1, e_2, A_0, A_1, \ldots, B_0, B_1, \ldots\}$ *be a basis for* (11) *where*

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots$$

and

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ldots .$$

The nonzero bracket operations are

$$\begin{array}{rclrcl} [A_0, e_1] & = & e_1, & [A_i, e_1] & = & B_{i-1}, \\ [A_0, e_2] & = & e_2, & [A_i, e_2] & = & A_{i-1}, \\ [B_0, e_2] & = & e_1, & [B_i, e_2] & = & B_{i-1}, \end{array}$$

for $i \geq 1$, *and*

$$\begin{array}{rcl} [A_i, A_j] & = & \dfrac{(i-j)(i+j+1)!}{(i+1)!(j+1)!} A_{i+j}, \\ [A_i, B_j] & = & \dfrac{(i-j)(i+j+1)!}{(i+1)!(j+1)!} B_{i+j}. \end{array}$$

Lemma 2.8. *Let* $\{e_1, e_2, A_0, A_1, \dots, B_0, B_1, \dots\}$ *be a basis for* (14), *where*

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \dots$$

and

$$B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \dots .$$

Then there exist nonzero bracket operations

$$\begin{aligned} [A_1, A_i] &= \alpha A_{i+1}, \\ [A_1, B_i] &= \beta B_{i+1}, \\ [B_1, B_i] &= \gamma A_{i+1}, \end{aligned}$$

where $i = 2, 3, \dots$ *and* $\alpha, \beta, \gamma \neq 0$.

Lemma 2.9. *Suppose* $\lambda \neq -1$, *and let* $\{e_1, e_2, A_0, A_1, \dots, B_0, B_1, \dots\}$ *be a basis for* (16), *where*

$$A_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix}, \dots$$

and

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \dots .$$

The nonzero bracket operations are

$$\begin{aligned} [A_0, e_1] &= \lambda e_1, & [A_i, e_1] &= \lambda B_{i-1}, \\ [A_0, e_2] &= (\lambda+1) e_2, & [A_i, e_2] &= A_{i-1}, \\ [B_0, e_2] &= e_1, & [B_i, e_2] &= B_{i-1}, \end{aligned}$$

for $i \geq 1$, *and*

$$\begin{aligned} [A_i, A_j] &= \frac{(\lambda+1)(i-j)(i+j+1)!}{(i+1)!(j+1)!} A_{i+j}, \\ [A_i, B_j] &= \frac{(\lambda(i-j) - (j+1))(i+j+1)!}{(i+1)!(j+1)!} B_{i+j}. \end{aligned}$$

The proofs of (1) through (18) follow directly from the lemmas. The proof of (19) and (20) are special cases of Lemma 2.9. To prove (21) through (24), the following lemma is required.

Lemma 2.10. *Consider the basis* $\{e_1, e_2, A_k^j, B_k\}$ *for* (21), *where*

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \dots ,$$

$$A_1^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A_1^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \dots A_3^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \dots .$$

Then G_k has basis $\{A_1^k, \dots, A_{k+2}^k, B_k\}$. The nonzero relations for this algebra are

$$[A_1^0, e_1] = [A_2^0, e_2] = e_1,$$

and for $k \geq 1$

$$\begin{aligned} [A_i^k, e_1] &= A_i^{k-1}, \ 1 \leq i \leq k+1, \\ [A_i^k, e_2] &= A_{i-1}^{k-1}, \ 2 \leq i \leq k+2, \\ [A_{i+1}^i, A_{j+2}^j] &= \frac{(i+1)(i+j+1)!}{(i+1)!(j+1)!} A_{i+j+2}^{i+j}, \\ [A_1^1, A_j^{i-1}] &= \alpha A_j^i, \ \textit{for some } \alpha \neq 0. \end{aligned}$$

3. The Spencer Cohomology

For any graded algebra $\prod G_p$, define $C^{i,j}$ to be the space of skew-symmetric multilinear maps $c : \bigwedge^j G_{-1} \to G_{i-1}$. If we define the coboundary operator

$$\partial : C^{i,j} \to C^{i-1,j+1}$$

by

$$(\partial c)(v_1, \dots, v_{j+1}) = \sum_k (-1)^k [c(v_1, \dots, \widehat{v_k}, \dots, v_{j+1}), v_k],$$

then $\partial^2 = 0$. The resulting cohomology groups are known as the *Spencer cohomology groups*, which we will denote by $H^{i,j}$ for $i, j \geq 0$. For $A \in G_0$ define a map $c \mapsto c^A$ from $C^{i,j}$ to itself by

$$c^A(v_1, \dots, v_j) = [A, c(v_1, \dots, v_j)] - \sum_k c(v_1, \dots, [A, v_k], \dots, v_j).$$

Then $(\partial c)^A = \partial(c^A)$. Consequently, G_0 acts on $H^{i,j}$, which we shall denote by $\xi \mapsto \xi^A$. An element $\xi \in H^{i,j}$ is *invariant* if $\xi^A = 0$ for all $A \in G_0$. The set of invariant elements of a cohomology group $H^{i,j}$ is denoted by $(H^{i,j})^I$. If $\eta \in \mathrm{Hom}(G_i, C^{j,l})$ and $\xi \in C^{i+1,k}$, define $\xi \cdot \eta \in C^{j,k+l}$ by

$$\begin{aligned} &\xi \cdot \eta(v_1, \dots, v_{l+l}) \\ &\qquad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \eta(\xi(v_{\sigma(1)}, \dots, v_{\sigma(k)}))(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}). \end{aligned}$$

In [6] it was shown that $\xi \cdot \eta \in H^{j,k+l}$.

The following proposition is due to Kobayashi and Nagano [7].

Proposition 3.1. *Let $G = \prod G_p$ be a graded Lie algebra. Then the following statements are true.*

1. $H^{0,0} = G_{-1}$.
2. $H^{i,0} = 0$ *for* $i \geq 1$.

3. $H^{0,1} = \mathfrak{gl}(G_{-1})/G_0$.
4. $H^{i,1} = \Lambda^1 G_{i-1}/G_i$ *for* $i \geq 1$. *In particular,* $H^{i,1} = 0$ *if and only if* $\Lambda^1 G_{i-1} = G_i$.

Let $L_p = G_p \times G_{p+1} \times \dots$, and $[\ ,\ ]$ be the usual Lie bracket on a graded algebra $\prod G_p$. An *n-bracket* on $\prod G_p$ is a skew-bilinear map

$$[\ ,\ ]'_n : \prod G_p \times \prod G_p \to \prod G_p$$

satisfying the following conditions.

1. For $X \in L_i$, $Y \in L_j$, $[X, Y]'_n - [X, Y] \in L_{i+j+1}$.
2. If X, Y, $Z \in \prod G_p$, then

$$[X, [Y, Z]'_n]'_n + [Y, [Z, X]'_n]'_n + [Z, [X, Y]'_n]'_n \in L_{n-1}.$$

If $[X, Y]'_n - [X, Y] \in L_{n-1}$ for X, $Y \in \prod G_p$, then $[\ ,\ ]'_n$ is a *flat n-bracket.*

If $[\ ,\ ]'$ is 0-bracket, we can define an element $\overline{c}$ in $C^{0,2}$ by

$$\overline{c}(u, v) = [u, v]' \bmod L_0,$$

for $u, v \in G_{-1}$. By definition $C^{-1,3} = 0$; therefore, $\partial\overline{c} = 0$. We will let $c \in H^{0,2}$ be the element in cohomology represented by $\overline{c}$. Similarly, if we are given a flat n-bracket with $n \geq 1$, we can define elements $c \in H^{n,2}$ and $\eta_i \in \operatorname{Hom}(G_i, H^{n,1})$ for $i = 0, \dots, n-1$. We now state several theorems from [6].

Theorem 3.2. *Let $[\ ,\ ]'$ be a 0-bracket on $\prod G_p$, and suppose that*

1. $c \cdot c = c^2 = 0$;
2. $c \in (H^{0,2})^I$.

If $H^{k,1} = H^{k,2} = H^{k,3} = 0$ for $k \geq 0$, then there exists a complete filtered Lie algebra L with Lie algebra bracket $[\ ,\ ]_L$ on $\prod G_p$ extending $[\ ,\ ]'$ such that $\prod G_p$ under the usual graded bracket is the associated graded algebra of L.

Theorem 3.3. *Let $[\ ,\ ]$ be a n-bracket on $\prod G_p$ with $n \geq 1$, and suppose that the following equations are satisfied.*

1. $\eta_0[A, B] = \eta_0(B)^A - \eta_0(A)^B$ *for* A, $B \in G_0$.
2. $\eta_i[A, B] = \eta_i(B)^A$ *for* $A \in G_0$, $B \in G_i$ *with* $i = 1, \dots, n-1$.
3. $\eta_i[A, B] = 0$ *for* $A \in G_p$, $B \in G_q$ *with* $p + q = i$, $p, q \geq 1$.
4. $c^A = \eta_0(A) \cdot \eta_{n-1}$ *for* $A \in G_0$.
5. $c \cdot \eta_{n-1} = 0$.
6. $\partial A \cdot \eta_{n-1} = 0$ *for* $A \in G_n$.

7. $\partial A \cdot \eta_{n-1} = -\eta_i(A) \cdot \eta_{n-1}$ *for* $A \in G_i$, *where* $i = 1, \dots, n-1$.

If $H^{k,1} = H^{k,2} = H^{k,3} = 0$ *for* $k > n$, *then there exists a complete filtered Lie algebra* L *with Lie algebra bracket* $[\ ,\ ]_L$ *on* $\prod G_p$ *extending* $[\ ,\ ]'$ *such that* $\prod G_p$ *under the usual graded bracket is the associated graded algebra of* L.

Let L and M be complete filtered Lie algebras with associated graded algebras isomorphic to $\prod G_p$ and denote the bracket operations on L and M by $[\ ,\ ]_L$ and $[\ ,\ ]_M$, respectively. An n*-isomorphism* or n*-map* is a linear map $\psi : L \to M$ such that

1. $\psi(L_p) \subset M_p$;

2. $L_p \xrightarrow{\psi} M_p \to M_p/M_{p+1} = G_p$ is the map $L_p \to L_p/L_{p+1} = G_p$;

3. $[\psi(X), \psi(Y)]_M - \psi([X,Y]_L) \in M_{n-1}$ for $X, Y \in L$.

If an n-map $\prod G_p \to L$ exists, we can define $c^L \in H^{n,2}$ and $\eta_i^L \in \mathrm{Hom}(G_i, H^{n,1})$. These elements satisfy the structure equations in either Theorem 3.2 or Theorem 3.3 depending on whether $n = 0$ or $n \geq 1$.

If $\alpha \in GL(G_{-1})$, then α acts on G_p via

$$A^\alpha(v_1, \dots, v_p) = \alpha A(\alpha^{-1}v_1, \dots, \alpha^{-1}v_p) \text{ for } A \in G_p \text{ and } v_i \in G_{-1}$$

which results in an automorphism of $\prod G_p$. Hence, there is a natural action of $\mathrm{Aut}(\prod G_p)$ on the cohomology groups $H^{i,j}$ that sends invariant elements to invariant elements. We denote this action by α_* for $\alpha \in \mathrm{Aut}(\prod G_p)$. Furthermore, if $\eta \in \mathrm{Hom}(G_p, H^{i,j})$, then the induced action α^* on η is $\alpha^*(\eta)(A) = \alpha_*\eta(\alpha^{-1}A)$ for $A \in G_p$.

Theorem 3.4. *Let* L *and* M *be complete filtered Lie algebra with graded algebra* $\prod G_p$ *and let* $\psi : L \to M$ *be an* n*-map satisfying the following conditions.*

1. $(H^{k,2})^I = 0$ *for* $k > n$.

2. *For* $k > n$,

$$\frac{\{\eta : G_0 \to H^{k,1} : \eta[A,B] = \eta(B)^A - \eta(A)^B\}}{\{\eta : G_0 \to H^{k,1} : \eta(A) = \xi^A \text{ for some } \xi \in H^{k,1}\}} = 0.$$

3. $\mathrm{Hom}_{G_0}(G_i, H^{k,1}) = 0$ *for* $n < k$ *and* $1 \leq i < k$.

If $n = 0$ *and there exists an* $\alpha \in \mathrm{Aut}(\prod G_p)$ *such that* $\alpha_* c^L = c^M$, *then* $L \cong M$. *If* $n \geq 1$ *and there exist* n*-maps* $\phi_L : \prod G_p \to L$ *and* $\phi_M : \prod G_p \to M$, *and for some* $\alpha \in \mathrm{Aut}(\prod G_p)$, $\alpha_* c^L = c^M$ *and* $\alpha^* \eta_i^L = \eta_i^M$ *for* $i = 0, \dots, n-1$, *then* $L \cong M$.

4. The Group of n-maps

The n-maps from $\prod G_p$ to itself act on the cohomological elements c and η_i. These n-maps form a group $\mathcal{H}$. There exists a series of subgroups of $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_n,$$

where $\tau \in \mathcal{H}_i$ whenever $\tau(v) = v + \tau_i(v) + \tau_{i+1}(v) + \cdots$ and $\tau_p \in \mathrm{Hom}(G_{-1}, G_p)$. In addition, $\mathcal{H}_{i+1}$ is normal in $\mathcal{H}_i$. Let $c^L, \eta_0^L, \ldots, \eta_{n-1}^L$ be the elements in cohomology defined by the n-map $\phi : \prod G_p \to L$. The group $\mathcal{H}$ acts on c^L and η_i^L via the n-map $\phi\sigma$ and gives elements $(c^L)^\sigma$ and $(\eta_i^L)^\sigma$, where $\sigma \in \mathcal{H}$.

Proposition 4.1. *Let $\phi : \prod G_p \to L$ be an n-map that defines cohomological elements c^L and η_i^L for $i = 0, \ldots, n-1$. If $\sigma \in \mathcal{H}_n$, then the following statements are true.*

1. *If $\sigma(v) = v + \sigma_0(v) + \sigma_1(v) + \cdots$, $v \in G_{-1}$ and $\sigma_i \in \mathrm{Hom}(G_{-1}, G_i)$, then*
$$(c^L)^\sigma = c^\sigma + c^L + \sum_{k=0}^{n-1} \sigma_k \cdot \eta_k^L.$$
2. *If $0 \le p \le n-1$ and $A \in G_p$, then*
$$(\eta_p^L)^\sigma(A) = \eta_p^\sigma(A) + \eta_p^L(A) + \sum_{k=p+1}^{n-1} \eta_k^L(\sigma_k(A)),$$
where $\sigma(A) = A + \sigma_{p+1}(A) + \sigma_{p+2}(A) + \cdots$, $\sigma_i(A) \in G_i$.

The action of $\mathcal{H}_n$ on the elements c^L and η_i^L is trivial. Let

$$\sigma(v) = v + \sigma_{n-1}(v) + \sigma_n(v) + \cdots$$

be a representative for $\overline{\sigma} \in \mathcal{H}_{n-1}/\mathcal{H}_n$ where $\sigma_i \in \mathrm{Hom}(G_{-1}, G_i)$. Since $\partial\sigma_{n-1} = 0$, there is a well-defined natural map $\theta : \mathcal{H}_{n-1}/\mathcal{H}_n \to H^{n,1}$. Furthermore, θ is surjective.

Proposition 4.2. *Let $\overline{\sigma}, \overline{\tau}$ in $\mathcal{H}_{n-1}/\mathcal{H}_n$ have representatives $\sigma, \tau \in \mathcal{H}_{n-1}$, respectively. If $\theta(\overline{\sigma}) = \theta(\overline{\tau})$, then σ and τ act the same on the elements c^L and η_i^L, $0 \le i < n-1$. In addition, if $\overline{\sigma}$ induces $\sigma_{n-1} \in \mathcal{H}_{n-1}$, then*

1. $(c^L)^\sigma = \begin{cases} c^L + [\sigma_0, \sigma_0] + \sigma \cdot \eta_0^L, & n = 1 \\ c^L + \sigma_{n-1} \cdot \eta_{n-1}^L, & n \ge 2; \end{cases}$
2. $(\eta_0^L)^\sigma(A) = \sigma_{n-1}^A + \eta_0^L(A)$;
3. $(\eta_i^L)^\sigma(A) = \eta_i^L(A)$, $i = 1, \ldots, n-1$.

The action of the groups $\mathcal{H}_{p-1}/\mathcal{H}_p$ on $c^L, \eta_0^L, \ldots, \eta_{n-1}^L$ for $1 \le p < n$ is partially determined by the adjoint map $\mathrm{Ad}_X : \prod G_p \to \prod G_p$ defined by

$$\mathrm{Ad}_X Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots,$$

where $X \in G_p$, $p \ge 1$. The map Ad_X is both an n-map and an automorphism of $\prod G_p$. The set Ad_{G_p} of all Ad_X where $X \in G_p$ is a subgroup of $\mathcal{H}_{p-1}$, and the subgroup $\langle \mathcal{H}_p \cup \mathrm{Ad}_{G_p} \rangle$ of $\mathcal{H}_{p-1}$ generated by $\mathcal{H}_p$ and Ad_{G_p} is normal in $\mathcal{H}_{p-1}$.

Proposition 4.3. *Let $X \in G_p$, $p = 1, \ldots, n-1$. Then*

1. $(\eta_i^L)^{\mathrm{Ad}_X}(A) = \eta_i^L(A)$ *for* $i = 1, \ldots, n-1$;
2. $(\eta_0^L)^{\mathrm{Ad}_X}(A) = \eta_0^L(A) + \eta_p^L(X)^A$;
3. $(c^L)^{\mathrm{Ad}_X} = c^L + \partial X \cdot \eta_{p-1}^L$.

Define a map $\theta : \mathcal{H}_{p-1}/\mathcal{H}_p \to (H^{p,1})^I$ for $p = 1, \ldots, n-1$ as follows. Let $\sigma(v) = v + \sigma_{p-1} + \cdots$ be a representative for $\overline{\sigma} \in \mathcal{H}_{p-1}/\mathcal{H}_p$, then $\partial \sigma_{p-1} = 0$. Let $\overline{\sigma}_{p-1} \in (H^{p,1})^I$ be the element in cohomology represented by σ_p.

Proposition 4.4. *For $p = 1, \ldots, n-1$*

$$\theta : \mathcal{H}_{p-1}/\langle \mathcal{H}_p \cup \mathrm{Ad}_X \rangle \to (H^{p,1})^I$$

is an injection.

The map $\theta : \mathcal{H}_{p-1}/\langle \mathcal{H}_p \cup \mathrm{Ad}_X \rangle \to (H^{p,1})^I$ is generally not surjective; however, an element $\xi \in (H^{p,1})^I$ is the image of some element in $\mathcal{H}_{p-1}/\langle \mathcal{H}_p \cup \mathrm{Ad}_X \rangle$ under the map θ exactly when ξ is given by an n-derivation on $\prod G_p$. An *n-derivation* is a linear map $D : \prod G_p \to \prod G_p$ such that

1. $D(G_i) \subset G_{i+1} \times G_{i+2} \times \cdots$;
2. $D[X,Y] - [DX,Y] - [X,DY] \in G_{n-1} \times G_n \times \cdots$, for $X, Y \in \prod G_p$.

Suppose $\sigma \in \mathcal{H}_i$ $(0 \le i < n-1)$ and $\sigma(v) = v + \sigma_i(v) + \sigma_{i+1}(v) + \cdots$. Then there exists an n-derivation D such that $D(v) = \sigma_i(v)$. Conversely, the map $\exp D$ is an n-map. The following theorem gives a method of calculating the action of n-maps on $\prod G_p$ [6].

Theorem 4.5. *An element $\overline{D} \in (H^{p,1})^I$ is the image of some element in*

$$\mathcal{H}_{p-1}/\langle \mathcal{H}_p \cup \mathrm{Ad}_X \rangle$$

under the map

$$\theta : \mathcal{H}_{p-1}/\langle \mathcal{H}_p \cup \mathrm{Ad}_X \rangle \to (H^{p,1})^I$$

exactly when D induces an n-derivation on $\prod G_p$.

5. Algebras with $\dim G_{-1} = 2$

We are now ready to classify all complete filtered Lie algebras L with graded algebra $\prod G_p$ and dim $G_{-1} = 2$. We first decide the cases where L is *flat*; i.e, $L \cong \prod G_p$. The following propositions shall prove useful. The proofs of the propositions can be found in Koch's paper [9].

Proposition 5.1. Koch *Let L be a complete filtered Lie algebra with graded algebra $\prod G_p$ such that the following conditions are satisfied.*

1. $(H^{i,2})^I = 0$ *for* $i \geq 0$.
2. *For* $j > 0$,

$$\frac{\{\eta : G_0 \to H^{j,1} : \eta[A,B] = \eta(B)^A - \eta(A)^B\}}{\{\eta : G_0 \to H^{j,1} : \eta(A) = \xi^A \text{ for some } \xi \in H^{j,1}\}} = 0.$$

3. $\operatorname{Hom}_{G_0}(G_i, H^{j,1}) = 0$ *for* $1 \leq i < j$.

Then $L \cong \prod G_p$.

Proposition 5.2. Gunning *Let L be a complete filtered Lie algebra with graded algebra $\prod G_p$ where G_0 contains the identity map, then $L \cong \prod G_p$.*

Proposition 5.3. *If $(H^{i,2})^I = 0$ and $H^{i,1} = 0$ for $i \geq 1$, then $L \cong \prod G_p$.*

The algebras (3) ($\lambda = 1$), (8)–(15), (25)–(34), (37), and (38) have no complete filtered Lie algebras that are not isomorphic to their associated graded algebras since in each algebra G_0 contains the identity. Singer and Sternberg [12] proved that (35) and (36) are flat. To analyze the remaining cases, it is necessary to compute the cohomology groups of each graded algebra in question. We remark here that $H^{i,3} = 0$ for $i \geq 0$ since $\dim G_{-1} = 2$.

Proposition 5.4. *Table 2 is a complete list of all nonzero cohomology groups $H^{i,1}$, $(i \geq 1)$ and $H^{i,2}$, $(i \geq 0)$ together with the generators for each of the cohomology groups for the graded algebras* (1)–(7) *and* (16)–(24) *of Table* 1.

We will compute the cohomology for (5) as an example. Using Lemma 2.5, we may take $\{e_1, e_2, A_0, \ldots, a_n\}$ as a basis for this algebra. The nonzero bracket operations are $[A_0, e_1] = e_1$ and $[A_i, e_1] = A_{i-1}$, where $1 \leq i \leq n$. Since $\dim V = 2$, $H^{i,j} = 0$ for $j \geq 3$, and $H^{i,1} = 0$ for $i \neq n+1$ by Proposition 3.1. To compute $H^{n+1,1}$, consider the sequence

$$C^{n+2,0} \to C^{n+1,1} \to C^{n,2}.$$

If $\xi \in C^{n+1,1}$ is the linear map from V to G_n defined by $\xi(e_1) = aA_n$ and $\xi(e_2) = bA_n$, then

$$\partial\xi(e_1, e_2) = [\xi(e_1), e_2] - [\xi(e_2), e_1] = -bA_{n-1}.$$

Hence, the kernel of $\partial\xi$ consists of linear maps of the form $\xi(e_1) = aA_n$ and $\xi(e_2) = 0$. Since $C^{n+2,0} = 0$, $H^{n+1,1} = \mathbb{R}$. To see that there are no invariant elements in $H^{n+1,1}$, observe that

$$\xi^{A_0}(e_1) = [A_0, \xi(e_1)] - \xi([A_0, e_1]) = -aA_n.$$

To compute $H^{0,2}$, consider the sequence

$$C^{1,1} \to C^{0,2} \to 0.$$

We may take $\xi \in C^{1,1}$ to be the linear map defined by $\xi(e_1) = aA_0$ and $\xi(e_2) = bA_0$. Then

$$\partial\xi(e_1, e_2) = [\xi(e_1), e_2] - [\xi(e_2), e_1] = -be_1.$$

Thus, $H^{0,2} = \mathbb{R}$ with representative $(e_1, e_2) \mapsto ae_2$. Since

$$\xi^{A_0}(e_1, e_2) = [A_0, \xi(e_1, e_2)] - \xi([A_0, e_1], e_2) - \xi(e_1, [A_0, e_2]) = -\xi(e_1, e_2),$$

there are no invariant elements in $H^{0,2}$. The computation of $H^{n+1,2}$ and $(H^{n+1,2})^I$ follows in a similar manner.

Table 2. Cohomology Groups of $\prod G_0$ with $G_0 \subset \mathfrak{gl}(2, \mathbb{R})$.

	Cohomology Group	**Generators**
(1)	$H^{0,2} = \mathbb{R}$	$(e_1, e_2) \mapsto ae_1$
	$(H^{0,2})^I = H^{0,2}$	
(2) $n \geq 0$	$H^{0,2} = \mathbb{R}$	$(e_1, e_2) \mapsto ae_1$
	$(H^{0,2})^I = H^{0,2}$	
	$H^{n+1,1} = \mathbb{R}$	$e_1 \mapsto 0, e_2 \mapsto aA_n$
	$(H^{n+1,1})^I = H^{n+1,1}$	
	$H^{n+1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto aA_n$
	$(H^{n+1,2})^I = H^{n+1,2}$	
(3) $\lambda \neq 0$	$H^{1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto aA_0$
	$(H^{1,2})^I = 0$ where $\lambda \neq -1$	
	$(H^{1,2})^I = H^{1,2}$ where $\lambda = -1$	
(4)	$H^{0,2} = \mathbb{R}$	$(e_1, e_2) \mapsto ae_2$
	$(H^{0,2})^I = 0$	
(5) $n \geq 0$	$H^{0,2} = \mathbb{R}$	$(e_1, e_2) \mapsto ae_2$
	$(H^{0,2})^I = 0$	
	$H^{n+1,1} = \mathbb{R}$	$e_1 \mapsto aA_n, e_2 \mapsto 0$
	$(H^{n+1,1})^I = 0$	
	$H^{n+1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto aA_n$
	$(H^{n+1,2})^I = 0$	
(6)	$H^{1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto aA_0$
	$(H^{1,2})^I = 0$	
(7)	$H^{1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto aA_0$
	$(H^{1,2})^I = 0$ where $\lambda \neq 0$	
	$(H^{1,2})^I = H^{1,2}$ where $\lambda = 0$	
(16) All cohomology groups vanish.		
(17) $\lambda \neq -1$ and $\dim G_k = 1$ for $k \geq 1$		
	$H^{1,1} = \mathbb{R}$	$e_1 \mapsto \lambda aB_0, e_2 \mapsto aA_0$
	$(H^{1,1})^I = 0$	
	$H^{1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto aA_0$
	$(H^{1,2})^I = 0$ where $\lambda \neq -1/2$	
	$(H^{1,2})^I = H^{1,2}$ where $\lambda = -1/2$	

$G_k = 0$ for $k \geq 1$		
	$H^{1,1} = \mathbb{R}^2$	$e_1 \mapsto \lambda a B_0$,
		$e_2 \mapsto a A_0 + b B_0$
	$(H^{1,1})^I = 0$ where $\lambda \neq -2$	
	$(H^{1,1})^I = \mathbb{R}$ where $\lambda = -2$	$e_1 \mapsto 0, e_2 \mapsto b B_0$
	$H^{1,2} = \mathbb{R}^2$	$(e_1, e_2) \mapsto a A_0 + b B_0$
	$(H^{1,2})^I = 0$	
$n \geq 1$ and $G_k = 0$ for $k > n$		
	$H^{1,1} = \mathbb{R}$	$e_1 \mapsto a\lambda B_0, e_2 \mapsto a A_0$
	$(H^{1,1})^I = 0$	
	$H^{1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a A_0$
	$(H^{1,2})^I = 0$ where $\lambda \neq -1/2$	
	$(H^{1,2})^I = H^{1,2}$ where $\lambda = -1/2$	
	$H^{n+1,1} = \mathbb{R}$	$e_1 \mapsto 0, e_2 \mapsto a B_n$
	$(H^{n+1,1})^I = 0$	
	where $\lambda \neq -(n+2)/(n+1)$	
	$(H^{n+1,1})^I = H^{n+1,1}$	
	where $\lambda = -(n+2)/(n+1)$	
	$H^{n+1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a B_n$
	$(H^{n+1,2})^I = 0$	
(18)	$H^{2,1} = \mathbb{R}$	$e_1 \mapsto a\lambda B_1, e_2 \mapsto a A_1$
	$(H^{2,1})^I = 0$	
	$H^{2,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a A_1$
	$(H^{2,2})^I = 0$ where $\lambda \neq -2/3$	
	$(H^{2,2})^I = H^{2,2}$ where $\lambda = -2/3$	
(19)	$H^{2,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a A_1$
	$(H^{2,2})^I = 0$	
(20) $\lambda = -(n+1)/(n-1)$		
	$H^{2,1} = \mathbb{R}$	$e_1 \mapsto a\lambda B_1, e_2 \mapsto a A_1$
	$(H^{2,1})^I = 0$	
	$H^{2,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a A_1$
	$(H^{2,2})^I = 0$	
	$H^{n+1,1} = \mathbb{R}$	$e_1 \mapsto 0, e_2 \mapsto a B_n$
	$(H^{n+1,1})^I = 0$	
	$H^{n+1,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a B_n$
	$(H^{n+1,2})^I = 0$	
(21) All cohomology groups vanish.		
(22)	$H^{0,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a e_1$
	$(H^{0,2})^I = 0$	
	$H^{2,1} = \mathbb{R}$	$e_1 \mapsto a A_1^1, e_2 \mapsto 0$
	$(H^{2,1})^I = 0$	
(23) $\dim G_k = 2$	$H^{0,2} = \mathbb{R}$	$(e_1, e_2) \mapsto a e_1$

$$
\begin{array}{ll}
(H^{0,2})^I = 0 & \\
H^{1,1} = \mathbb{R} & e_1 \mapsto aA_1^0, e_2 \mapsto 0 \\
(H^{1,1})^I = 0 &
\end{array}
$$

$\dim G_k = 1$ for $k > n \geq 1$

$$
\begin{array}{ll}
H^{0,2} = \mathbb{R} & (e_1, e_2) \mapsto ae_1 \\
(H^{0,2})^I = 0 & \\
H^{1,1} = \mathbb{R} & e_1 \mapsto aA_1^0, e_2 \mapsto 0 \\
(H^{1,1})^I = 0 & \\
H^{n+1,1} = \mathbb{R} & e_1 \mapsto aA_{n+1}^n, e_2 \mapsto aA_{n+2}^n \\
(H^{n+1,1})^I = H^{n+1,1} & \\
H^{n+1,2} = \mathbb{R} & (e_1, e_2) \mapsto aA_{n+1}^n \\
(H^{n+1,2})^I = 0 &
\end{array}
$$

(24) $\dim G_k = 1$ for $k \geq 1$

$$
\begin{array}{ll}
H^{0,2} = \mathbb{R} & (e_1, e_2) \mapsto ae_1 \\
(H^{0,2})^I = 0 & \\
H^{1,1} = \mathbb{R}^2 & e_1 \mapsto aA_1^0 + bA_2^0, \\
 & e_2 \mapsto bA_1^0 \\
(H^{1,1})^I = \mathbb{R} & e_1 \mapsto bA_2^0, e_2 \mapsto bA_1^0 \\
H^{1,2} = \mathbb{R} & (e_1, e_2) \mapsto aA_1^0 \\
(H^{1,1})^I = 0 &
\end{array}
$$

$G_k = 0$ for $k > n \geq 1$ and $\dim G_k = 1,\ n = 1, \ldots, n$

$$
\begin{array}{ll}
H^{0,2} = \mathbb{R} & (e_1, e_2) \mapsto ae_1 \\
(H^{0,2})^I = 0 & \\
H^{1,1} = \mathbb{R}^2 & e_1 \mapsto aA_1^0 + bA_2^0, \\
 & e_2 \mapsto bA_1^0 \\
(H^{1,1})^I = \mathbb{R} & e_1 \mapsto bA_2^0, e_2 \mapsto bA_1^0 \\
H^{1,2} = \mathbb{R} & (e_1, e_2) \mapsto bA_1^0 \\
(H^{1,1})^I = 0 & \\
H^{n+1,1} = \mathbb{R} & e_1 \mapsto 0, e_2 \mapsto aA_{n+2}^n \\
(H^{n+1,1})^I = 0 & \\
H^{n+1,2} = \mathbb{R} & (e_1, e_2) \mapsto aA_{n+2}^n \\
(H^{n+1,2})^I = 0 &
\end{array}
$$

By Proposition 5.3, $L \cong \prod G_p$ for the algebras (3), $(\lambda \neq -1, 0)$, (4), (6), (7) $(\lambda \neq 0)$, (16), (19), and (21), since the appropriate cohomology groups vanish. A straightforward but lengthy computation shows that the algebras (5), (20), and (22) satisfy the hypothesis of Proposition 5.1; therefore, these algebras are also flat.

The remaining algebras to be considered are (1)–(3), (7), (17), (18), (23), and (24). If the cohomological elements $c, \eta_0. \ldots, \eta_{n-1}$ are known modulo the actions of $\mathrm{Aut}(\prod G_p)$ and $\mathcal{H}$, then we may determine all complete filtered Lie algebras with dim $G_{-1} = 2$ provided that all higher obstructions vanish.

Lemma 5.5. *There exist exactly two complete filtered Lie algebras L having graded algebra* (1)*. These algebras are determined by $c \in H^{0,2}$ with $c = 0$ if L is graded and $c \neq 0$ if L is nongraded.*

Proof. The only nonzero cohomology group is $H^{0,2}$. Let $c \in H^{0,2}$ have generator $c(e_1, e_2) = ae_1$. The hypothesis of Theorem 3.2 are satisfied; hence, c induces a complete filtered Lie algebra bracket on $\prod G_p$. The group of n-maps acts trivially on c since $H^{i,0} = 0$ for $i \geq 0$. An automorphism α on G_{-1} is given by a matrix of the form

$$\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}.$$

An easy computation yields $\alpha_* c(e_1, e_2) = (a/t)e_1$. An appropriate choice of α will send c to any other nonzero element in $H^{0,2}$. Therefore, if $c \neq 0$, there exists exactly one nongraded algebra L. ■

The proofs of the next two lemmas are similar to the proof of Lemma 5.5.

Lemma 5.6. *The two complete filtered Lie algebras having graded algebra* (3) $(\lambda = -1)$ *are characterized by $c \neq 0$ and $c = 0$ (graded case), where $c \in H^{1,2}$.*

Lemma 5.7. *For the algebra* (7) $(\lambda = 0)$*, let $c \in H^{1,2}$ have the generator $c(e_1, e_2) = \beta A_0$. Then there are three distinct complete filtered Lie algebras determined by $\beta > 0$, $\beta < 0$, and $\beta = 0$ (graded case).*

Lemma 5.8. *The complete filtered Lie algebras having graded algebra* (2) *are parameterized by $(\beta_{-1}, \ldots, \beta_n)$, $\beta_i \in \mathbb{R}$. Furthermore, $L_{\beta_{-1},\ldots,\beta_n} \cong L_{\gamma_{-1},\ldots,\gamma_n}$ if there exists a $\lambda \in \mathbb{R}$ such that*

$$(\beta_{-1}, \ldots, \beta_n) = (\lambda^{n+2}\gamma_{-1}, \ldots, \lambda\gamma_n).$$

Proof. Let $c \in H^{n+1,2}$ and $\eta_i \in \mathrm{Hom}(G_i, H^{n+1,1})$ for $0 \leq i \leq n$ be given by

$$\begin{aligned} c(e_1, e_2) &= \beta_{-1} A_n, \\ \eta_i(A_i, e_1) &= 0, \\ \eta_i(A_i, e_2) &= \beta_i A_n. \end{aligned}$$

One quickly checks that the hypothesis of Theorem 3.3 hold. Applying Proposition 4.2, we see that the n-maps have no effect on the $c, \eta_0, \ldots, \eta_{n-1}$.

It remains to show how $\mathrm{Aut}(\prod G_p)$ acts on the $c, \eta_0, \ldots, \eta_{n-1}$. Let α be as in Lemma 5.5. Then

$$\begin{aligned} \alpha_* c(e_1, e_2) &= (\beta_{-1}/t) A_n, \\ \alpha^* \eta_i(A_i, e_2) &= (\beta_i / t^{n-i-1}) A_n. \end{aligned}$$

Therefore, if $L = L_{\beta_{-1},\ldots,\beta_n}$ is the algebra determined by the c and η_i, then any other algebra determined by

$$\begin{aligned} \overline{c}(e_1, e_2) &= \lambda^{n+2} \beta_{-1} A_n, \\ \overline{\eta}_i(A_i, e_1) &= 0, \\ \overline{\eta}_i(A_i, e_2) &= \lambda^{n-i+1} \beta_i A_n, \end{aligned}$$

for some $\lambda \in \mathbb{R}$, must be isomorphic to L. ■

Lemma 5.9. *Let $\prod G_p$ be the graded algebra*

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

as in (17)*. If $\lambda \neq -1,\ 0,\ -1/2$, then $\prod G_p$ is flat. If $\lambda = -1/2$, then there is exactly one nongraded algebra characterized by $c \neq 0$, $c \in H^{1,2}$. If $\lambda = 0$ there is exactly one nongraded algebra characterized by $\eta_0(B_0) \neq 0$, $\eta_0 \in \mathrm{Hom}(G_0, H^{1,1})$.*

Proof. Let

$$\begin{gathered} c(e_1, e_2) = \beta A_0, \\ \eta_0(A_0, e_1) = \lambda\gamma B_0, \quad \eta_0(A_0, e_2) = \gamma A_0, \\ \eta_0(B_0, e_1) = \lambda\delta B_0, \quad \eta_0(B_0, e_2) = \delta A_0. \end{gathered}$$

If $\lambda = -1/2$ and we apply the structural equations of Theorem 3.3, we may assume that $c(e_1, e_2) = \beta A_0$. If $\lambda \neq -1, 0$, or $-1/2$, then $\eta_0(A_0, e_1) = \lambda\gamma B_0$ and $\eta_0(A_0, e_2) = \gamma A_0$. If $\lambda = 0$, then $\eta_0(A_0, e_2) = \gamma A_0$ and $\eta_0(B_0, e_2) = \delta A_0$. The actions of the n-maps in the first case show that there are no nongraded algebras if $\lambda \neq 1, 0$, or $-1/2$. If $\lambda = -1/2$, the n-maps act trivially. If $\lambda = 0$, we may assume that $\gamma = 0$. Finally, notice that $\alpha \in \mathrm{Aut}(\prod G_p)$ is given on G_{-1} by a matrix of the form

$$\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}.$$

If either $\lambda = 0$ or $-1/2$, any nonzero element may be sent to any other nonzero element by the appropriate choice of α. ∎

Lemma 5.10. *Let*

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - (0) - \cdots$$

be as in (17)*. If $\lambda \neq 0$ or -2, then all algebras are graded. If $\lambda = 0$, then there exists one nongraded algebra characterized by $\eta_0(B_0) \neq 0$. If $\lambda = -2$, there is exactly one nongraded algebra characterized by $\eta_0(A_0) \neq 0$.*

Proof. Let

$$\begin{gathered} c(e_1, e_2) = \alpha A_0 + \beta B_0, \\ \eta_0(A_0, e_1) = \lambda\gamma B_0, \quad \eta_0(A_0, e_2) = \gamma A_0 + \delta B_0, \\ \eta_0(B_0, e_1) = \lambda\sigma B_0, \quad \eta_0(B_0, e_2) = \sigma A_0 + \tau B_0. \end{gathered}$$

The equations of Theorem 3.3 allow for two cases. If $\lambda = 0$, then

$$\begin{aligned} \eta_0(A_0, e_2) &= \gamma A_0 + \delta B_0, \\ \eta_0(B_0, e_2) &= \sigma A_0 + \gamma B_0. \end{aligned}$$

If $\lambda \neq 0$, then

$$\begin{gathered} c(e_1, e_2) = \frac{\lambda\gamma^2}{(\lambda+1)^2} B_0, \\ \eta_0(A_0, e_1) = \lambda\gamma B_0, \quad \eta_0(A_0, e_2) = \gamma A_0 + \delta B_0, \\ \eta_0(B_0, e_1) = 0, \quad \eta_0(B_0, e_2) = \frac{\lambda - 1}{\lambda + 1}\gamma B_0. \end{gathered}$$

Applying Proposition 4.2, we may assume that $\eta_0(B_0, e_2) = \sigma A_0$ if $\lambda = 0$. If $\lambda \neq 0$, the c and the η_i's vanish except for the case $\lambda = -2$, where $\eta_0(A_0, e_2) = \delta B_0$. The automorphism group of $\prod G_p$ sends any nonzero element to any other nonzero element. ■

The proofs of the next four lemmas are similar to the proofs above.

Lemma 5.11. *Let*

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

be the algebra given in (17)*, where* $n \geq 1$ *and* $G_k = 0$ *for* $k > n$*. If* $\lambda \neq -(n+2)/(n+1)$*, then* $L \cong \prod G_p$*. If* $\lambda = -(n+2)/(n+1)$*, then there exists exactly one nongraded algebra determined by* $\eta_0(A_0) \neq 0$*.*

Lemma 5.12. *Algebra* (18) *is flat if* $\lambda \neq -1$, 0, *or* $-2/3$*. If* $\lambda = 0$ *or* $\lambda = -2/3$*, then there is exactly one nongraded example in each case that is determined by* $\eta_0(B_0) \neq 0$ *and* $c \neq 0$*, respectively.*

We remark that the algebra given by

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

in (23) is graded by Proposition 5.1.

Lemma 5.13. *Let*

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & a & b \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 & 0 & b \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} - \cdots$$

be the algebra in (23) *with dim* $G_k = 1$ *for* $k > n$*. The nongraded algebras are parameterized by* $(\beta_0, \ldots \beta_n)$, $\beta_i \in \mathbb{R}$*; where* $L_{\beta_0,\ldots,\beta_n} \cong L_{\gamma_0,\ldots,\gamma_n}$ *if there exists* $\lambda \in \mathbb{R}$ *such that*

$$(\beta_0, \ldots, \beta_n) = (\lambda^{n+1}\gamma_0, \ldots, \lambda\gamma_n).$$

Lemma 5.14. *Given the algebra*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

in (24)*, there exists one nongraded algebra determined by* $\eta_0(A_1^0) \neq 0$*.*

Lemma 5.15. *Let*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

be as in (24) *where* $G_k = 0$ *for* $k > n$*. The nongraded algebras are parameterized by* $(\beta_0, \ldots \beta_n)$, $\beta_i \in \mathbb{R}$*; and* $L_{\beta_0,\ldots,\beta_n} \cong L_{\gamma_0,\ldots,\gamma_n}$ *if there exists* $\lambda \in \mathbb{R}$ *such that* $(\beta_0, \ldots, \beta_n) = (\lambda^{n+1}\gamma_0, \ldots, \lambda\gamma_n)$*.*

Proof. Let $c(e_1, e_2) = \alpha A^n_{n+2}$, $\eta_0(A_0, e_2) = \beta A^n_{n+2}$, and $\eta_i(B_i, e_2) = \gamma_i A^n_{n+2}$ for $0 \le i < n$. Equations (1) through (7) of Theorem 3.3 are satisfied. Let D be the n-derivation D given by

$$\begin{gathered} De_1 = A^0_2, \quad De_2 = A^0_1, \quad DA^0_1 = 0, \\ DA^i_{i+2} = (i+2)A^{i+1}_{i+3} \quad (0 \le i < n), \\ DA^n_{n+2} = 0. \end{gathered}$$

The n-map $\exp D$ is given by

$$\begin{gathered} (\exp D)(e_1) = e_1 + A^0_2 + \cdots A^n_{n+2}, \\ (\exp D)(e_2) = e_2 + A^0_1, \\ (\exp D)(A^0_1) = 0, \\ (\exp D)(A^i_{i+2}) = A^i_{i+2} + (i+2)A^{i+1}_{i+3} + \frac{(i+2)(i+3)}{2!}A^{i+2}_{i+4} + \cdots . \end{gathered}$$

The n-maps act on c as in Proposition 4.1, allowing us to assume that $c = 0$. The action of the n-maps on $H^{n+1,1}$ also allows us to assume that $\beta = 0$. Now apply $\mathrm{Aut}(\prod G_p)$ as in Lemma 5.8. ∎

Theorem 5.16. *Table 3 is a complete list of all nongraded algebras for* $\prod G_p$, *where* $\dim G_{-1} = 2$.

Table 3. Nongraded Algebras with $\dim G_{-1} = 2$.

(1) One nongraded algebra.

(2) Nongraded algebras $L_{\beta_{-1}\cdots\beta_n}$, $\beta_i \in \mathbb{R}$ where $L_{\beta_{-1}\cdots\beta_n} \cong L_{\gamma_{-1}\cdots\gamma_n}$ if there exists $\lambda \in \mathbb{R}$ such that $(\beta_{-1}, \dots, \beta_n) = (\lambda^{n+2}\gamma_{-1}, \dots, \lambda\gamma_n)$.

(3) One nongraded algebra $(\lambda = -1)$.

(7) Two nongraded algebras $(\lambda = 0)$.

(17) One nongraded algebra in each case $(\lambda = -1/2, 0)$.

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} - \cdots$$

One nongraded algebra in each case $(\lambda = -2, 0)$.

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - (0) - \cdots$$

One nongraded algebra in each case $(\lambda = -(n+2)/(n+1))$ and $G_k = 0$ for $k > n$.

$$\begin{pmatrix} \lambda a & b \\ 0 & (\lambda+1)a \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots$$

(18) One nongraded algebra in each case ($\lambda = 0, -2/3$).

(23) Let $\prod G_p$ be the algebra

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & a & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - \cdots,$$

where $\dim G_k = 1$ for $k > n$. Nongraded algebras $L_{\beta_0 \cdots \beta_n}$ exist, where $\beta_i \in \mathbb{R}$ and $L_{\beta_0 \cdots \beta_n} \cong L_{\gamma_0 \cdots \gamma_n}$ if there exists $\lambda \in \mathbb{R}$ such that

$$(\beta_0, \ldots, \beta_n) = (\lambda^{n+1}\gamma_0, \ldots, \lambda\gamma_n).$$

(24) One nongraded algebra for the graded algebra

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - \cdots$$

For the algebra

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \cdots - \begin{pmatrix} 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \end{pmatrix} - (0) - \cdots,$$

where $G_k = 0$ for $k > n$, there exist nongraded algebras $L_{\beta_0 \cdots \beta_n}$ with $\beta_i \in \mathbb{R}$ and $L_{\beta_0 \cdots \beta_n} \cong L_{\gamma_0 \cdots \gamma_n}$ if there exists a $\lambda \in \mathbb{R}$ such that $(\beta_0, \ldots, \beta_n) = (\lambda^{n+1}\gamma_0, \ldots, \lambda\gamma_n)$.

6. Conclusion

Cartan first classified the pseudogroups on $\mathbb{R}^2$ in [1] using a different approach, and Conn treats the structure of transitive Lie algebras in [2]. The methods used in this paper is that they can reasonably be applied to dimensions higher than two in many cases. Although the techniques here are useful in constructing examples of nongraded Lie algebras, they do not allow a complete classification. If higher obstructions in cohomology exist, then these techniques may fail. Volpert offers another method using the Spencer cohomology and spectral sequences to obtain examples of complete filtered Lie algebras in [14, 15]. Finally, complete filtered Lie algebras are the algebraic objects corresponding to pseudogroups and transitive differential geometry. The geometric meaning of the c's and the η_i's are only partially understood [3, 12, 13].

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