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# Vanishing Properties of Analytically Continued Matrix Coefficients

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**Abstract.** We consider (generalized) matrix coefficients associated to irreducible unitary representations of a simple Lie group $G$ which admit holomorphic continuation to a complex semigroup domain $S \subseteq G_{\mathbb{C}}$. Vanishing theorems for these analytically continued matrix coefficients, one of Howe-Moore type and one for cusp forms, are proved.

## Introduction

Recall the Howe-Moore Theorem (cf. [9]; see also [18] and [20]) on the vanishing of matrix coefficients:

**Theorem.** *Let $G$ be a semisimple Lie group with no compact simple factors and compact center. If $(\pi, \mathcal{H})$ is a non-trivial irreducible unitary representation of $G$, then for all $v, w \in \mathcal{H}$ one has*

$$\lim_{g \to \infty} \langle \pi(g).v, w \rangle = 0.$$

Now, if $G$ happens to be hermitian and $(\pi, \mathcal{H})$ is a unitary highest weight representation of $G$, then it was discovered by Olshanski and Stanton (cf. [16], [19]) that $(\pi, \mathcal{H})$ analytically extends to a complex $G \times G$-biinvariant domain $S \subseteq G_{\mathbb{C}}$. These domains turn out to be complex semigroups, so-called *complex Olshanski semigroups.* There is a maximal one $S_{\max}$ which is the compression semigroup of the bounded symmetric domain $G/K \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}P^+$. Here $G \subseteq P^- K_{\mathbb{C}} P^+$ denotes the Harish-Chandra decomposition. Hence one always has $S \subseteq P^- K_{\mathbb{C}} P^+$. Our interest however lies in the minimal complex Olshanski semigroup which is given by

$$S_{\min} = G \exp(i W_{\min})$$

with $W_{\min}$ a minimal $\mathrm{Ad}(G)$-invariant closed convex cone in $\mathrm{Lie}(G)$ of non-empty interior. Our first result is (cf. Theorem 2.5):

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**Theorem A.** (Vanishing at infinity of analytically continued matrix coefficients) *Let $G$ be a linear hermitian group and $(\pi_\lambda, \mathcal{H}_\lambda)$ a unitary highest weight representation of $G$ analytically continued to the minimal complex Olshanski semigroup $S_{\min}$. Then for all $v, w \in \mathcal{H}_\lambda$ we have that*

$$\lim_{\substack{s\to\infty \\ s\in S_{\min}}} \langle \pi_\lambda(s).v, w\rangle = 0,$$

*i.e, the analytically continued matrix coefficients $s \mapsto \langle \pi_\lambda(s).v, w\rangle$, $s \in S_{\min}$, vanish at infinity.*

It is interesting to observe that the proof of this theorem relies on geometric facts only: firstly that the middle projection $\kappa: P^- K_{\mathbb{C}} P^+ \to K_{\mathbb{C}}$ restricted to $S_{\min}$ is a proper mapping (cf. Proposition 1.2) and secondly an explicit description of $\kappa(S_{\min})$ (cf. Corollary 2.4). Since $G \subseteq S_{\min}$ is closed, our methods imply a simple new proof of the Howe-Moore Theorem for the special case of unitary highest weight representations.

Let now $\Gamma < G$ be a lattice and $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ a $\Gamma$-invariant distribution vector for $(\pi_\lambda, \mathcal{H}_\lambda)$. Then for all $K$-finite vectors $v$ of $(\pi_\lambda, \mathcal{H}_\lambda)$ the prescription

$$\theta_{v,\eta}: \Gamma\backslash G \to \mathbb{C}, \quad \Gamma g \mapsto \langle \pi_\lambda(g).v, \eta\rangle := \overline{\eta(\pi_\lambda(g).v)}$$

defines an automorphic form of $\Gamma\backslash G$. One can show that $\theta_{v,\eta}$ naturally extends to a function on $\Gamma\backslash S_{\min} \subseteq \Gamma\backslash G_{\mathbb{C}}$. We denote this extension by the same symbol. Then our next result is (cf. Theorem 3.3):

**Theorem B.** (Vanishing at infinity of analytically continued automorphic forms) *Let $\Gamma < G$ be a lattice and $\eta \in (\mathcal{H}_\lambda^{-\infty})$ a cuspidal element for a non-trivial unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of the hermitian Lie group $G$. Then for all $K$-finite vectors $v$ of $(\pi_\lambda, \mathcal{H}_\lambda)$ the analytically continued automorphic forms $\theta_{v,\eta}$ vanish at infinity:*

$$\lim_{\substack{\Gamma s\to\infty \\ \Gamma s\in \Gamma\backslash S_{\min}}} \theta_{v,\eta}(\Gamma s) = 0.$$

Theorem B has applications to complex analysis. For example it implies that the bounded holomorphic functions on $\Gamma\backslash \operatorname{int} S_{\min}$ separate the points (cf. [1]).

For $G = \mathrm{Sl}(2, \mathbb{R})$ the results in this paper were first proved in the diplome thesis of the second named author (cf. [17]).

It is our pleasure to thank the referee for his careful work.

## 1. Preliminaries on hermitian Lie groups

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{g}$ is called *hermitian* if $\mathfrak{g}$ is simple and $\mathfrak{z}(\mathfrak{k}) \neq \{0\}$. Here $\mathfrak{z}(\mathfrak{k})$ denotes the center of $\mathfrak{k}$.

Hermitian Lie algebras are classified. The complete list is as follows (cf. [6, p. 518]):

$$\mathfrak{su}(p,q), \quad \mathfrak{so}^*(2n), \quad \mathfrak{sp}(n,\mathbb{R}), \quad \mathfrak{so}(2,n), \quad \mathfrak{e}_{6(-14)}, \quad \mathfrak{e}_{7(-25)}.$$

That $\mathfrak{g}$ is hermitian implies in particular that $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}X_0$ is one dimensional, and after a renormalization of $X_0$ we can assume that

$$\operatorname{Spec}(\operatorname{ad} X_0) = \{-i, 0, i\}$$

(cf. [6, Ch. VIII]). If $\mathfrak{l}$ is a Lie algebra we denote by $\mathfrak{l}_\mathbb{C}$ its complexification. The spectral decomposition of $\operatorname{ad} X_0$ then reads as follows

$$\mathfrak{g}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^-$$

with $\mathfrak{p}^\pm = \{X \in \mathfrak{g}_\mathbb{C} \colon [X_0, X] = \mp iX\}$. Note that $\mathfrak{p}^\pm$ are abelian, $[\mathfrak{k}_\mathbb{C}, \mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm$ and $\mathfrak{p}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

We extend $\mathfrak{z}(\mathfrak{k})$ to a compact Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. We may assume that $\mathfrak{t} \subseteq \mathfrak{k}$. Let $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ denote the root system with respect to $\mathfrak{t}_\mathbb{C}$. Then

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\mathbb{C}^\alpha$$

with $\mathfrak{g}_\mathbb{C}^\alpha$ the root spaces.

A root $\alpha \in \Delta$ is called *compact* if $\alpha(X_0) = 0$ and *non-compact* otherwise. The collection of compact roots, resp. non-compact roots, is denoted by $\Delta_k$, resp. $\Delta_n$. Note that $\Delta = \Delta_k \dot\cup \Delta_n$ and that $\alpha \in \Delta_k$ if and only if $\mathfrak{g}_\mathbb{C}^\alpha \subseteq \mathfrak{k}_\mathbb{C}$ and $\alpha \in \Delta_n$ iff $\mathfrak{g}_\mathbb{C}^\alpha \subseteq \mathfrak{p}_\mathbb{C}$.

If $\Delta^+$ is a positive system of $\Delta$ we set $\Delta^- = -\Delta^+$, $\Delta_k^\pm = \Delta_k \cap \Delta^\pm$ and $\Delta_n^\pm = \Delta_k \cap \Delta^\pm$. We can choose $\Delta^+$ such that

$$\Delta_n^+ = \{\alpha \in \Delta \colon \alpha(X_0) = -i\}.$$

Note that $\mathfrak{p}^\pm = \oplus_{\alpha \in \Delta_n^\pm} \mathfrak{g}_\mathbb{C}^\alpha$.

If $\mathfrak{l}$ is a Lie algebra and $\mathfrak{a} < \mathfrak{l}$ is a subalgebra of $\mathfrak{l}$, then we define $\operatorname{Inn}_\mathfrak{g}(\mathfrak{a}) := \langle e^{\operatorname{ad} X} \colon X \in \mathfrak{a} \rangle$.

Define the *little Weyl group* of $(\mathfrak{g}, \mathfrak{t})$ by $\mathcal{W}_\mathfrak{k} := N_{\operatorname{Inn}_\mathfrak{k}(\mathfrak{t})} / Z_{\operatorname{Inn}_\mathfrak{k}(\mathfrak{t})}$. If $\alpha \in \Delta$ we write $\check\alpha \in i\mathfrak{t}$ for its *coroot*, i.e., $\check\alpha \in [\mathfrak{g}_\mathbb{C}^\alpha, \mathfrak{g}_\mathbb{C}^{-\alpha}] \subseteq \mathfrak{t}_\mathbb{C}$ with $\alpha(\check\alpha) = 2$.

If $X$ is a topological space and $Y \subseteq X$, then we write $\operatorname{cl} Y$ for the closure and $\operatorname{int} Y$ for the interior of $Y$. If $V$ is a vector space and $E \subseteq V$, then we write $\operatorname{conv} E$ for the convex hull of $E$ and $\operatorname{cone} E$ for the convex cone generated by $E$.

Define the *minimal cone in* $\mathfrak{t}$ by

$$C_{\min} := \operatorname{cl}(\operatorname{cone}\{-i\check\alpha \colon \alpha \in \Delta_n^+\}).$$

Note that $C_{\min}$ is a $\mathcal{W}_{\mathfrak{k}}$-invariant convex cone with non-empty interior in $\mathfrak{t}$. Define the *minimal cone in* $\mathfrak{g}$ by

$$W_{\min} := \mathrm{cl}\left(\mathrm{conv}(\mathrm{Inn}(\mathfrak{g}).\mathbb{R}^+ X_0)\right).$$

Note that $W_{\min}$ is a convex $\mathrm{Inn}(\mathfrak{g})$-invariant cone in $\mathfrak{g}$ with non-empty interior and $W_{\min} \cap \mathfrak{t} = C_{\min}$ (cf. [7, Sect. 7]). In the sequel we set $W := \mathrm{int}\, W_{\min}$. Then $\mathrm{cl}\, W = W_{\min}$.

We write $G$ for a linear connected Lie group with Lie algebra $\mathfrak{g}$. Then $G \subseteq G_{\mathbb{C}}$ with $G_{\mathbb{C}}$ the universal complexification of $G$. The prescription

$$S := G \exp(iW)$$

defines a subsemigroup of $G_{\mathbb{C}}$, a so-called *complex Olshanski semigroup*. The closure of $S$ is given by $\mathrm{cl}\, S = G \exp(i\, \mathrm{cl}\, W)$. This is a consequence of Lawson's Theorem which states that the *polar mapping*

$$G \times \mathrm{cl}\, W \to \mathrm{cl}\, S, \quad (g, X) \mapsto g \exp(iX)$$

is a homeomorphism (cf. [13] or [15, Th. XI.1.7]).

Write $G_{\mathbb{C}} \to G_{\mathbb{C}},\ g \mapsto \overline{g}$ for the complex conjugation of $G_{\mathbb{C}}$ with respect to the real form $G$. Then the prescription

$$\mathrm{cl}\, S \to \mathrm{cl}\, S, \quad s = g \exp(iX) \mapsto s^* := \overline{s}^{-1} = \exp(iX) g^{-1}$$

defines an involution on $\mathrm{cl}\, S$ which is antiholomorphic when restricted to the open subset $S$ of $G_{\mathbb{C}}$.

Write $K, K_{\mathbb{C}}, P^+$ and $P^-$ for the analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{k}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^+$ and $\mathfrak{p}^-$. A theorem of Harish-Chandra states that the multiplication mapping

$$P^- \times K_{\mathbb{C}} \times P^+ \to G_{\mathbb{C}}, \quad (p^-, k, p^+) \mapsto p^- k p^+$$

is a biholomorphism onto its open image and that $G \subseteq P^- K_{\mathbb{C}} P^+$ (cf. [6, Ch. VIII]). If $s \in P^- K_{\mathbb{C}} P^+$, then $s = l^-(s)\kappa(s)l^+(s)$ with holomorphic maps $l^{\pm}: P^- K_{\mathbb{C}} P^+ \to P^{\pm}$ and $\kappa: P^- K_{\mathbb{C}} P^+ \to K_{\mathbb{C}}$. The *Harish-Chandra realization* $\mathcal{D} \subseteq \mathfrak{p}^-$ of the hermitian symmetric space $G/K$ is the image of the injective holomorphic map

$$\zeta: G/K \to \mathfrak{p}^-, \quad gK \mapsto \log l^-(g).$$

Note that $\mathcal{D}$ is a bounded symmetric domain (cf. [6, Ch. VIII]). The *compression semigroup* of $\mathcal{D}$ is defined by

$$\begin{aligned}\mathrm{comp}(\mathcal{D}) &:= \{g \in G_{\mathbb{C}} : g.\mathcal{D} \subseteq \mathcal{D}\} \\ &= \{g \in G_{\mathbb{C}} : g \exp(\mathcal{D}) K_{\mathbb{C}} P^- \subseteq \exp(\mathcal{D}) K_{\mathbb{C}} P^-\}.\end{aligned}$$

Then the $G$-biinvariance of $\mathrm{comp}(\mathcal{D})$ together with $\exp(i\mathbb{R}^+ X_0) \subseteq \mathrm{comp}(\mathcal{D})$ imply that

$$\mathrm{cl}\, S \subseteq \mathrm{comp}(\mathcal{D}).$$

This was first realized by Olshanski (cf. [16] or [15, Th. XII.3.3]).

The idea behind the following Lemma is not new and can also be found in [8].

**Lemma 1.1.** *We have*

$$\mathrm{cl}\, S \subseteq \exp(\mathcal{D}) K_{\mathbb{C}} \overline{\exp(\mathcal{D})}$$

*with* $\overline{\exp(\mathcal{D})} \subseteq P^+$ *the complex conjugate of* $\exp(\mathcal{D})$.

**Proof.** Since $\mathrm{cl}\, S$ compresses $\mathcal{D}$, we conclude that

$$\mathrm{cl}\, S \subseteq \exp(\mathcal{D}) K_{\mathbb{C}} P^+.$$

Now $\mathrm{cl}(S)$ is $*$-invariant and so together with $\mathcal{D} = -\mathcal{D}$ we get that

$$\mathrm{cl}\, S = (\mathrm{cl}\, S)^* \subseteq P^- K_{\mathbb{C}} \overline{\exp(\mathcal{D})}.$$

Finally

$$\mathrm{cl}\, S \subseteq \exp(\mathcal{D}) K_{\mathbb{C}} P^+ \cap P^- K_{\mathbb{C}} \overline{\exp(\mathcal{D})} = \exp(\mathcal{D}) K_{\mathbb{C}} \overline{\exp(\mathcal{D})}.$$ ■

**Proposition 1.2.** *The middle projection restricted to* $\mathrm{cl}\, S$

$$\kappa\colon \mathrm{cl}\, S \to K_{\mathbb{C}}, \quad s \mapsto \kappa(s)$$

*is a proper mapping.*

**Proof.** Let $A \subseteq K_{\mathbb{C}}$ be a compact subset. Then $\kappa^{-1}(A)$ is closed in $\mathrm{cl}\, S$ by the continuity of $\kappa$. By Lemma 1.1 we have that $\kappa^{-1}(A) \subseteq \exp(\mathcal{D}) A \overline{\exp(\mathcal{D})}$ and the latter set is relatively compact in $G_{\mathbb{C}}$ by the boundedness of $\mathcal{D}$. Hence the assertion follows. ■

**Remark 1.3.** There are many other interesting complex Olshanski semigroups than the one associated to the minimal cone. There is a distinguished maximal cone $W_{\max}$ characterized by

$$C_{\max} := W_{\max} \cap \mathfrak{t} = \{X \in \mathfrak{t} \colon (\forall \alpha \in \Delta_n^+)\ \alpha(iX) \geq 0\}$$

and with it comes a continuous family of closed convex $\mathrm{Inn}(\mathfrak{g})$-invariant cones $W_0$ lying between $W_{\min}$ and $W_{\max}$:

$$W_{\min} \subseteq W_0 \subseteq W_{\max}.$$

To each $W_0$ one can associate a complex Olshanski semigroup

$$S_0 = G \exp(i\, \mathrm{int}\, W_0)$$

featuring the same properties as $S$. In particular Lemma 1.1 and Proposition 1.2 remain true for $\mathrm{cl}\, S_0$. One has $S_{\max} = G \exp(i W_{\max}) = \mathrm{comp}(\mathcal{D})$ (cf. [7, Th. 8.49]). However, for the applications we have in mind, namely vanishing properties of matrix coefficients on $S$ and $\Gamma \backslash S$, the assumption on the minimalilty of the cone is crucial. For more details we refer to [15, Sect. VII.3, Ch. X-XI]. ■

## 2. Matrix coefficients on $S$

In the sequel $(\pi_\lambda, \mathcal{H}_\lambda)$ denotes a unitary highest weight representation of $G$ with highest weight $\lambda \in i\mathfrak{t}^*$ and with respect to the positive system $\Delta^+$. We refer to [15, Ch. XI] for more on unitary highest weight representations.

Let $\mathcal{H}$ be a Hilbert space with bounded operators $B(\mathcal{H})$. By a *holomorphic representation* of $S$ we understand a holomorphic semigroup homomorphism

$$\pi\colon S \to B(\mathcal{H})$$

which in addition satisfies $\pi(s^*) = \pi(s)^*$ for all $s \in S$.

If $V$ is a finite dimensional real vector space, $V^*$ its dual and $C \subseteq V$ a subset, then we define the dual cone of $C$ by

$$C^\star := \{\alpha \in V^* \colon (\forall X \in C)\ \alpha(X) \geq 0\}.$$

Note that $C$ is a closed convex subcone of $V^*$.

The central ideas of part (ii) in the next theorem go back to Olshanski and Stanton (cf. [16], [19]); a very systematic approach to these results is due to Neeb (cf. [14]).

**Theorem 2.1.** *Let $G$ be a hermitian Lie group and $S$ an associated minimal complex Olshanski semigroup. Then for every non-trivial unitary highest weight representation of $G$ the following statements hold:*

(i) $\lambda \in i \operatorname{int} C^\star_{\min}$.

(ii) $(\pi_\lambda, \mathcal{H}_\lambda)$ *extends to a strongly continuous and contractive representation* $\pi_\lambda \colon \operatorname{cl} S \to B(\mathcal{H}_\lambda)$ *with* $\pi_\lambda|_S$ *a holomorphic representation.*

**Proof.** (i) [15, Th. IX.2.17].
(ii) This follows from (i) and [15, Th. XI.4.8]. ■

We now take a closer look at the inclusion $\operatorname{cl} S \subseteq P^+ K_{\mathbb{C}} P^-$ and prove a refinement of Lemma 1.1. This will be accomplished with tools provided by representation theory.

Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of $G$. In view of Theorem 2.1(ii) we henceforth consider $(\pi_\lambda, \mathcal{H}_\lambda)$ as a representation of $\operatorname{cl} S$. We denote by $V_\lambda \subseteq \mathcal{H}_\lambda$ the space of $K$-finite vectors. Since every vector in $V_\lambda$ is $\mathfrak{p}^+$-finite we have a natural representation $\sigma_\lambda$ of the semidirect product group $K_{\mathbb{C}} \rtimes P^+$ on $V_\lambda$ obtained by exponentiating the derived representation $d\pi_\lambda|_{\mathfrak{k}_{\mathbb{C}} \rtimes \mathfrak{p}^+}$.

If $v_\lambda \in V_\lambda$ is a highest weight vector, then we set

$$F(\lambda) := \operatorname{span}_{\mathbb{C}}\{\pi_\lambda(K).v_\lambda\}$$

for the finite dimensional subspace of the highest $K$-type.

**Lemma 2.2.** *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of $G$. Then we have for all $s \in \operatorname{cl} S$ and $v, w \in F(\lambda)$ that*

$$\langle \pi_\lambda(s).v, w\rangle = \langle \sigma_\lambda(\kappa(s)).v, w\rangle.$$

**Proof.** This follows from [11, Prop. 2.20]. ∎

We write $HW(G)$ for those $\lambda \in i\mathfrak{t}^*$ for which there exists a unitary highest weight representation of $G$ with respect to $\Delta^+$. Recall that $HW(G) \subseteq i \operatorname{int} C_{\min}^\star \cup \{0\}$ (cf. Theorem 2.1(i)). Moreover, from our knowledge on the unitarizable highest weight modules for $G$ we have

$$iHW(G)^\star = C_{\min} \tag{2.1}$$

(cf. [10, Lemma II.5]; this follows basically from the fact that $HW(G)$ contains a subset of the form $\Gamma \cap (x + i \operatorname{int} C_{\min}^\star)$ with $\Gamma \subseteq i\mathfrak{t}^*$ a vector lattice and $x \in i\mathfrak{t}^*$ a certain element). Write $W_K := \operatorname{Ad}(K).C_{\min}$ and note that $W_K$ is a convex cone, a consequence of Kostant's convexity theorem. Define now the semigroup

$$S_K := K \exp(iW_K) = K \exp(iC_{\min})K \subseteq K_{\mathbb{C}}$$

and note that

$$S_K \subseteq \operatorname{cl} S.$$

**Proposition 2.3.** *The following inclusion holds*

$$\operatorname{cl} S \subseteq \exp(\mathcal{D}) S_K \overline{\exp(\mathcal{D})}.$$

**Proof.** We define

$$U := \bigcap_{\lambda \in HW(G)} \{k \in K_{\mathbb{C}} : \sigma_\lambda(k)|_{F(\lambda)} \text{ is a contraction}\}.$$

Note that Lemma 2.2 together with Lemma 1.1 and the fact that the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $\operatorname{cl} S$ is contractive (cf. Theorem 2.1(ii)) imply that $\operatorname{cl} S \subseteq \exp(\mathcal{D}) U \overline{\exp(\mathcal{D})}$. Hence it is sufficient to show that $U = S_K$.

From the definition of $U$ it is clear that $U$ is $K$-biinvariant and so $U = K \exp(iC)K$ with $C \subseteq i\mathfrak{t}$ a convex cone (note that $\mathfrak{t}$ is abelian). By a theorem of Kostant we know that the $\mathfrak{t}_{\mathbb{C}}$-weight spectrum of $F(\lambda)$ is contained in $\operatorname{conv}(\mathcal{W}_{\mathfrak{k}}.\lambda)$. Thus we obtain that

$$U = K\Big(\bigcap_{\lambda \in HW(G)} \exp(\{X \in i\mathfrak{t} : (\forall w \in \mathcal{W}_{\mathfrak{k}})\ (w.\lambda)(X) \leq 0\})\Big)K,$$

and so (2.1) implies that $C = C_{\min}$, concluding the proof of the proposition. ∎

**Corollary 2.4.** *We have that $\kappa(\operatorname{cl} S) = S_K$.*

**Proof.** Since $S_K \subseteq \operatorname{cl} S \cap K_{\mathbb{C}}$ the inclusion "$\supseteq$" is clear. The converse inclusion follows from Proposition 2.3. ∎

We now come to the main result of this Section.

**Theorem 2.5.** (Vanishing at infinity of analytically continued matrix coefficients) *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of $G$ analytically continued to $\operatorname{cl} S$. Then for all $v, w \in \mathcal{H}_\lambda$ we have that*

$$\lim_{\substack{s\to\infty \\ s\in \operatorname{cl} S}} \langle \pi_\lambda(s).v, w\rangle = 0,$$

*i.e, the analytically continued matrix coefficients $\langle \pi_\lambda(s).v, w\rangle$, $s \in \operatorname{cl} S$, vanish at infinity.*

**Proof.** Since $V_\lambda \subseteq \mathcal{H}_\lambda$ is a dense subspace and $\|\pi_\lambda(s)\| \leq 1$ for all $s \in \operatorname{cl} S$, it is sufficient to prove the theorem for $v, w \in V_\lambda$. For $v, w \in V_\lambda$ the proof of [11, Prop. 2.20] shows that

$$\langle \pi_\lambda(s).v, w\rangle = \langle \sigma_\lambda(\kappa(s))\sigma_\lambda(l^+(s)).v, \sigma_\lambda(\overline{l^-(s)}^{-1}).w\rangle.$$

Write $l^+(s) = \exp(X)$, $\overline{l^-(s)}^{-1} = \exp(Y)$ for elements $X, Y \in \overline{\mathcal{D}} \subseteq \mathfrak{p}^+$ (cf. Lemma 1.1). Hence there exists an $N \in \mathbb{N}$, independent from $s \in \operatorname{cl} S$, such that

$$\langle \pi_\lambda(s).v, w\rangle = \sum_{j,k=1}^{N} \frac{1}{j!k!} \langle \sigma_\lambda(\kappa(s)) d\pi_\lambda(X)^j.v, d\pi_\lambda(Y)^k.w\rangle.$$

Note that

$$\sup_{\substack{1\leq j,k\leq N \\ s\in \operatorname{cl} S}} \{\|d\pi_\lambda(X)^j.v\|, \|d\pi_\lambda(Y)^k.w\|\} < \infty$$

since $\overline{\mathcal{D}}$ is bounded. Hence it is sufficient to show that

$$\langle \sigma_\lambda(\kappa(s)).v, w\rangle \to 0 \tag{2.2}$$

for $s \to \infty$ in $\operatorname{cl} S$ and $v, w \in V_\lambda$. As $\kappa\colon \operatorname{cl}(S) \to K_{\mathbb{C}}$ is proper by Proposition 1.2, Corollary 2.4 implies that (2.2) is equivalent to

$$\lim_{\substack{s\to\infty \\ s\in S_K}} \langle \sigma_\lambda(s).v, w\rangle = 0 \tag{2.3}$$

for all $v, w \in V_\lambda$.

Now we make a final reduction from which the theorem will follow. Write $C_{\min}^+ := \{X \in C_{\min}\colon (\forall \alpha \in \Delta^+)\ i\alpha(X) \geq 0\}$ and note that $C_{\min}^+$ is a fundamental domain in $C_{\min}$ for the $\mathcal{W}_{\mathfrak{k}}$-action (see also Remark 1.3 for the inclusion $C_{\min} \subseteq C_{\max}$ which is needed here).

Since $S_K = K\exp(iC_{\min})K$, we obtain that $S_K = K\exp(iC_{\min}^+)K$. Hence the fact that $K$ is compact, and $v, w$ are $K$-finite implies that (2.3) is equivalent to

$$\lim_{\substack{X\to\infty \\ X\in C_{\min}^+}} \langle \sigma_\lambda(\exp(iX)).v, w\rangle = 0 \tag{2.4}$$

for all $v, w \in V_\lambda$. W.l.o.g. we may assume that $v, w$ are $\mathfrak{t}_{\mathbb{C}}$-weight vectors. Recall that

$$\operatorname{Spec}(d\pi_\lambda|_{\mathfrak{t}_{\mathbb{C}}}) \subseteq \lambda - \mathbb{N}_0[\Delta^+].$$

The fact that $\lambda(iX) < 0$ for all $X \in C_{\min}\backslash\{0\}$ (cf. Theorem 2.1(i)) proves (2.4) and hence the theorem. ∎

## 3. Analytic continuation of holomorphic automorphic forms

Let $\mathcal{H}_\lambda^\infty$ be the $G$-Fréchet module of smooth vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$. Then the *strong antidual* (the space of antilinear continuous functionals equipped with the strong topology) of $\mathcal{H}_\lambda^\infty$ is denoted by $\mathcal{H}_\lambda^{-\infty}$ and we refer to it as the space of *distribution vectors* of $(\pi_\lambda, \mathcal{H}_\lambda)$. Recall the chain of continuous inclusions

$$\mathcal{H}_\lambda^\infty \hookrightarrow \mathcal{H}_\lambda \hookrightarrow \mathcal{H}_\lambda^{-\infty}.$$

For a discrete subgroup $\Gamma < G$ we write $(\mathcal{H}_\lambda^{-\infty})^\Gamma$ for the $\Gamma$-invariants of $\mathcal{H}_\lambda^{-\infty}$. If $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ and $v \in \mathcal{H}_\lambda^\infty$, then we consider the general matrix coefficient

$$\theta_{v,\eta}: \Gamma\backslash G \to \mathbb{C}, \quad \Gamma g \mapsto \langle \pi_\lambda(g).v, \eta\rangle := \overline{\eta(\pi_\lambda(g).v)}.$$

Note that $\theta_{v,\eta} \in C^\infty(\Gamma\backslash G)$.

Since $\Gamma$ acts properly discontinuously on $G_\mathbb{C}$, we get Hausdorff quotients $\Gamma\backslash S, \Gamma\backslash \operatorname{cl} S \subseteq \Gamma\backslash G_\mathbb{C}$. Note that $\Gamma\backslash S$ is also a complex submanifold of $\Gamma\backslash G_\mathbb{C}$.

In view of the results of [12, App.], we have $\pi_\lambda(\operatorname{cl} S).\mathcal{H}_\lambda^\infty \subseteq \mathcal{H}_\lambda^\infty$ and so the functions $\theta_{v,\eta}$ naturally extend to functions on $\Gamma\backslash \operatorname{cl} S$. We denote these extensions also by $\theta_{v,\eta}$. Note that $\theta_{v,\eta}|_{\Gamma\backslash S}$ is a holomorphic map since $\pi_\lambda(S).\mathcal{H}_\lambda^{-\infty} \subseteq \mathcal{H}_\lambda$ (cf. [12, App.]).

**Remark 3.1.** If $v \in V_\lambda$ is a $K$-finite vector of $(\pi_\lambda, \mathcal{H}_\lambda)$, then $\theta_{v,\eta}|_{\Gamma\backslash G}$ is an automorphic form in the sense of Borel and Wallach (cf. [Wal92, 11.9.1]).

If $v \in F(\lambda)$, then $\theta_{v,\eta}$ is a so-called *holomorphic automorphic form* (cf. [2, §6]). ∎

From now on $\Gamma < G$ denotes a lattice, i.e, $\Gamma$ is a discrete subgroup with $12(\Gamma\backslash G) < \infty$. We call an element $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ *cuspidal* if for all $v \in V_\lambda$ the automorphic form $\theta_{v,\eta}|_{\Gamma\backslash G}$ is a cusp form (cf. [5, Ch. I, §4] for the definition of cusp forms).

**Remark 3.2.** The definition of cusp forms is technical and we restrained to give it here and refered to [5] instead. However, some remarks are appropriate.
(a) In [5] automorphic forms are defined for arithmetic lattices $\Gamma < G$ only. In view of more recent results, this is no major constraint anymore: Margulis' "arithmeticity theorem" (cf. [21, Th. 6.1.2]) implies that every lattice is arithmetic if $\operatorname{rank}_\mathbb{R}(G) \geq 2$; if $\operatorname{rank}_\mathbb{R}(G) = 1$, then the difficulties (in particular the existence of a Siegel set) can be overcome by the work of Garland and Raghunathan (cf. [4]).
(b) If $\eta \in (\mathcal{H}_\lambda^{-\infty})^\Gamma$ such that $\theta_{v,\eta}|_{\Gamma\backslash G}$ belongs to $L^2(\Gamma\backslash G)$ for all $v \in V_\lambda$, then $\eta$ is cuspidal. This is a special feature related to holomorphic automorphic forms; a conceptual proof of this fact for the group $G = \mathrm{Sl}(2,\mathbb{R})$ is for example given in [3, Cor. 7.10].
(c) In [1, Th. 3.11] it is shown that the Poincaré series $P(v_\lambda)$ of $v_\lambda$

$$P(v_\lambda) = \sum_{\gamma\in\Gamma} \pi_\lambda(\gamma).v_\lambda$$

converges for almost all parameters $\lambda$ in the module of hyperfunction vectors $\mathcal{H}_\lambda^{-\omega}$ to a non-zero $\Gamma$-fixed element. Since convergent Poincaré series define cuspidal elements (cf. [3, Th. 8.9]), the existence of sufficiently many non-trivial cuspidal elements is hence guaranteed. ■

**Theorem 3.3.** (Vanishing at infinity of analytically continued automorphic forms) *Let $\Gamma < G$ be a lattice and $\eta \in (\mathcal{H}_\lambda^{-\omega})^\Gamma$ a cuspidal element for a non-trivial unitary highest weight representation of the hermitian Lie group $G$. Then for all $K$-finite vectors $v \in V_\lambda$ the analytically continued automorphic forms $\theta_{v,\eta}$ vanish at infinity:*

$$\lim_{\substack{\Gamma s \to \infty \\ \Gamma s \in \Gamma \backslash \operatorname{cl} S}} \theta_{v,\eta}(\Gamma s) = 0. \qquad ■$$

**Remark 3.4.** (a) For $\Gamma < G$ cocompact Theorem 3.2 was proved in [1] with different methods coming from representation theory.
(b) Theorem 3.2 together with [1, Th. 4.7] implies in particular that the bounded holomorphic functions on $\Gamma\backslash S$ separate the points. Here it might by interesting to observe that the surrounding complex manifold $\Gamma\backslash G_\mathbb{C}$ admits no holomorphic functions except the constants: $\mathrm{Hol}(\Gamma\backslash G_\mathbb{C}) = \mathbb{C}\mathbf{1}$. For more information we refer to [1]. ■

**Proof of Theorem 3.3.** First we reduce the assertion of the theorem to the case where $v = v_\lambda$ is a highest weight vector. Assume that $\theta_{v_\lambda,\eta}$ vanishes at infinity on $\Gamma\backslash \operatorname{cl} S$. Then it follows that $\theta_{v,\eta}$ vanishes at infinity for all $v \in E_\lambda := \operatorname{span}_\mathbb{C}\{\pi_\lambda(G).v_\lambda\}$. Note that $E_\lambda$ is dense in $\mathcal{H}_\lambda$ since $(\pi_\lambda, \mathcal{H}_\lambda)$ is irreducible.

If $(\chi, U_\chi)$ is an irreducible representation of $K$, then we write $V_\lambda^{[\chi]}$ for the $\chi$-isotypical part of the $K$-module $V_\lambda$. By the density of $E_\lambda \subseteq \mathcal{H}_\lambda$ we conclude that the orthogonal projection

$$P_\chi : E_\lambda \to V_\lambda^{[\chi]}, \quad v \mapsto \frac{1}{\dim U_\chi} \int_K \overline{\operatorname{tr}\chi(k)}\pi(k).v \; dk$$

is onto. In particular, if $v \in V_\lambda^{[\chi]}$ with $v = P_\chi(w)$ for some $w \in E_\lambda$, then we have

$$\theta_{v,\eta}(\Gamma s) = \frac{1}{\dim U_\lambda} \int_K \overline{\operatorname{tr}\chi(k)}\theta_{\pi(k).w,\eta}(s) \; dk.$$

Hence the compactness of $K$ implies that $\theta_{v,\eta}$ vanishes at infinity, completing the proof of our reduction.

We now show that $\theta_{v_\lambda,\eta}$ vanishes at infinity. First we need some notation. Write

$$p_{F(\lambda)} : \mathcal{H}_\lambda^{-\infty} \to F(\lambda)$$

for the projection onto the highest $K$-type along the other $K$-types. Define the function

$$f : G \to F(\lambda), \quad g \mapsto p_{F(\lambda)}(\pi_\lambda(g^{-1}).\eta).$$

Note that $f$ is smooth, left $\Gamma$-invariant and that

$$\theta_{v_\lambda,\eta}(\Gamma g) = \langle v_\lambda, f(g)\rangle \qquad (g \in G).$$

Further we define

$$\mu_\lambda(s) := \sigma_\lambda(\kappa(s))\,|_{F(\lambda)} \in \mathrm{Gl}(F(\lambda)) \qquad (s \in \operatorname{cl} S).$$

Then on $\mathcal{D} \cong G/K$ the prescription

$$F(gK) := \mu_\lambda(g^{-1})^{-1} f(g) \qquad (g \in G) \tag{3.1}$$

defines an anti-holomorphic function on $\mathcal{D}$ (cf. [2, §6]).

We claim that $F$ is bounded. Let $\|\cdot\|$ be a norm on $G_{\mathbb{C}}$. Denote by $\mathcal{S} \subseteq G$ a Siegel set for $\Gamma$. Recall that a Siegel set has the properties that $\Gamma\mathcal{S} = G$ and $|\Gamma\mathcal{S} \cap \mathcal{S}| < \infty$. Then the fact that $\theta_{v,\eta}$ is a cusp form for all $v \in F(\lambda)$ implies that there exists for all $N \in \mathbb{N}$ a constant $C = C_N > 0$ such that

$$(\forall g \in \mathcal{S}) \qquad |\theta_{v,\eta}(\Gamma g)| \leq C_N \|v\| \cdot \|g\|^{-N} \tag{3.2}$$

(cf. [3, Th. 7.5] for $G = \mathrm{Sl}(2,\mathbb{R})$ and [5, Ch. I, Lemma 10] for the general case). By Lemma 1.1 there exists constants $C_1, C_2 > 0$ such that $C_1\|g\| \leq \|\kappa(g^{-1})^{-1}\| \leq C_2\|g\|$ for all $g \in G$. Hence there exists an $M \in \mathbb{N}$ and a constant $C > 0$ such that $\|\mu_\lambda(g^{-1})^{-1}\| \leq C\|g\|^M$. In view of (3.1) and (3.2), our claim now follows.

From (3.1) we get that

$$f(g) = \mu_\lambda(g^{-1}) F(gK)$$

and so

$$\theta_{v_\lambda,\eta}(\Gamma g) = \langle v_\lambda, \mu_\lambda(g^{-1}) F(gK)\rangle. \tag{3.3}$$

Write $\widetilde{F} \colon \operatorname{cl} S \to F(\lambda),\ s \mapsto F(s.K)$ and note that $\widetilde{F}$ is anti-holomorphic on $S$ (Recall that $\operatorname{cl} S.\mathcal{D} \subseteq \mathcal{D}$). Thus analytic continuation of (3.3) yields

$$\theta_{v_\lambda,\eta}(\Gamma s) = \langle v_\lambda, \mu_\lambda(s^*).\widetilde{F}(s)\rangle \tag{3.4}$$

Since $F$ is bounded, $\widetilde{F}$ is bounded. By (3.4) it hence suffices to show $\mu_\lambda(s^*) \to 0$ for $s \to \infty$ in $\operatorname{cl} S$. But since $s \mapsto s^*$ is a homeomorphism of $\operatorname{cl} S$ this now follows from Proposition 1.2 and Proposition 2.3. ∎

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