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Casimir Operators on Pseudodifferential Operators of Several Variables

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Abstract. We study the actions of the Lie algebra $n\mathfrak{sl}(2,\mathbb{C})$ of $SL(2,\mathbb{C})^n$ and the associated Casimir operator on the space of pseudodifferential operators of n variables. We describe the effect of the Casimir operator on a pseudodifferential operator in connection with the symbol map and construct an $n\mathfrak{sl}(2,\mathbb{C})$-invariant lifting of the symbol map.

1. Introduction

It is well-known that pseudodifferential operators of a single variable play an important role in the theory of nonlinear integrable partial differential equations, also known as soliton equations, as well as in conformal field theory. The algebra of pseudodifferential operators generalizes the algebra of differential operators and admits various other important algebraic structures (see e.g. [6]). In a recent paper [3] (see also [14]), Cohen, Manin and Zagier investigated connections among pseudodifferential operators, modular forms, and formal power series called Jacobi-like forms. Among other things, given a discrete subgroup Γ of $SL(2,\mathbb{R})$ acting on the Poincaré upper half plane $\mathcal{H}$ as usual, they constructed a natural correspondence between Γ-invariant pseudodifferential operators on $\mathcal{H}$ and sequences of modular forms for Γ. Since the product of two Γ-invariant pseudodifferential operators are again Γ-invariant, such a correspondence determines a family of noncommutative products of modular forms known as the Rankin-Cohen brackets (cf. [2], [12]). The construction of the Rankin-Cohen brackets can also be extended to the case of Siegel modular forms as was done in [1] and [4]. On the other hand, in [10] Olver and Sanders studied highly interesting connections of the Rankin-Cohen brackets for modular forms with various topics in pure and applied mathematics including transvectants, the Heisenberg group, solitons, Hirota operators and coherent states (see also [8], [9]).

Pseudodifferential operators of several variables were systematically introduced recently by Parshin in [11], where he discussed some of their algebraic structures as well as their role in soliton theory. As is expected, the close link between pseudodifferential operators and modular forms can be extended to the case of sev-

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eral variables. Indeed, the group $SL(2,\mathbb{C})^n$ acts on the space of pseudodifferential operators of n variables, and a pseudodifferential operator on the n-fold product $\mathcal{H}^n$ of the Poincaré upper half plane $\mathcal{H}$ invariant under the action of a discrete subgroup $\Gamma_n \subset SL(2,\mathbb{R})^n$ can be identified with a sequence of Hilbert modular forms for Γ_n, and the Rankin-Cohen brackets for Hilbert modular forms can also be constructed (see [7]).

In this paper we study the actions of the Lie algebra $n\mathfrak{sl}(2,\mathbb{C})$ of the Lie group $SL(2,\mathbb{C})^n$ and the associated Casimir operator on the space of pseudodifferential operators of n variables. We describe the effect of the Casimir operator on a pseudodifferential operator in connection with the symbol map, which associates the coefficient of the highest order term to each pseudodifferential operator. We also construct an $n\mathfrak{sl}(2,\mathbb{C})$-equivariant lifting of the symbol map.

2. Pseudodifferential operators of several variables

In this section we review some of the properties of pseudodifferential operators of several variables introduced by Parshin [11]. Let $(z_1,\dots,z_n)$ be the standard coordinate system for $\mathbb{C}^n$, and let $\partial_1,\dots,\partial_n$ be the associated partial differentiation operators given by

$$\partial_1 = \frac{\partial}{\partial z_1},\dots,\partial_n = \frac{\partial}{\partial z_n}.$$

We denote by $\mathcal{F}$ the ring of complex-valued C^∞ functions $f(z) = f(z_1,\dots,z_n)$ on $\mathbb{C}^n$.

For convenience we often use the multi-index notation throughout the paper. Thus, given $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$ and $u = (u_1,\dots,u_n) \in \mathbb{C}^n$, we have

$$\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}, \quad u^\alpha = u_1^{\alpha_1}\cdots u_n^{\alpha_n}. \tag{1}$$

If $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}_+^n$ with $\mathbb{Z}_+$ denoting the set of nonnegative integers, we write

$$\beta! = \beta_1!\cdots\beta_n!, \quad \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n},$$

where for $1 \le i \le n$ we have

$$\binom{\alpha_i}{0} = 1, \quad \binom{\alpha_i}{\beta_i} = \frac{\alpha_i(\alpha_i-1)\cdots(\alpha_i-\beta_i+1)}{\beta_i!}$$

for $\beta_i > 0$. Furthermore, for $\mu = (\mu_1,\dots,\mu_n), \nu = (\nu_1,\dots,\nu_n) \in \mathbb{Z}^n$ we write $\mu \le \nu$ if $\mu_i \le \nu_i$ for each $i = 1,\dots,n$, and also write $\boldsymbol{c} = (c,\dots,c) \in \mathbb{Z}^n$ if $c \in \mathbb{Z}$.

Definition 2.1. A *pseudodifferential operator of n variables* is a formal series of the form

$$L = \sum_{\alpha\le\nu} f_\alpha(z)\partial^\alpha \tag{2}$$

for some $\nu \in \mathbb{Z}^n$, where $z = (z_1,\dots,z_n) \in \mathbb{C}^n$ and $f_\alpha \in \mathcal{F}$ for all $\alpha \le \nu$. We shall denote by ΨDO the complex vector space consisting of all pseudodifferential operators of n variables.

Definition 2.2. (i) The *order* of an element $L \in \Psi\mathrm{DO}$, denoted by $\mathrm{ord}(L)$, is the smallest integer r such that $L = \sum_{i \le r} a_i \partial_n^i$ with $a_r \neq 0$.

(ii) The *highest term* of $L \in \Psi\mathrm{DO}$, denoted by $\mathrm{HT}(L)$, is the term in L defined inductively by

$$\mathrm{HT}(L) = (\mathrm{HT}(a_r))\partial_n^r$$

for $L = \sum_{i \le r} a_i \partial_n^i$ with $\mathrm{ord}(L) = r$.

If the highest term of L is of the form $\mathrm{HT}(L) = f(z)\partial_1^{\eta_1} \cdots \partial_n^{\eta_n}$, then we set

$$\nu(L) = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n.$$

We denote by $\prec$ the lexicographic type of order on $\mathbb{Z}^n$ such that

$$\nu(L) = (\eta_1, \dots, \eta_n) \prec 0$$

if and only if

$$\eta_n < 0, \quad \text{or} \quad \eta_n = 0 \text{ and } \eta_{n-1} < 0, \quad \text{or } \dots, \text{ etc.}$$

and use $\preceq$ to mean $\prec$ or $=$. Given an element $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}^n$, we consider the subspaces $\Psi\mathrm{DO}_\omega$ and $\Psi\mathrm{DO}_\omega^*$ of $\Psi\mathrm{DO}$ defined by

$$\Psi\mathrm{DO}_\omega = \{L \in \Psi\mathrm{DO} \mid \nu(L) \preceq \omega\}, \quad \Psi\mathrm{DO}_\omega^* = \{L \in \Psi\mathrm{DO} \mid \nu(L) \prec \omega\}.$$

Let Ξ_ω be the *symbol map* sending a pseudodifferential operator $L \in \Psi\mathrm{DO}_\omega$ to the coefficient of its highest term, that is, $\Xi_\omega(L) = f_\omega(z)$ if $\mathrm{HT}(L) = f_\omega(z)\partial^\omega$. Then we see that the kernel of Ξ_ω is $\Psi\mathrm{DO}_\omega^*$, and therefore we obtain a short exact sequence

$$0 \to \Psi\mathrm{DO}_\omega^* \to \Psi\mathrm{DO}_\omega \xrightarrow{\Xi_\omega} \mathcal{F} \to 0 \tag{3}$$

of complex vector spaces.

3. Casimir operators

Let $\mathfrak{sl}(2,\mathbb{C})$ be the Lie algebra of the Lie group $SL(2,\mathbb{C})$, and let $\{X, Y, H\}$ be the standard basis for $\mathfrak{sl}(2,\mathbb{C})$ given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

satisfying

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

We denote by $n\mathfrak{sl}(2,\mathbb{C})$ the direct sum of n copies of $\mathfrak{sl}(2,\mathbb{C})$, which is the Lie algebra of $SL(2,\mathbb{C})^n$. For $1 \le i \le n$ let

$$\varepsilon_i : \mathfrak{sl}(2,\mathbb{C}) \to n\mathfrak{sl}(2,\mathbb{C}) \tag{5}$$

be the natural inclusion map sending an element of $\mathfrak{sl}(2,\mathbb{C})$ to the i-th component of $n\mathfrak{sl}(2,\mathbb{C})$, and set

$$X_i = \varepsilon_i(X), \quad Y_i = \varepsilon_i(Y), \quad H_i = \varepsilon_i(H).$$

Then we see that the set

$$\{X_i, Y_i, H_i \mid 1 \le i \le n\} \tag{6}$$

is a basis for $n\mathfrak{sl}(2,\mathbb{C})$. Let End($\Psi$DO) be the space of complex linear endomorphisms of ΨDO, and define the complex linear map $\sigma : n\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\Psi\mathrm{DO})$ by

$$\sigma(X_i) = \sqrt{-1}z_i^2\partial_i, \quad \sigma(Y_i) = \sqrt{-1}\partial_i, \quad \sigma(H_i) = 2z_i\partial_i \tag{7}$$

for $1 \le i \le n$. As usual End(ΨDO) has the structure of a complex Lie algebra whose bracket operation is given by

$$[\psi_1, \psi_2] = \psi_1\psi_2 - \psi_2\psi_1$$

for all $\psi_1, \psi_2 \in \Psi\mathrm{DO}$. We denote this Lie algebra by $\mathfrak{gl}(\Psi\mathrm{DO})$.

Lemma 3.1. *The linear map σ given by* (7) *determines a Lie algebra homomorphism from $n\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{gl}(\Psi\mathrm{DO})$.*

Proof. It suffices to check the condition for the basis elements for $n\mathfrak{sl}(2,\mathbb{C})$ in (6). Using (7), for each i we obtain

$$\begin{aligned}
[\sigma(H_i), \sigma(X_i)] &= [2z_i\partial_i, \sqrt{-1}z_i^2\partial_i] \\
&= 2\sqrt{-1}(z_i(2z_i\partial_i + z_i^2\partial_i^2) - z_i^2(\partial_i + z_i\partial_i^2)) \\
&= 2\sqrt{-1}z_i^2\partial_i = 2\sigma(X_i) = \sigma([H_i, X_i]), \\
[\sigma(H_i), \sigma(Y_i)] &= [2z_i\partial_i, \sqrt{-1}\partial_i] \\
&= 2\sqrt{-1}(z_i\partial_i^2 - \partial_i - z_i\partial_i^2) \\
&= -2\sqrt{-1}\partial_i = -2\sigma(Y_i) = \sigma([H_i, Y_i]), \\
[\sigma(X_i), \sigma(Y_i)] &= [\sqrt{-1}z_i^2\partial_i, \sqrt{-1}\partial_i] \\
&= -(z_i^2\partial_i^2 - 2z_i\partial_i - z_i^2\partial_i^2) \\
&= 2z_i\partial_i = \sigma(H_i) = \sigma([X_i, Y_i]),
\end{aligned}$$

and hence the lemma follows. ∎

By Lemma 3.1 the composition of σ with the adjoint representation of the Lie algebra $\mathfrak{gl}(\Psi\mathrm{DO})$ determines a representation of the Lie algebra $n\mathfrak{sl}(2,\mathbb{C})$ in the complex vector space ΨDO. The associated Casimir element C is given by

$$C = \sum_{i=1}^{n}(\sigma(H_i)^2/2 + \sigma(X_i)\sigma(Y_i) + \sigma(Y_i)\sigma(X_i)) \tag{8}$$

(see e.g. [5, §6.2]). For $1 \le i \le n$ and $1 \le j \le 3$ we set

$$L_{i,j} = z_i^{j-1}\partial_i, \tag{9}$$

which we regard as operators acting on ΨDO by commutation. Then by (7) and (8) we see that the Casimir operator can be written in the form

$$C = \sum_{i=1}^{n}(2L_{i,2}^2 - L_{i,1}L_{i,3} - L_{i,3}L_{i,1}) \tag{10}$$

and that $C(\Psi\mathrm{DO}_\eta) \subset \Psi\mathrm{DO}_\eta$ for each $\eta \in \mathbb{Z}^n$.

Theorem 3.2. *Given an element* $\psi = \sum_{\nu \geq \mathbf{0}} f_\nu \partial^{\eta-\nu} \in \Psi\mathrm{DO}_\eta$ *with*

$$\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$$

we have

$$\Xi_\eta(C\psi) = 2\sum_{i=1}^n \eta_i(\eta_i+1)\Xi_\eta(\psi),$$

where Ξ_η *is the symbol map in* (3).

Proof. First, we consider an element of the form $h\partial^\omega \in \Psi\mathrm{DO}$ with $\omega = (\omega_1, \dots, \omega_n)$. Then for each $i \in \{1, \dots, n\}$ we have

$$\begin{aligned}
L_{i,1}(h\partial^\omega) &= [\partial_i, h\partial^\omega] = \partial_i(h\partial^\omega) - h\partial^\omega\partial_i \\
&= (\partial_i h)\partial^\omega + h\partial^{\omega+\boldsymbol{e}_i} - h\partial^{\omega+\boldsymbol{e}_i} = (\partial_i h)\partial^\omega \\
L_{i,2}(h\partial^\omega) &= [z_i\partial_i, h\partial^\omega] = z_i\partial_i(h\partial^\omega) - h\partial^\omega(z_i\partial_i) \\
&= z_i(\partial_i h)\partial^\omega + z_i h\partial^{\omega+\boldsymbol{e}_i} - hz_i\partial^{\omega+\boldsymbol{e}_i} - \omega_i h\partial^\omega = (z_i(\partial_i h) - \omega_i h)\partial^\omega \\
L_{i,3}(h\partial^\omega) &= [z_i^2\partial_i, h\partial^\omega] = z_i^2\partial_i(h\partial^\omega) - h\partial^\omega(z_i^2\partial_i) \\
&= z_i^2(\partial_i h)\partial^\omega + z_i^2 h\partial^{\omega+\boldsymbol{e}_i} - h(z_i^2\partial^{\omega+\boldsymbol{e}_i} + 2\omega_i z_i\partial^\omega + \omega_i(\omega_i-1)\partial^{\omega-\boldsymbol{e}_i}) \\
&= (z_i^2(\partial_i h) - 2\omega_i z_i h)\partial^\omega - \omega_i(\omega_i-1)h\partial^{\omega-\boldsymbol{e}_i},
\end{aligned}$$

where $\boldsymbol{e}_i$ denotes the element of $\mathbb{Z}^n$ with 1 in the i-th entry and 0 elsewhere. Thus we see that

$$\begin{aligned}
L_{i,2}^2(h\partial^\omega) &= z_i(\partial_i(z_i(\partial_i h) - \omega_i h))\partial^\omega - \omega_i(z_i(\partial_i h) - \omega_i h)\partial^\omega \\
&= z_i((\partial_i h) + z_i(\partial_i^2 h) - \omega_i(\partial_i h))\partial^\omega - \omega_i z_i(\partial_i h)\partial^\omega + \omega_i^2 h\partial^\omega \\
&= z_i(\partial_i h)\partial^\omega + z_i^2(\partial_i^2 h)\partial^\omega - \omega_i z_i(\partial_i h)\partial^\omega - \omega_i z_i(\partial_i h)\partial^\omega + \omega_i^2 h\partial^\omega, \\
L_{i,1}L_{i,3}(h\partial^\omega) &= (\partial_i(z_i^2(\partial_i h) - 2\omega_i z_i h))\partial^\omega - \omega_i(\omega_i-1)(\partial_i h)\partial^{\omega-\boldsymbol{e}_i} \\
&= (2z_i(\partial_i h) + z_i^2(\partial_i^2 h) - 2\omega_i h - 2\omega_i z_i(\partial_i h))\partial^\omega \\
&\qquad - \omega_i(\omega_i-1)(\partial_i h)\partial^{\omega-\boldsymbol{e}_i}, \\
L_{i,3}L_{i,1}(h\partial^\omega) &= (z_i^2(\partial_i^2 h) - 2\omega_i z_i(\partial_i h))\partial^\omega - \omega_i(\omega_i-1)(\partial_i h)\partial^{\omega-\boldsymbol{e}_i}.
\end{aligned}$$

Using these relations and (10), it follows that

$$C(h\partial^\omega) = 2\sum_{i=1}^n \Big(\omega_i(\omega_i+1)h\partial^\omega + \omega_i(\omega_i-1)(\partial_i h)\partial^{\omega-\boldsymbol{e}_i}\Big).$$

Thus, if $\psi = \sum_{\nu \geq \mathbf{0}} f_\nu \partial^{\eta-\nu} \in \Psi\mathrm{DO}_\eta$, we have

$$\begin{aligned}
C\psi - 2\sum_{i=1}^n \eta_i(\eta_i+1)\psi &= 2\sum_{\nu\geq\mathbf{0}}\sum_{i=1}^n \Big((\eta_i-\nu_i)(\eta_i-\nu_i+1) - \eta_i(\eta_i+1)\Big) f_\nu\partial^{\eta-\nu} \\
&\qquad + 2\sum_{\nu\geq\mathbf{0}}\sum_{i=1}^n (\eta_i-\nu_i)(\eta_i-\nu_i-1)(\partial_i f_\nu)\partial^{\eta-\nu-\boldsymbol{e}_i} \\
&= 2\sum_{i=1}^n\sum_{\nu\geq\boldsymbol{e}_i} \nu_i(\nu_i-2\eta_i-1) f_\nu\partial^{\eta-\nu} \\
&\qquad + 2\sum_{\nu\geq\mathbf{0}}\sum_{i=1}^n (\eta_i-\nu_i)(\eta_i-\nu_i-1)(\partial_i f_\nu)\partial^{\eta-\nu-\boldsymbol{e}_i}.
\end{aligned}$$

Hence we obtain

$$\Xi_\eta\Big(C\psi - 2\sum_{i=1}^{n} \eta_i(\eta_i + 1)\psi\Big) = 0,$$

and therefore the theorem follows. ∎

Remark 3.3. Given $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$, if $\Psi\mathrm{DO}^*_\eta$ is as in (3), we have $C(\Psi\mathrm{DO}^*_\eta) \subset \Psi\mathrm{DO}^*_\eta$; hence the Casimir operator C acts on the quotient space $\Psi\mathrm{DO}_\eta/\Psi\mathrm{DO}^*_\eta$ by

$$C(\Psi\mathrm{DO}^*_\eta + \psi) = \Psi\mathrm{DO}^*_\eta + C\psi \tag{11}$$

for all $\psi \in \Psi\mathrm{DO}_\eta$. Using (11) and Theorem 3.2, we see that

$$\begin{aligned} C(\Psi\mathrm{DO}^*_\eta + \psi) &= \Psi\mathrm{DO}^*_\eta + \Xi_\eta(C\psi) \\ &= \Psi\mathrm{DO}^*_\eta + 2\sum_{i=1}^{n} \eta_i(\eta_i + 1)\Xi_\eta(\psi) \\ &= 2\sum_{i=1}^{n} \eta_i(\eta_i + 1)(\Psi\mathrm{DO}^*_\eta + \psi), \end{aligned}$$

and therefore it follows that the Casimir operator C acts on $\Psi\mathrm{DO}_\eta/\Psi\mathrm{DO}^*_\eta$ as multiplication by $2\sum_{i=1}^{n} \eta_i(\eta_i + 1)$.

4. The lifting map

Since $\mathcal{F}$ may be regarded as a subspace of $\Psi\mathrm{DO}$, the action of $n\mathfrak{sl}(2,\mathbb{C})$ on $\Psi\mathrm{DO}$ determined by σ in (5) induces an action of $n\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{F}$. Relative to such actions we see easily that the symbol map $\Xi_\eta : \Psi\mathrm{DO}_\eta \to \mathcal{F}$ with $\eta \in \mathbb{Z}^n$ is $n\mathfrak{sl}(2,\mathbb{C})$-equivariant, that is,

$$\Xi_\eta(\sigma(W)\psi) = \sigma(W)\Xi_\eta(\psi)$$

for all $\psi \in \Psi\mathrm{DO}_\eta$ and $W \in n\mathfrak{sl}(2,\mathbb{C})$. In this section we construct a lifting of the symbol map Ξ_η that is $n\mathfrak{sl}(2,\mathbb{C})$-equivariant.

Via the inclusion map $\varepsilon_i : \mathfrak{sl}(2,\mathbb{C}) \to n\mathfrak{sl}(2,\mathbb{C})$ in (5), the representation σ of $n\mathfrak{sl}(2,\mathbb{C})$ on $\Psi\mathrm{DO}$ induces the representation $\sigma_i = \sigma \circ \varepsilon_i$ of $\mathfrak{sl}(2,\mathbb{C})$ in $\Psi\mathrm{DO}$ for each $i \in \{1, \dots, n\}$. If X, Y, H and X_i, Y_i, H_i are as in (4) and (6), the Casimir element C_i associated to σ_i is given by

$$\begin{aligned} C_i &= \sigma_i(H)^2/2 + \sigma_i(X)\sigma_i(Y) + \sigma_i(Y)\sigma_i(X) \\ &= \sigma(H_i)^2/2 + \sigma(X_i)\sigma(Y_i) + \sigma(Y_i)\sigma(X_i) \\ &= 2L_{i,2}^2 - L_{i,1}L_{i,3} - L_{i,3}L_{i,1}, \end{aligned} \tag{12}$$

where the $L_{i,j}$ are as in (9) for $1 \le j \le 3$.

Lemma 4.1. *Given $i \in \{1, \dots, n\}$, let C_i be the Casimir operator in* (12), *and let $\psi = \sum_{\nu \geq \mathbf{0}} f_\nu \partial^{\eta-\nu} \in \Psi\mathrm{DO}_\eta$ with $\eta \in \mathbb{Z}^n$. Then we have*

$$\begin{aligned}(C_i - 2\eta_i(\eta_i+1))\psi = 2\sum_{\nu \geq \boldsymbol{e}_i} \Big(&\nu_i(\nu_i - 2\eta_i - 1) f_\nu \\ &+ (\eta_i - \nu_i + 1)(\eta_i - \nu_i)(\partial_i f_{\nu - \boldsymbol{e}_i})\Big)\partial^{\eta-\nu}. \end{aligned} \tag{13}$$

Proof. Let $\psi = \sum_{\nu \geq \mathbf{0}} f_\nu \partial^{\eta-\nu} \in \Psi\mathrm{DO}_\eta$ with $\eta \in \mathbb{Z}^n$. Using the expressions of $L_{i,j}$ used in the proof of Theorem (3.2), we see that

$$\begin{aligned} C_i\psi &= \sum_{\nu \geq \mathbf{0}} \Big(2(\eta_i - \nu_i)(\eta_i - \nu_i + 1) f_\nu \partial^{\eta-\nu} \\ &\qquad + 2(\eta_i - \nu_i)(\eta_i - \nu_i - 1)(\partial_i f_\nu)\partial^{\eta - \nu - \boldsymbol{e}_i}\Big) \\ &= 2\sum_{\nu \geq \mathbf{0}} (\eta_i - \nu_i)(\eta_i - \nu_i + 1) f_\nu \partial^{\eta-\nu} \\ &\qquad + 2\sum_{\nu \geq \boldsymbol{e}_i} (\eta_i - \nu_i + 1)(\eta_i - \nu_i)(\partial_i f_{\nu - \boldsymbol{e}_i})\partial^{\eta-\nu} \\ &= 2\eta_i(\eta_i+1)\sum_{\nu \geq \mathbf{0}} f_\nu \partial^{\eta-\nu} \\ &\qquad + 2\sum_{\nu \geq \boldsymbol{e}_i} ((\eta_i - \nu_i)(\eta_i - \nu_i + 1) - \eta_i(\eta_i+1)) f_\nu \partial^{\eta-\nu} \\ &\qquad + 2\sum_{\nu \geq \boldsymbol{e}_i} (\eta_i - \nu_i + 1)(\eta_i + \nu_i)(\partial_i f_{\nu - \boldsymbol{e}_i})\partial^{\eta-\nu} \\ &= 2\eta_i(\eta_i+1)\psi + 2\sum_{\nu \geq \boldsymbol{e}_i}\Big(\nu_i(\nu_i - 2\eta_i - 1) f_\nu \\ &\qquad + (\eta_i - \nu_i + 1)(\eta_i + \nu_i)(\partial_i f_{\nu - \boldsymbol{e}_i})\Big)\partial^{\eta-\nu}. \end{aligned}$$

Hence the lemma follows. ■

Given $\omega \geq \mathbf{0}$ with $\omega \neq \mathbf{0}$, we define the linear maps $\mathcal{L}_\omega : \mathcal{F} \to \Psi\mathrm{DO}_\omega$, $\mathcal{L}_{-\omega} : \mathcal{F} \to \Psi\mathrm{DO}_{-\omega}$, and $\mathcal{L}_0 : \mathcal{F} \to \mathcal{F}$ by

$$\mathcal{L}_\omega(f) = \frac{\omega!(\omega - \mathbf{1})!}{(2\omega)!} \sum_{\mathbf{0} \leq \nu \leq \omega - \mathbf{1}} \frac{(2\omega - \nu)!}{\nu!(\omega-\nu)!(\omega - \nu - \mathbf{1})!}(\partial^\nu f)\partial^{\omega-\nu}, \tag{14}$$

$$\mathcal{L}_0(f) = f, \tag{15}$$

$$\mathcal{L}_{-\omega}(f) = \binom{2\omega - \mathbf{1}}{\omega} \sum_{\nu \geq \mathbf{0}} (-1)^{|\nu|} \frac{(\nu+\omega)!(\nu + \omega - \mathbf{1})!}{\nu!(\nu + 2\omega - \mathbf{1})!}(\partial^\nu f)\partial^{-\omega-\nu}, \tag{16}$$

for all $f \in \mathcal{F}$. Then we see easily that

$$\Xi_\eta \circ \mathcal{L}_\eta(f) = f$$

for each $f \in \mathcal{F}$ and $\eta \in \mathbb{Z}^n$; hence $\mathcal{L}_\eta$ is a lifting of Ξ_η.

Example 4.2. *We consider the case, where $n = 2$, $z_1 = z$, $z_2 = w$ and $\omega = (2,3) \in \mathbb{Z}^2_+$. Then, given $f = f(z,w) \in \mathcal{F}$, by* (14) *we have*

$$\begin{aligned}\mathcal{L}_\omega(f) &= \frac{2!3!1!2!}{4!6!}\sum_{i=0}^{1}\sum_{j=0}^{2}\frac{(4-i)!(6-j)!}{i!j!(2-i)!(3-j)!(1-i)!(2-j)!}\frac{\partial^{i+j} f}{\partial z^6 \partial w^j}(z,w)\partial_z^{2-i}\partial_w^{3-j}\\ &= f(z,w)\partial_z^2\partial_w^3 + \frac{\partial f}{\partial w}(z,w)\partial_z^2\partial_w^2 + \frac{1}{5}\frac{\partial^2 f}{\partial w^2}(z,w)\partial_z^2\partial_w + \frac{1}{2}\frac{\partial f}{\partial z}(z,w)\partial_z\partial_w^3\\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial z\partial w}(z,w)\partial_z\partial_w^2 + \frac{1}{10}\frac{\partial^3 f}{\partial z\partial w^2}(z,w)\partial_z\partial_w,\end{aligned}$$

where $\partial_z = \partial/\partial z$ and $\partial_w = \partial/\partial w$. Thus $\mathcal{L}_\omega(f)$ is in fact a differential operator rather than a pseudodifferential operator. On the other hand, using (16) *and*

$$\binom{2\omega - \mathbf{1}}{\omega} = \binom{(3,5)}{(2,3)} = \binom{3}{2}\binom{5}{3} = 30,$$

we see that $\mathcal{L}_{-\omega}(f)$ is a pseudodifferential operator given by

$$\mathcal{L}_{-\omega}(f) = 30\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\frac{(i+2)!(j+3)!i+1)!(j+2)!}{i!j!(i+3)!(j+5)!}\frac{\partial^{i+j} f}{\partial z^6\partial w^j}(z,w)\partial_z^{-i}\partial_w^{-j}.$$

Theorem 4.3. *For each $\eta \in \mathbb{Z}^n$ the linear map $\mathcal{L}_\eta : \mathcal{F} \to \Psi\mathrm{DO}_\eta$ given by* (14) *is $n\mathfrak{sl}(2,\mathbb{C})$-equivariant, that is,*

$$\mathcal{L}_\eta(\sigma(W)f) = \sigma(W)\mathcal{L}_\eta(f) \tag{17}$$

for all $W \in n\mathfrak{sl}(2,\mathbb{C})$ and $f \in \mathcal{F}$, where σ is as in (7).

Proof. We shall show that a lifting $\mathcal{L}_\eta$ of Ξ_η satisfying (17) must be given by the formulas (14), (15), and (16). If $\mathcal{L}_\eta$ is such a map, using the embeddings $\varepsilon_i : \mathfrak{sl}(2,\mathbb{C}) \to n\mathfrak{sl}(2,\mathbb{C})$, we see that $\mathcal{L}_\eta$ is $\mathfrak{sl}(2,\mathbb{C})$-equivariant via $\sigma_i = \sigma \circ \varepsilon_i$ for each $i \in \{1,\dots,n\}$. Using Lemma 4.1 and an argument similar to the one in Remark 3.3, we also see that the Casimir operator C_i operates on the quotient space $\Psi\mathrm{DO}_\eta/\Psi\mathrm{DO}^*_\eta$ as multiplication by $2\eta_i(\eta_i+1)$. Since $\mathcal{F}$ is isomorphic to $\Psi\mathrm{DO}_\eta/\Psi\mathrm{DO}^*_\eta$, we have

$$C_i(\mathcal{L}_\eta(f)) = \mathcal{L}_\eta(C_i f) = 2\eta_i(\eta_i+1)\mathcal{L}_\eta(f) \tag{18}$$

for $1 \le i \le n$ and $f \in \mathcal{F}$. Let $\omega \ge \mathbf{0}$ with $\omega \ne \mathbf{0}$, and set

$$\mathcal{L}_\omega(f) = \sum_{\nu \ge \mathbf{0}} f_\nu \partial^{\omega-\nu}.$$

If $\nu \ge \boldsymbol{e}_i$, then by (13) and (18) we have

$$\nu_i(\nu_i - 2\omega_i - 1)f_\nu + (\omega_i - \nu_i + 1)(\omega_i - \nu_i)(\partial_i f_{\nu-\boldsymbol{e}_i}) = 0.$$

Hence we obtain

$$f_\nu = -\frac{(\omega_i - \nu_i + 1)(\omega_i - \nu_i)}{\nu_i(\nu_i - 2\omega_i - 1)}\partial_i f_{\nu-\boldsymbol{e}_i}$$

for all $\nu \geq \boldsymbol{e}_i$. Thus, if $\nu_i \leq \omega_i - 1$, by iteration we have

$$\begin{aligned} f_\nu &= (-1)^{\nu_i} \frac{(\omega_i - \nu_i + 1) \cdots \omega_i \cdot (\omega_i - \nu_i) \cdots (\omega_i - 1)}{\nu_i!(\nu_i - 2\omega_i - 1) \cdots (-2\omega_i)} \partial_i^{\nu_i} f_{\nu - \nu_i \boldsymbol{e}_i} \\ &= \frac{(\omega_i!/(\omega_i - \nu_i)!)((\omega_i - 1)!/(\omega_i - \nu_i - 1)!)}{\nu_i!(2\omega_i)!/(2\omega_i - \nu_i)!} \partial_i^{\nu_i} f_{\nu - \nu_i \boldsymbol{e}_i} \\ &= \frac{\omega_i!(\omega_i - 1)!}{(2\omega_i)!} \cdot \frac{(2\omega_i - \nu_i)!}{\nu_i!(\omega_i - \nu_i)!(\omega_i - \nu_i - 1)!} \partial_i^{\nu_i} f_{\nu - \nu_i \boldsymbol{e}_i}. \end{aligned}$$

If $\nu_i > \omega_i - 1$, then we have $f_\nu = 0$. Hence we obtain

$$f_\nu = \frac{\omega!(\omega - \mathbf{1})!}{(2\omega)!} \cdot \frac{(2\omega - \nu)!}{\nu!(\omega - \nu)!(\omega - \nu - \mathbf{1})!} \partial^\nu f_{\mathbf{0}}$$

if $\mathbf{0} \leq \nu \leq \omega - \mathbf{1}$, and $f_\nu = 0$ otherwise. Thus it follows that

$$\mathcal{L}_\omega(f) = \frac{\omega!(\omega - \mathbf{1})!}{(2\omega)!} \sum_{\mathbf{0} \leq \nu \leq \omega - \mathbf{1}} \frac{(2\omega - \nu)!}{\nu!(\omega - \nu)!(\omega - \nu - \mathbf{1})!} (\partial^\nu f_{\mathbf{0}}) \partial^{\omega - \nu}.$$

However, since $\mathcal{L}_\omega$ is a lifting of Ξ_ω, we have $f_{\mathbf{0}} = \Xi_\omega(\mathcal{L}_\omega f) = f$, and therefore we obtain (14). On the other hand, if $\mathcal{L}_{-\omega}(f) = \sum_{\nu \geq \mathbf{0}} h_\nu \partial^{-\omega - \nu} \in \Psi\mathrm{DO}_{-w}$, then by (13) and (18) we have

$$\nu_i(\nu_i + 2\omega_i - 1) h_\nu + (\omega_i + \nu_i - 1)(\omega_i + \nu_i)(\partial_i h_{\nu - \boldsymbol{e}_i}) = 0$$

for $1 \leq i \leq n$ and $f \in \mathcal{F}$. Hence we obtain

$$h_\nu = -\frac{(\omega_i + \nu_i - 1)(\omega_i + \nu_i)}{\nu_i(\nu_i + 2\omega_i - 1)} \partial_i h_{\nu - \boldsymbol{e}_i}$$

for all $\nu \geq \boldsymbol{e}_i$. Thus by iteration we see that

$$\begin{aligned} h_\nu &= (-1)^{\nu_i} \frac{((\nu_i + \omega_i)!/\omega_i!)((\nu_i + \omega_i - 1)!/(\omega_i - 1)!)}{\nu_i!(\nu_i + 2\omega_i - 1)!/(2\omega_i - 1)!} \partial_i^{\nu_i} h_{\nu - \nu_i \boldsymbol{e}_i} \\ &= (-1)^{\nu_i} \binom{2\omega_i - 1}{\omega_i} \frac{(\nu_i + \omega_i)!(\nu_i + \omega_i - 1)!}{\nu_i!(\nu_i + 2\omega_i - 1)!} \partial_i^{\nu_i} h_{\nu - \nu_i \boldsymbol{e}_i}. \end{aligned}$$

Applying this for each i, we obtain

$$h_\nu = (-1)^{|\nu|} \binom{2\omega - \mathbf{1}}{\omega} \frac{(\nu + \omega)!(\nu + \omega - \mathbf{1})!}{\nu!(\nu + 2\omega - \mathbf{1})!} \partial^\nu h_{\mathbf{0}}.$$

Hence it follows that

$$\mathcal{L}_{-\omega}(f) = \binom{2\omega - \mathbf{1}}{\omega} \sum_{\nu \geq \mathbf{0}} (-1)^{|\nu|} \frac{(\nu + \omega)!(\nu + \omega - \mathbf{1})!}{\nu!(\nu + 2\omega - \mathbf{1})!} (\partial^\nu h_{\mathbf{0}}) \partial^{-\omega - \nu},$$

and therefore we obtain (16) by using $h_{\mathbf{0}} = \Xi_{-\omega}(\mathcal{L}_{-\omega}(f)) = f$. Since a lifting $\mathcal{L}_{\mathbf{0}}$ with $\mathcal{L}_{\mathbf{0}}(\mathcal{F}) \subset \mathcal{F}$ should obviously be the identity map, the proof of the theorem is complete. ∎

Remark 4.4. Let Γ be a discrete subgroup of $SL(2,\mathbb{C})^n$. Then for each $\eta \in \mathbb{Z}^n$ the exact sequence in (3) induces the short exact sequence

$$0 \to \Psi\mathrm{DO}_\eta^{*\Gamma} \to \Psi\mathrm{DO}_\eta^\Gamma \xrightarrow{\Xi_\eta} \mathcal{F}^\Gamma \to 0, \tag{19}$$

where $(\cdot)^\Gamma$ denotes the subset of Γ-fixed elements. By Theorem 4.3 the linear map $\mathcal{L}_\eta : \mathcal{F} \to \Psi\mathrm{DO}_\eta$ is also equivariant with respect to the actions of the Lie group $SL(2,\mathbb{C})^n$ of the Lie algebra $n\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{F}$ and on $\Psi\mathrm{DO}_\eta$. Thus it follows that the short exact sequence (19) splits. If $\eta \geq \mathbf{0}$, then $\mathcal{F}^\Gamma$ is the space of Hilbert modular forms for Γ of weight -2η, which was considered in [7].

5. Concluding remarks

We have discussed the action of the Lie group $SL(2,\mathbb{R})^n$ on pseudodifferential operators of n variables in terms of the corresponding Lie algebra action. As was described in the introduction, if Γ_n is a discrete subgroup of $SL(2,\mathbb{R})^n$, each Γ_n-invariant pseudodifferential operators of n variables can be identified with a sequence of Hilbert modular forms, which are essentially modular forms of several variables, for Γ_n. Using this correspondence and the role of pseudodifferential operators in soliton theory, we see that there is at least an indirect link between soliton equations and modular forms. In fact, Olver and Sanders [10] also discussed a connection between the Rankin-Cohen brackets for modular forms and solitons via Hirota operators. It would be interesting to search for more direct connections between modular forms and soliton equations and their solutions.

In another direction, we can consider nonholomorphic modular forms. In a recent monograph [13], Unterberger, among other things, discussed the Rankin-Cohen brackets for nonholomorphic modular forms and their relation with quantization theory through harmonic analysis. It might be worth studying actions of $SL(2,\mathbb{R})^n$ or its Lie algebra on pseudodifferential operators of n variables with nonholomorphic coefficients and exploring the possibility of relating them with nonholomorphic modular forms.

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