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Lie Algebras Graded by the Root System BC$_1$

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Abstract. We classify the Lie algebras that are graded by the nonreduced root system BC$_1$ and determine their central extensions, derivations, and invariant forms.

1. Introduction

Lie algebras graded by reduced root systems

1.1 Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over a field $\mathbf{F}$ of characteristic zero with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ relative to a split Cartan subalgebra $\mathfrak{h}$. Such a Lie algebra $\mathfrak{g}$ is the $\mathbf{F}$-analogue of a finite-dimensional simple complex Lie algebra. Motivated by the study of intersection matrix Lie algebras which arose in Slodowy's work [33] on singularities, Berman and Moody [19] initiated the investigation of Lie algebras graded by the root system Δ. Following them we say

Definition 1.2. A Lie algebra L over $\mathbf{F}$ is *graded by the (reduced) root system* Δ or is Δ-*graded* if

(i) L contains as a subalgebra a finite-dimensional split simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;

(ii) $L = \bigoplus_{\mu\in\Delta\cup\{0\}} L_\mu$, where $L_\mu = \{x \in L \mid [h,x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$ for $\mu \in \Delta \cup \{0\}$; and

(iii) $L_0 = \sum_{\mu\in\Delta} [L_\mu, L_{-\mu}]$.

The subalgebra $\mathfrak{g}$ is called the grading subalgebra of L.

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1.3 The Δ-graded Lie algebras have been determined up to central extensions by Berman and Moody [19] for $\Delta = \mathrm{A}_r, (r \geq 2)$, D_r, E_6, E_7, and E_8; by Benkart and Zelmanov ([36], [14], [15]) for $\Delta = \mathrm{A}_1$, B_r, C_r, F_4, and G_2 (see also [7] for C_2); and by Neher [29], who studied the Δ-graded Lie algebras for $\Delta \neq \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$ by Jordan algebra methods. Their central extensions have been described by Allison, Benkart, Gao [7] (see also [16], [17], and [18] for earlier work on types A_r, D_r, E_6, E_7, E_8). As a result, the Δ-graded Lie algebras are classified completely up to isomorphism. The derivations and invariant forms of all Δ-graded Lie algebras have been computed in [12], and a criterion for their simplicity can be found in [13].

In this work we address the problem of studying the Lie algebras graded by the nonreduced root systems BC_r, and in particular, by the root system BC_1.

Lie algebras graded by the nonreduced root systems BC_r

1.4 Assume $\mathfrak{g}$ is a split "simple"* Lie algebra whose root system relative to a split Cartan subalgebra $\mathfrak{h}$ is of type B_r, C_r, or D_r for some $r \geq 1$. Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta_X} \mathfrak{g}_\mu$ where $X = \mathrm{B}, \mathrm{C}$, or D, and

$$\begin{aligned} \Delta_B &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm\epsilon_i \mid i = 1, \ldots, r\}, \\ \Delta_C &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm 2\epsilon_i \mid i = 1, \ldots, r\}, \\ \Delta_D &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\}. \end{aligned} \tag{1.5}$$

The set

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq r\} \cup \{\pm\epsilon_i,\ \pm 2\epsilon_i \mid i = 1, \ldots, r\}, \tag{1.6}$$

which is contained in the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$, is a root system of type BC_r in the sense of Bourbaki [20, Chap. VI]. It is nonreduced since both ϵ_i and $2\epsilon_i$ are roots. The following notion of a Lie algebra graded by the root system BC_r was introduced in [8]:

Definition 1.7. A Lie algebra L over a field $\mathbf{F}$ of characteristic zero is *graded by the root system* BC_r or is BC_r*-graded* if

(i) L contains as a subalgebra a finite-dimensional split "simple" Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Delta_X} \mathfrak{g}_\mu$ whose root system relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ is Δ_X, $X = \mathrm{B}, \mathrm{C}$, or D as in (1.5);

(ii) $L = \bigoplus_{\mu \in \Delta \cup \{0\}} L_\mu$, where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$ for $\mu \in \Delta \cup \{0\}$ and Δ is the root system BC_r in (1.6); and

(iii) $L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$.

It follows from (ii) that $[L_\mu, L_\nu] \subseteq L_{\mu+\nu}$ if $\mu + \nu \in \Delta \cup \{0\}$ and $[L_\mu, L_\nu] = (0)$ otherwise.

1.8 We refer to $\mathfrak{g}$ as the *grading subalgebra* of L and say L is BC_r*-graded with grading subalgebra* $\mathfrak{g}$ *of type* X_r (where X = B, C, or D) to mean that the

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* In all cases but two, $\mathfrak{g}$ is simple. When $\mathfrak{g}$ is of type $\mathrm{D}_2 = \mathrm{A}_1 \times \mathrm{A}_1$ then $\mathfrak{g} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and when $\mathfrak{g}$ is of type D_1, then $\mathfrak{g} = \mathfrak{h}$, which is 1-dimensional.

root system of $\mathfrak{g}$ is of type X_r. The classification of finite-dimensional simple Lie algebras over nonalgebraically closed fields of characteristic zero which contain an $\mathfrak{sl}_2$ subalgebra (or equivalently, an ad-nilpotent element) involves Lie algebras that are BC_r-graded relative to a split Cartan subalgebra. For such Lie algebras, it is possible to select a grading subalgebra of any one of the types B_r, C_r, or D_r. In [32], Seligman works with a grading subalgebra of type C_r. However, for (possibly infinite-dimensional and possibly non-simple) general BC_r-graded Lie algebras, the existence of a grading subalgebra of type B_r or D_r does not guarantee the existence of a grading subalgebra of type C_r. For that reason we have broadened the concept of a BC_r-graded Lie algebra here and previously in [8] to include those of types B_r and D_r.

1.9 When viewed as a $\mathfrak{g}$-module under the adjoint action, a BC_r-graded Lie algebra L decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules. There is one possible isotypic component corresponding to each root length and one corresponding to 0 (the sum of the trivial $\mathfrak{g}$-modules). Thus, if $r \geq 2$ and $\mathfrak{g}$ is not of type D_2, there are up to four isotypic components which can be parametrized by subspaces A, B, C, D, so that the decomposition is given by

$$L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D. \tag{1.10}$$

Here V is an n-dimensional space with a nondegenerate bilinear form $(\,|\,)$ which is symmetric of maximal Witt index or is skew-symmetric. We write $(u|v) = \zeta(v|u)$ where $\zeta = 1$ if the form is symmetric, and $\zeta = -1$ if it is skew-symmetric. Then $\mathfrak{g}$ can be realized as the space

$$\mathfrak{g} = \{x \in \operatorname{End} V \mid (xu|v) = -(u|xv) \text{ for all } u, v \in V\} \tag{1.11}$$

of all skew-symmetric transformations relative to the form. If $\zeta = -1$ (necessarily $n = 2r$), then $\mathfrak{g}$ is a split simple Lie algebra of type C_r. If $\zeta = 1$, then $\mathfrak{g}$ is of type B_r when $n = 2r+1$, and $\mathfrak{g}$ is of type D_r when $n = 2r$. The $\mathfrak{g}$-module $\mathfrak{s}$ can be identified with the space,

$$\mathfrak{s} = \{s \in \operatorname{End} V \mid (su|v) = (u|sv) \text{ for all } u, v \in V, \text{ and } \operatorname{tr}(s) = 0\}, \tag{1.12}$$

of all symmetric transformations of trace zero relative to the form. In the exceptional case that L is BC_2-graded of type D_2, then

$$L = (\mathfrak{g}^{(1)} \otimes A^{(1)}) \oplus (\mathfrak{g}^{(2)} \otimes A^{(2)}) \oplus (\mathfrak{s} \otimes B) \oplus (V \otimes C) \oplus D, \tag{1.13 D_2}$$

where $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, V is a 4-dimensional space with a nondegenerate symmetric bilinear form $(\,|\,)$ of maximal Witt index, and $\mathfrak{g}$, $\mathfrak{s}$ are as above.

1.14 Any Lie algebra having a decomposition as in (1.10) and satisfying (iii) of Definition 1.7 is BC_r-graded. In particular, a Lie algebra graded by the root system B_r, C_r, or D_r is also graded by BC_r. In the first case, $\mathfrak{g}$ is a split simple Lie algebra of type B_r and $B = (0)$; while in the second, $\mathfrak{g}$ is a split simple Lie algebra of type C_r and $C = (0)$; and in the third, $\mathfrak{g}$ is a split simple Lie algebra of type D_r and $B = (0) = C$. The twisted affine Lie algebras of type $\mathrm{A}_{2r}^{(2)}$, $r \geq 1$, which have a realization as $(\mathfrak{g}\otimes\mathbf{F}[t^{\pm 2}])\oplus(\mathfrak{s}\otimes t\mathbf{F}[t^{\pm 2}])\oplus\mathbf{F}z$, where $\mathfrak{g}$ is of type B_r, $\mathbf{F}[t^{\pm 1}]$ is the algebra of Laurent polynomials in t, and $\mathbf{F}z$ is a nonsplit center, are BC_r-graded of type B_r (see [25], Chap. 8). The twisted affine algebras of type $\mathrm{A}_{2r-1}^{(2)}$, $r \geq 2$, can be realized as $(\mathfrak{g} \otimes \mathbf{F}[t^{\pm 2}]) \oplus (\mathfrak{s} \otimes t\mathbf{F}[t^{\pm 2}]) \oplus \mathbf{F}z$, where $\mathfrak{g}$ is of type C_r and z is central, so they are C_r-graded, hence BC_r-graded. Similarly, the twisted affine Lie algebras of type $\mathrm{D}_{r+1}^{(2)}$ can be identified with $(\mathfrak{g} \otimes \mathbf{F}[t^{\pm 2}]) \oplus (V \otimes t\mathbf{F}[t^{\pm 2}]) \oplus \mathbf{F}z$, where $\mathfrak{g}$ is of type B_r, so they are B_r-graded and hence, BC_r-graded.

The extended affine Lie algebras (previously termed quasisimple Lie algebras) are natural generalizations of the affine Lie algebras. The cores of extended affine Lie algebras of type BC_r are Lie algebras which are BC_r-graded, (see [6], [24]). Thus, the results of [8] and of our present study on BC_1-graded Lie algebras are expected to shed light on the extended affine Lie algebras of type BC_r and to facilitate their classification. As we have mentioned earlier, BC_r-graded Lie algebras arise naturally in the classification of finite-dimensional simple Lie algebras over nonalgebraically closed fields of characteristic zero (see [32] and in particular, see [1], [5] for BC_1-graded algebras). The "odd symplectic" Lie algebras of Gelfand and Zelevinsky [23] and of Proctor [30] and certain of the intersection matrix algebras of Slodowy [33] provide further interesting examples of BC_r-graded Lie algebras. Consequently, the notion of a BC_r-graded Lie algebra is an important unifying concept encompassing a diverse array of Lie algebras.

1.15 The classification problem for BC_r-graded Lie algebras amounts to determining the coordinating spaces A, B, C, D (or $A^{(1)}$, $A^{(2)}$, B, C, D) and the multiplication in L. This is accomplished in [8] for $r \geq 2$. The BC_1 case differs from the higher rank cases in that (as we discuss below) the Lie algebras graded by BC_1 have many additional degeneracies, and because of that, they require more general coordinates. Indeed, the phenomenon that small rank algebras exhibit degeneracies already is very apparent in the work [8], where roughly half the monograph is devoted to the Lie algebras graded by the root system BC_2.

1.16 In this paper we classify the BC_1-graded Lie algebras and determine their derivations, central extensions, and invariant forms. An important role is played by 5-graded Lie algebras. There are several algebraic structures connected with 5-graded Lie algebras - structurable algebras, J-ternary algebras, and Kantor pairs. All are generalizations of Jordan algebras. For each of them there is a Lie algebra construction which generalizes the well-known Tits-Kantor-Koecher construction and which produces 5-graded Lie algebras. The need for more general coordinates for BC_1-graded Lie algebras requires us to introduce a new class of algebraic structures, termed *Jordan-Kantor pairs*, and to give an even more general Tits-Kantor-Koecher construction of an associated Lie algebra. Simple $\mathbf{Z}$-graded Lie algebras have been studied extensively (e.g., [26], [35], [34]). In [35] they are shown to arise from associative algebras or to be finite-dimensional exceptional Lie algebras (possibly after a change of the base field). The 5-graded Lie algebras we encounter in this work are not assumed to be simple or finite-dimensional, and because of this, their description involves the more general algebraic structures defined in (3.1).

1.17 Our main results are

(i) an introduction of the notion of a Jordan-Kantor pair (in (3.1)) and its Tits-Kantor-Koecher Lie algebra (in Section 4), and an investigation of their properties;

(ii) the classification of BC_1 graded Lie algebras of type D_1 (Theorem 5.24), of type C_1 (Theorem 6.34), and of type B_1 (Theorem 7.20);

(iii) a determination of the derivations of all BC_1-graded Lie algebras (Theorem 8.6);

(iv) a description of the invariant forms on all BC_1-graded Lie algebras (Theorem 9.10).

1.18 Some results in this paper have been presented at the International Conference on Jordan Structures, Universidad de Málaga, Spain, June 1997, and the Workshop in Algebra, Ottawa-Carleton Institute of Mathematics and Statistics, Canada, September 1996. We take this opportunity to thank the organizers of those conferences. Also our special thanks go to Bruce Allison for several helpful discussions.

2. BC_1-graded Lie algebras.

2.1 Suppose L is a BC_1-graded Lie algebra with grading subalgebra $\mathfrak{g}$, and consider the decomposition of L into irreducible $\mathfrak{g}$-modules relative to $\operatorname{ad}\mathfrak{g}$. Assume $V, \mathfrak{g}, \mathfrak{s}$ are as in Section 1 (see (1.11) and (1.12)). When the grading subalgebra $\mathfrak{g}$ is of type B_1, $\mathfrak{g} \cong V$ as $\mathfrak{g}$-modules. When $\mathfrak{g}$ is of type C_1, then $\mathfrak{s} = (0)$; and when $\mathfrak{g}$ is of type D_1, $\mathfrak{g} = \mathbf{F}h$. This means that any BC_1-graded Lie algebra has one of the following forms,

$$L = \begin{cases} (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus D & \text{type } \mathrm{B}_1, \\ (\mathfrak{g} \otimes A) \oplus (V \otimes C) \oplus D & \text{type } \mathrm{C}_1, \\ L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 & \text{type } \mathrm{D}_1. \end{cases} \tag{2.2}$$

For D_1, $\mathfrak{g} = \mathbf{F}h$, and so the decomposition into $\mathfrak{g}$-modules is simply the decomposition into eigenspaces relative to $\operatorname{ad} h$. For all types, we identify the Lie algebra $\mathfrak{g}$ with $\mathfrak{g} \otimes 1$ so that $[x, y]$ corresponds to $[x \otimes 1, y \otimes 1] = [x, y] \otimes 1$.

2.3 When the grading subalgebra is of type B_1 or C_1, then $\mathfrak{g} \cong \mathfrak{sl}_2$, and L decomposes into irreducible $\mathfrak{g}$-modules of dimensions 1, 3, 5 in the B-case and of dimensions 1, 2, 3 in the C-case. In writing (2.2), we have collected the isomorphic $\mathfrak{g}$-summands, so in the B-case, $\mathfrak{s} \otimes B$ is the sum of all the 5-dimensional $\mathfrak{g}$-modules, and in the C-case, $V \otimes C$ is the sum of all the 2-dimensional $\mathfrak{g}$-modules. For both cases, $\mathfrak{g} \otimes A$ is the sum of all the 3-dimensional $\mathfrak{g}$-modules, and D is the sum of the 1-dimensional $\mathfrak{g}$-modules. The following two results of Allison describe Lie algebras which have such decomposition properties in terms of J-ternary and structurable algebras. The definition of J-ternary algebra can be found in (6.2) below and of structurable algebra in (7.6). In stating these results we use the notion of an $\mathfrak{sl}_2$-*triple* to mean a triple of elements e, f, h which satisfy the standard $\mathfrak{sl}_2$-relations, $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.

Proposition 2.4. *[2, Thm. 1] (see also [11]) Assume $\mathfrak{L}$ is a Lie algebra which contains a subalgebra $\mathfrak{T} = \langle e, f, h\rangle$ generated by an $\mathfrak{sl}_2$-triple such that $\mathfrak{T}$ centralizes no proper ideals of $\mathfrak{L}$. Assume $\mathfrak{L}$ as an $\mathfrak{T}$-module is the direct sum of irreducible $\mathfrak{T}$-modules of dimensions 1, 2, 3, and $\mathfrak{L}$ is generated by the -2, -1, 1, and 2 eigenspaces for* $\operatorname{ad} h$. *Then $\mathfrak{L} \cong K(\mathfrak{J}, \mathfrak{M}) = \widetilde{\mathfrak{J}} \oplus \widetilde{\mathfrak{M}} \oplus \mathfrak{instr}(\mathfrak{J}, \mathfrak{M}) \oplus \mathfrak{M} \oplus \mathfrak{J}$, the Tits-Kantor-Koecher Lie algebra of a J-ternary algebra $(\mathfrak{J}, \mathfrak{M})$.*

Proposition 2.5. *([4]) Assume $\mathfrak{L}$ is a Lie algebra which contains a subalgebra $\mathfrak{T} = \langle e, f, h\rangle$ generated by an $\mathfrak{sl}_2$-triple such that $\mathfrak{T}$ centralizes no proper ideals of $\mathfrak{L}$. Assume $\mathfrak{L}$ as a $\mathfrak{T}$-module is the direct sum of irreducible $\mathfrak{T}$-modules of dimensions 1, 3, 5, and $\mathfrak{L}$ is generated by the -2 and 2 eigenspaces for* $\operatorname{ad} h$. *Then $\mathfrak{L} \cong \mathcal{K}(X, {}^-) = \widetilde{S} \oplus \widetilde{X} \oplus \mathfrak{instr}(X, {}^-) \oplus X \oplus S$, the Tits-Kantor-Koecher Lie algebra of a structurable algebra $(X, {}^-)$ with involution "$^-$" and skew-hermitian elements S relative to $^-$.*

Using those propositions we may conclude the following about a BC_1-graded Lie algebra L modulo its center $Z(L)$.

Theorem 2.6. *Let L be a BC_1-graded Lie algebra with grading subalgebra $\mathfrak{g}$.*

(i) *When $\mathfrak{g}$ is of type B_1, then $L = (\mathfrak{g} \otimes A) \oplus (\mathfrak{s} \otimes B) \oplus D$ and $L/Z(L) \cong K(\mathfrak{a}, \eta) = \widetilde{B} \oplus \widetilde{\mathfrak{a}} \oplus \mathfrak{instr}(\mathfrak{a}, \eta) \oplus \mathfrak{a} \oplus B$, where $\mathfrak{a} = A \oplus B$ is a structurable algebra with involution η, and A is the set of symmetric elements and B the set of skew-hermitian elements relative to η.*

(ii) *When $\mathfrak{g}$ is of type C_1, then $L = (\mathfrak{g} \otimes A) \oplus (V \otimes C) \oplus D$ and $L/Z(L) \cong K(A, C) = \widetilde{A} \oplus \widetilde{C} \oplus \mathfrak{instr}(A, C) \oplus C \oplus A$ where (A, C) is a J-ternary algebra.*

2.7 Since $Z(L)$ is contained in D, we can use the weight space decomposition of $L/Z(L)$ relative to $\operatorname{ad} h$ to realize the isomorphisms in (i) and (ii) more conceptually. Indeed, we have the following identifications:

$$\text{(i)} \qquad \begin{array}{cc} \mathfrak{s}_{-2\epsilon_1} \otimes B \to \widetilde{B}, & \mathfrak{s}_{2\epsilon_1} \otimes B \to B, \\ \big(\mathfrak{g}_{-\epsilon_1} \otimes A\big) \oplus \big(\mathfrak{s}_{-\epsilon_1} \otimes B\big) \to \widetilde{\mathfrak{a}}, & \big(\mathfrak{g}_{\epsilon_1} \otimes A\big) \oplus \big(\mathfrak{s}_{\epsilon_1} \otimes B\big) \to \mathfrak{a}, \\ \multicolumn{2}{c}{\big(\mathfrak{g}_0 \otimes A\big) \oplus \big(\mathfrak{s}_0 \otimes B\big) \oplus D/Z(L) \to \mathfrak{instr}(\mathfrak{a}, \eta),} \end{array}$$

$$\text{(ii)} \qquad \begin{array}{cc} \mathfrak{g}_{-2\epsilon_1} \otimes A \to \widetilde{A}, & \mathfrak{g}_{2\epsilon_1} \otimes A \to A, \\ V_{-\epsilon_1} \otimes C \to \widetilde{C}, & V_{\epsilon_1} \otimes C \to C, \\ \multicolumn{2}{c}{\big(\mathfrak{g}_0 \otimes A\big) \oplus D/Z(L) \to \mathfrak{instr}(A, C).} \end{array}$$

2.8 In Sections 3 and 4, we study Jordan-Kantor pairs, BC_1-graded Lie algebras of type D_1, and more generally, 5-graded Lie algebras. Section 5 investigates central extensions of 5-graded Lie algebras culminating in a classification of the BC_1-graded Lie algebras of type D_1. The derivations and invariant forms of these Lie algebras (in fact of all BC_1-graded Lie algebras) are computed in Sections 8 and 9. In Sections 6 and 7, we return to BC_1-graded Lie algebras of type B_1 and C_1, J-ternary algebras, and structurable algebras. There we apply the results on BC_1-graded Lie algebras of type D_1 to gain insight into these Lie algebras and determine their classification. This is possible because of the following result which relates BC_1-graded Lie algebras of various types. This theorem employs the notion of a *grading element* in a $\mathbf{Z}$-graded Lie algebra $L = \bigoplus_{i\in\mathbf{Z}} L_i$. By that we mean an element h such that $\mathrm{ad}\, h\,|_{L_i} = i\ \mathrm{Id}|_{L_i}$.

Theorem 2.9. *For a Lie algebra L, the following are equivalent:*

(i) $L = \bigoplus_{i=-2}^{2} L_i$ *is a 5-graded Lie algebra with a grading element $h \in L_0$ and with $L_0 = [L_{-1}, L_1] + [L_{-2}, L_2]$;*

(ii) *L is a BC_1-graded Lie algebra of type D_1;*

(iii) *L is a BC_1-graded Lie algebra;*

(iv) *L is Δ-graded Lie algebra.*

Moreover, in this case

(a) $L = \bigoplus_{i=-2}^{2} L_i$ *is a BC_1-graded Lie algebra of type B_1 if and only if $L_{-1}\oplus L_0\oplus L_1$ contains an $\mathfrak{sl}_2$-triple $\langle e, f, h\rangle$ with $(1/2)h$ being a grading element of L.*

(b) $L = \bigoplus_{i=-2}^{2} L_i$ *is a BC_1-graded Lie algebra of type C_1 if and only if $L_{-2} \oplus L_0 \oplus L_2$ contains an $\mathfrak{sl}_2$-triple $\langle e, f, h\rangle$ with h being a grading element of L.*

Proof. The equivalence of (i) and (ii) is apparent. For (iii) $\Rightarrow$ (i) observe that if L is BC_1-graded of type B_1, then L decomposes into modules for $\mathfrak{g} \cong \mathfrak{sl}_2 = \langle e, f, h\rangle$ of dimensions 1, 3, 5. From the representation theory of $\mathfrak{sl}_2$ we know then that $(1/2)\,\mathrm{ad}\, h$ has eigenvalues $-2, -1, 0, 1, 2$ on L. This gives L a 5-grading relative to the grading element $(1/2)h$. The condition $L_0 = [L_{-1}, L_1] + [L_{-2}, L_2]$ is (iii) of Definition 1.7. Therefore, L is BC_1-graded of type D_1. The case when L is BC_1-graded of type C_1 can be done similarly.

For the proof of (iv) $\Rightarrow$ (ii), suppose L is a Δ-graded Lie algebra with grading subalgebra $\mathfrak{g}$. If L is a BC_1-graded Lie algebra of type D_1, the result is evident. Assume otherwise, so that the grading subalgebra $\mathfrak{g}$ is a split finite-dimensional semisimple algebra. In [3, Thm. 10] it is shown that there exists an $\mathfrak{sl}_2$-triple $\langle e, f, h\rangle$ in $\mathfrak{g}$ such that eigenvalues of $\mathrm{ad}\, h$ on $\mathfrak{g}$ belong to $\{-4, -2, 0, 2, 4\}$, and the spaces $\mathfrak{g}_i$, $i \neq 0$, generate $\mathfrak{g}$. (This was done in [3] for $\mathfrak{g}$ simple. But when $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$ is of type D_2, then $\mathfrak{g}^{(i)} \cong \mathfrak{sl}_2 = \langle e_i, f_i, h_i\rangle$, $i = 1, 2$, and $e = e_1 + e_2, f = f_1 + f_2, h = h_1 + h_2$ have the desired properties.) For this choice of h, the Lie algebra L is graded by the eigenspaces of $(1/2)\,\mathrm{ad}\, h$: $L = \bigoplus_{i=-2}^{2} L_i$. The subalgebra M of L generated by the spaces L_i, $i \neq 0$ is an ideal of L which contains $\mathfrak{g}$. Now it follows that $L \subseteq M$ and that L is BC_1-graded of type D_1. ■

3. Jordan-Kantor pairs

3.1 All vector spaces are assumed to be over a field $\mathbf{F}$ of characteristic zero. This restriction on the characteristic is to insure that our BC_1-graded algebras have a decomposition into $\mathfrak{sl}_2$-modules as in (2.2). However, the results in our paper which make no reference to BC_1-graded algebras are valid under the weaker assumption that the characteristic of $\mathbf{F}$ is not equal to 2 or 3.

Henceforth $\sigma \in \{-,+\}$ with the obvious interpretation of $-\sigma$. Recall that a pair of spaces $J = (J_-, J_+)$ is said to be a (linear) *Jordan pair* provided there is a trilinear product $\{a,b,c\} = D_{a,b}c \in J_\sigma$ for $a,c \in J_\sigma$ and $b \in J_{-\sigma}$ such that for all $d \in J_{-\sigma}$,

$$\textbf{(JP)} \qquad \{a,b,c\} = \{c,b,a\} \ \text{ and } \ [D_{a,b}, D_{c,d}] = D_{D_{a,b}c,d} - D_{c,D_{b,a}d}.$$

The equivalence of this definition with the definition of Jordan pair given by Loos in [28] follows from [28, Prop. 2.2].

Suppose that $M = (M_-, M_+)$ is a special J-bimodule relative to the action $\circ$, i.e., $a \circ x \in M_\sigma$ for $a \in J_\sigma$, $x \in M_{-\sigma}$ and

$$\textbf{(JK1)} \qquad \{a,b,c\} \circ y = a \circ (b \circ (c \circ y)) + c \circ (b \circ (a \circ y))$$

for all $a,c \in J_\sigma$, $b \in J_{-\sigma}$, $y \in M_{-\sigma}$. Assume that $M = (M_-, M_+)$ possesses a triple product $\{x,y,z\} = V_{x,y}z \in M_\sigma$ for $x,z \in M_\sigma$ and $y \in M_{-\sigma}$ such that

$$\textbf{(JK2)} \qquad [V_{x,y}, V_{z,w}] = V_{V_{x,y}z,w} - V_{z,V_{y,x}w}$$

for all $x,z \in M_\sigma$, $y,w \in M_{-\sigma}$. In addition, we suppose that there are anticommutative bilinear maps $\kappa : M_\sigma \times M_\sigma \to J_\sigma$ which satisfy the following identities:

$$\textbf{(JK3)} \qquad \kappa(x,z) \circ y = \{x,y,z\} - \{z,y,x\},$$

$$\textbf{(JK4)} \qquad \kappa(x,z) \circ (b \circ u) = \{z, b \circ x, u\} - \{x, b \circ z, u\},$$

$$\textbf{(JK5)} \qquad \kappa(b \circ x, y) \circ z = b \circ \{x,y,z\} + \{y, x, b \circ z\},$$

$$\textbf{(JK6)} \qquad \{a, b, \kappa(x,z)\} = \kappa(a \circ (b \circ x), z) + \kappa(x, a \circ (b \circ z)),$$

$$\textbf{(JK7)} \qquad \{a, \kappa(y,w), c\} = \kappa(a \circ w, c \circ y) + \kappa(c \circ w, a \circ y),$$

$$\textbf{(JK8)} \qquad \kappa(\kappa(z,u) \circ y, x) = \kappa(\{x,y,z\}, u) + \kappa(z, \{x,y,u\}),$$

for all $x,z,u \in M_\sigma$, $y,w \in M_{-\sigma}$, $a,c \in J_\sigma$, $b \in J_{-\sigma}$.

A pair (J,M) satisfying (JP) and (JK1-JK8) is said to be *a Jordan-Kantor pair* (or simply a JKP for short).

Remark 3.2. As we will see in Section 7, the pair $M = (M_-, M_+)$ with its products is a Kantor pair. Thus, any JKP (J,M) consists of a Jordan pair J and a Kantor pair M related in a certain way. Later we will discuss how structurable algebras, Kantor pairs, and J-ternary algebras all can be regarded as JKPs.

3.3 The following useful identity can be derived from relations (JK3-5):

$$\textbf{(JK9)} \qquad a \circ (b \circ \{x,y,z\}) = \{a \circ (b \circ x), y, z\} - \{x, b \circ (a \circ y), z\} + \{x, y, a \circ (b \circ z)\}.$$

To see this, first observe that (JK5) and (JK3) give $b \circ \{x,y,z\} = \{b \circ x, z, y\} - \{y, z, b \circ x\} - \{y, x, b \circ z\}$. Applying this identity repeatedly shows that

$$\begin{aligned} a \circ (b \circ \{x, y, z\}) &= a \circ \{b \circ x, z, y\} - a \circ \{y, z, b \circ x\} - a \circ \{y, x, b \circ z\} \\ &= \{a \circ (b \circ x), y, z\} - \{z, y, a \circ (b \circ x)\} - \{z, b \circ x, a \circ y\} \\ &\quad - \{a \circ y, b \circ x, z\} + \{z, b \circ x, a \circ y\} + \{z, y, a \circ (b \circ x)\} \\ &\quad\quad - \{a \circ y, b \circ z, x\} + \{x, b \circ z, a \circ y\} + \{x, y, a \circ (b \circ z)\} \\ &= \{a \circ (b \circ x), y, z\} - [\{a \circ y, b \circ x, z\} + \{a \circ y, b \circ z, x\} \\ &\quad - \{x, b \circ z, a \circ y\}] + \{x, y, a \circ (b \circ z)\}. \end{aligned}$$

Finally, note that $\{a\circ y, b\circ x, z\} + \{a\circ y, b\circ z, x\} - \{x, b\circ z, a\circ y\} = \{x, b\circ(a\circ y), z\}$ according to (JK4) and (JK3).

The structure and inner structure algebras of a JKP

3.4 We construct a Lie algebra out of a Jordan-Kantor pair following the usual pattern of first constructing the structure and inner structure algebras. They are Lie subalgebras of the Lie algebra

$$\mathcal{E}(J, M) = \mathrm{End}(J_-) \oplus \mathrm{End}(M_-) \oplus \mathrm{End}(M_+) \oplus \mathrm{End}(J_+)$$

under the commutator product $[T, T'] = TT' - T'T$. It has been customary to write (S, S^ϵ) for elements of $\mathrm{End}(J_-) \oplus \mathrm{End}(J_+)$. Here we adopt a nonconventional but convenient notation T for a pair (S, S^ϵ). Then $S = T|_{J_-}$ and $S^\epsilon = T|_{J_+}$ are the restrictions of T to J_- and J_+ respectively. In other words, we identify $\mathcal{E}(J, M) = \mathrm{End}(J_-) \oplus \mathrm{End}(M_-) \oplus \mathrm{End}(M_+) \oplus \mathrm{End}(J_+)$ with its natural diagonal embedding into $\mathrm{End}(J_- \oplus M_- \oplus M_+ \oplus J_+)$.

3.5 Define the *structure algebra* $\mathfrak{str}(J, M)$ of the JKP (J, M) to be the set of all endomorphisms $T \in \mathcal{E}(J, M)$ such that

(**STR1**) $\quad [T, D_{a,b}] = D_{Ta,b} + D_{a,Tb}, \qquad [T, V_{x,y}] = V_{Tx,y} + V_{x,Ty},$

(**STR2**) $\quad T(a \circ y) = Ta \circ y + a \circ Ty, \qquad T\kappa(x, z) = \kappa(Tx, z) + \kappa(x, Tz),$

for all $x, z \in M_\sigma,\ y \in M_{-\sigma},\ a \in J_\sigma,\ b \in J_{-\sigma}$. It is easy to see that $\mathfrak{str}(J, M)$ is closed under the commutator product and hence is a Lie subalgebra of $\mathcal{E}(J, M)$.

For $a \in J_+, b \in J_-$, there is a unique element $\delta(a, b)$ of $\mathcal{E}(J, M)$ such that

$$\begin{aligned} \delta(a, b)c &= \{a, b, c\}, & \delta(a, b)d &= -\{b, a, d\}, \\ \delta(a, b)x &= a \circ (b \circ x), & \delta(a, b)y &= -b \circ (a \circ y), \end{aligned}$$

for all $c \in J_+,\ d \in J_-,\ x, \in M_+,\ y \in M_-$. For convenience, we put $\delta(b, a) = -\delta(a, b)$. This eliminates the (+/−)-asymmetry in the formulas above and allows them to be written as

$$\delta(a, b)c = \{a, b, c\} \quad \text{and} \quad \delta(a, b)x = a \circ (b \circ x) \tag{3.6}$$

for all $a, c \in J_\sigma,\ b \in J_{-\sigma}$ and $x \in M_\sigma$.

Similarly, for $x \in M_+, y \in M_-$ we define $\upsilon(x,y)$ of $\mathcal{E}(J,M)$ by

$$\upsilon(x,y)z = \{x,y,z\}, \qquad \upsilon(x,y)w = -\{y,x,w\},$$
$$\upsilon(x,y)a = \kappa(a \circ y, x), \qquad \upsilon(x,y)b = -\kappa(b \circ x, y),$$

for $z \in M_+,\ w \in M_-,\ a \in J_+,\ b \in J_-$. Again, by setting $\upsilon(y,x) = -\upsilon(x,y)$ we have

$$\upsilon(x,y)z = \{x,y,z\} \ \text{ and } \ \upsilon(x,y)a = \kappa(a \circ y, x) \tag{3.7}$$

for all $x,z \in M_\sigma,\ y,w \in M_{-\sigma},\ a \in J_\sigma,\ b \in J_{-\sigma}$.

Lemma 3.8. *The endomorphisms $\delta(a,b),\ \upsilon(x,y)$ belong to $\mathfrak{str}(J,M)$ for all $a \in J_\sigma, b \in J_{-\sigma},\ x \in M_\sigma, y \in M_{-\sigma}$.*

Proof. We need to verify that the identities (STR1–STR2) are satisfied for these endomorphisms. We begin with $\delta(a,b)$.

Since $\delta(a,b)|_{J_\sigma} = D_{a,b}$, the first part of (STR1) is merely the second identity in (JP). The second part is equivalent to (JK9). Equation (JK1) is equivalent to the first equation in (STR2) and (JK6) to the second equation in (STR2) for $T = \delta(a,b)$.

Now when $T = \upsilon(x,y)$, the second part of (STR1) is exactly (JK2). The first identity in (STR2) for $\upsilon(x,y)$ is equivalent to (JK5) and the second to (JK8). The only identity which requires some calculation is the first part of (STR1). For any $c \in J_\sigma$, $[\upsilon(x,y), D_{a,b}](c) = D_{\upsilon(x,y)a,b}(c) + D_{a,\upsilon(x,y)b}(c)$ is equivalent to saying

$$\textbf{(JK7}'\textbf{)}\quad \kappa(\{a,b,c\} \circ y, x) = \{\kappa(a \circ y, x), b, c\} - \{a, \kappa(b \circ x, y), c\} + \{a, b, \kappa(c \circ y, x)\}.$$

By (JK1), that relation is equivalent to

$$\begin{aligned}\kappa(a \circ (b \circ (c \circ y)), x) &+ \kappa(c \circ (b \circ (a \circ y)), x)\\ &= \{\kappa(a \circ y, x), b, c\} - \{a, \kappa(b \circ x, y), c\} + \{a, b, \kappa(c \circ y, x)\},\end{aligned}$$

which is equivalent to

$$\{a, \kappa(b \circ x, y), c\} = \kappa(c \circ y, a \circ (b \circ x)) + k(a \circ y, c \circ (b \circ x))$$

by (JK6). This is exactly (JK7) with $b \circ x$ as w.* ■

Lemma 3.9. *If $D \in \mathfrak{str}(J,M)$, then*

$$[D, \delta(a,b)] = \delta(Da,b) + \delta(a,Db) \qquad [D, \upsilon(x,y)] = \upsilon(Dx,y) + \upsilon(x,Dy)$$

for any $a \in J_\sigma,\ b \in J_{-\sigma},\ x \in M_\sigma,\ y \in M_{-\sigma}$.

Proof. This follows directly from (STR1-STR2) and the definitions of $\delta(a,b)$ and $\upsilon(x,y)$. ■

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* If $J_{-\sigma} M_\sigma = M_{-\sigma}$ holds, then relation (JK7′) is equivalent to (JK7). In this case, the two conditions (STR1) and (STR2) are equivalent to (JK1-2) and (JK9), (JK5-8), and the last part of (JP).

3.10 The subalgebra $\mathfrak{instr}(J,M)$ of $\mathfrak{str}(J,M)$ generated by the elements $\delta(a,b)$ and $\upsilon(x,y)$ is said to be the *inner structure algebra* of the JKP (J,M). Lemma 3.9 states that the subspaces $\delta(J_+,J_-)$ and $\upsilon(M_+,M_-)$ are ideals of the Lie algebra $\mathfrak{str}(J,M)$, and that

$$\mathfrak{instr}(J,M)=\delta(J_+,J_-)+\upsilon(M_+,M_-). \tag{3.11}$$

The structure algebra $\mathfrak{str}(J,M)$ also contains the element $h=(-2\,\mathrm{Id}, -\mathrm{Id},\mathrm{Id},2\,\mathrm{Id})\in\mathcal{E}(J,M)=\mathrm{End}(J_-)\oplus\mathrm{End}(M_-)\oplus\mathrm{End}(M_+)\oplus\mathrm{End}(J_+)$. The endomorphism h lies in the center of $\mathfrak{str}(J,M)$ and is called the *grading derivation* of (J,M).

Invariant forms of Jordan-Kantor pairs

3.12 Let $P=(J,M)$ be a JKP. An *invariant form* on P consists of a pair of bilinear forms $(\ ,\ ):J_+\times J_-\to\mathbf{F}$, $(\ ,\ ):M_+\times M_-\to\mathbf{F}$ for which the following identities hold:

(**IF1**) $(D_{a,b}c,d)=(c,D_{b,a}d)$,

(**IF2**) $(V_{x,y}z,w)=(z,V_{y,x}w)$,

(**IF3**) $(\kappa(x,z),b)=(z,b\circ x)$,

(**IF4**) $(a,\kappa(y,w))=(w,a\circ y)$,

for all $a,c\in J_+$, $b,d\in J_-$, $x,z\in M_+$, $y,w\in M_-$. We extend the form by setting $(b,a)=(a,b)$ and $(y,x)=(x,y)$ for $a\in J_+$, $b\in J_-$, $x\in M_+$, and $y\in M_-$. It follows from (IF3) and $\kappa(x,z)=-\kappa(z,x)$ that

(**IF5**) $(x,b\circ z)=-(z,b\circ x)$.

3.13 We note that this definition implies that $\delta(a,b)$ and $\upsilon(x,y)$ are skew-symmetric transformations on the vector space $J_-\oplus M_-\oplus M_+\oplus J_+$ with respect to the symmetric bilinear form g defined by

$$g(p,q)=\begin{cases}(p,q) & \text{if } p\in J_\sigma \text{ and } q\in J_{-\sigma},\\ & \text{or if } p\in M_\sigma \text{ and } q\in M_{-\sigma},\\ 0 & \text{otherwise}\end{cases}$$

Often we identify g with the invariant form on (J,M).

4. The Lie algebra $\mathfrak{L}(J,M,\mathfrak{D})$

4.1 Let (J,M) be a JKP and assume $\mathfrak{D}$ is a Lie subalgebra of $\mathfrak{str}(J,M)$ which contains $\mathfrak{instr}(J,M)$. On the space

$$\mathfrak{L}(J,M,\mathfrak{D})=J_-\oplus M_-\oplus\mathfrak{D}\oplus M_+\oplus J_+$$

define an anticommutative product $[\ ,\ ]$ by the following rules:

$$\begin{aligned}&[D,D'] \quad \text{is the product in } \mathfrak{D},\\ &[D,(x,a)_\sigma]=(D(x),D(a))_\sigma,\\ &[(x,a)_\sigma,(z,c)_\sigma]=(0,\kappa(x,z))_\sigma,\\ &[(x,a)_\sigma,(y,b)_{-\sigma}]=(-b\circ x,0)_{-\sigma}+\delta(a,b)+\upsilon(x,y)+(a\circ y,0)_\sigma,\end{aligned} \tag{4.2}$$

for all pairs $(x,a)_\sigma$, $(z,c)_\sigma\in M_\sigma\oplus J_\sigma$, $(y,b)_{-\sigma}\in M_{-\sigma}\oplus J_{-\sigma}$, and $D,D'\in\mathfrak{D}$.

Theorem 4.3. *The space $\mathfrak{L} = \mathfrak{L}(J, M, \mathfrak{D})$ with the product in (4.2) is a Lie algebra. It has a 5-grading $\mathfrak{L} = \mathfrak{L}_{-2} \oplus \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2$, where $\mathfrak{L}_{-2} = J_-$, $\mathfrak{L}_{-1} = M_-$, $\mathfrak{L}_0 = \mathfrak{D}$, $\mathfrak{L}_1 = M_+$, $\mathfrak{L}_2 = J_+$, and $\ker(\mathrm{ad}_\#) = 0$ where the map $\mathrm{ad}_\# : \mathfrak{L}_0 \to \mathcal{E}(J, M) = \mathrm{End}(\mathfrak{L}_{-2}) \oplus \mathrm{End}(\mathfrak{L}_{-1}) \oplus \mathrm{End}(\mathfrak{L}_1) \oplus \mathrm{End}(\mathfrak{L}_2)$ is defined by $\mathrm{ad}_\#(l_0) = \mathrm{ad}(l_0)|_{\mathfrak{L}_{-2}\oplus\mathfrak{L}_{-1}\oplus\mathfrak{L}_1\oplus\mathfrak{L}_2}$ for all $l_0 \in \mathfrak{L}_0$.*

Proof. To prove the first assertion it suffices to verify the Jacobi identity $\mathrm{j}(X_1, X_2, X_3) = [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0$ for arbitrary elements $X_1, X_2, X_3 \in (M_\sigma \oplus J_\sigma) \bigcup \mathfrak{D} \bigcup (M_{-\sigma} \oplus J_{-\sigma})$.

If two or more of these elements come from $\mathfrak{D}$, the identity is true because $\mathfrak{D}$ is a Lie algebra of endomorphisms.

If $X_1 = (x, a)_\sigma, X_2 = (z, c)_\sigma \in M_\sigma \oplus J_\sigma$ and $X_3 = D \in \mathfrak{D}$,

$$\mathrm{j}(X_1, X_2, X_3) = -(0, D(\kappa(x, z)))_\sigma + (0, \kappa(D(z), x))_\sigma + (0, \kappa(D(x), z))_\sigma = 0$$

because of (STR2) and the anticommutativity of κ.

For $X_1 = (x, a)_\sigma \in M_\sigma \oplus J_\sigma$, $X_2 = (y, b)_{-\sigma} \in M_{-\sigma} \oplus J_{-\sigma}$, and $X_3 = D \in \mathfrak{D}$,

$$\begin{aligned}\mathrm{j}(X_1, X_2, X_3) = {}& \Big(D(b \circ x) - D(b) \circ x - b \circ D(x), 0\Big)_{-\sigma} \\ &+ \Big([\delta(a, b), D] + \delta(D(a), b) + \delta(a, D(b)) \\ &\quad + [\upsilon(x, y), D] + \upsilon(D(x), y) + \upsilon(x, D(y))\Big) \\ &\quad\quad + \Big(- D(a \circ y) + D(a) \circ y + a \circ D(y), 0\Big)_\sigma = 0\end{aligned}$$

due to (STR2) and Lemma 3.9.

Finally, if $X_1 = (x, a)_\sigma, X_3 = (z, c)_\sigma \in M_\sigma \oplus J_\sigma$, and $X_2 = (y, b)_{-\sigma} \in M_{-\sigma} \oplus J_{-\sigma}$, then

$$\begin{aligned}\mathrm{j}(X_1, X_2, X_3) = {}& \upsilon(z, b \circ x) - \upsilon(x, b \circ z) + \delta(\kappa(z, x), b) \\ &+ \Big(c \circ (b \circ x) - \delta(c, b)x + \delta(a, b)z - a \circ (b \circ z) \\ &\quad + \upsilon(x, y)z - \upsilon(z, y)x + \kappa(z, x) \circ y, \delta(a, b)c - \delta(c, b)a \\ &\quad\quad + \upsilon(x, y)c - \kappa(c \circ y, x) + \kappa(a \circ y, z) - \upsilon(z, y)a\Big)_\sigma = 0.\end{aligned}$$

Here we have made use of the relations $\delta(a, b)c - \delta(c, b)a = 0$, $c \circ (b \circ x) - \delta(c, b)x = 0$, $\delta(a, b)z - a \circ (b \circ z) = 0$, $\upsilon(x, y)c - \kappa(c \circ y, x) = 0$, and $\kappa(a \circ y, z) - \upsilon(z, y)a = 0$ which follow from the definitions of δ and υ, and of the equation $\upsilon(x, y)z - \upsilon(z, y)x + \kappa(z, x) \circ y = 0$ which is (JK4). The relation

$$\upsilon(z, b \circ x) - \upsilon(x, b \circ z) + \delta(\kappa(z, x), b) = 0, \tag{4.4}$$

which is needed for the remaining terms, is equivalent to (JK5). Thus, we have checked the Jacobi identity for all possible choices of X_1, X_2, X_3.

The assertion about the grading of $\mathfrak{L}$ follows immediately from (4.2). The last statement of the theorem follows from the fact that elements of $\mathfrak{D}$ are endomorphisms and from the formulas in (4.2). ∎

It is convenient in what follows to adopt the following conventions: if $P = (J, M)$ is a Jordan-Kantor pair, then

$$\mathfrak{L}(P, \mathfrak{D}) := \mathfrak{L}(J, M, \mathfrak{D})$$
$$\mathfrak{L}(P) \text{ (or } \mathfrak{L}(J, M)) := \mathfrak{L}(J, M, \mathfrak{D}) \quad \text{when} \quad \mathfrak{D} = \mathfrak{instr}(J, M).$$

A converse of the last theorem is the essence of

Theorem 4.5. *Let L be a Lie algebra with a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$. Then*

(i) *$J = (L_{-2}, L_2)$ is a Jordan pair with the respect to the product $\{a, b, c\} = [[a, b], c]$;*

(ii) *$M = (L_{-1}, L_1)$ is a special J-bimodule under the action $a \circ x = [a, x]$;*

(iii) *the pair (J, M) together with the triple product $\{x, y, z\} = [[x, y], z]$ and the map $\kappa(x, z) = [x, z]$ form a JKP denoted by $\mathcal{P}(L)$;*

(iv) *if $\ker(\mathrm{ad}_\#) = 0$ where the map $\mathrm{ad}_\# : L_0 \to \mathcal{E}(J, M) = \mathrm{End}(L_{-2}) \oplus \mathrm{End}(L_{-1}) \oplus \mathrm{End}(L_1) \oplus \mathrm{End}(L_2)$ is defined by*

$$\mathrm{ad}_\# \, l_0 = \mathrm{ad} \, l_0 \,|_{L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2}$$

for $l_0 \in L_0$, then L is isomorphic to the Lie algebra $\mathfrak{L}(\mathcal{P}(L), \mathfrak{D})$ for some Lie subalgebra $\mathfrak{D}$ of $\mathfrak{str}(\mathcal{P}(L))$ which contains $\mathfrak{instr}(\mathcal{P}(L))$. If in addition $L_0 = [L_{-1}, L_1] + [L_{-2}, L_2]$, then $\mathfrak{D} = \mathfrak{instr}(\mathcal{P}(L))$.

Proof. Assertions (i) and (ii) follow directly from the Jacobi identity and properties of the grading. For example, if $a, c \in L_{2\sigma}, b \in L_{-2\sigma}$, $x \in L_{-\sigma}$, then $[[[a, b], c], x] = [a, [b, [c, x]]] + [c, [b, [a, x]]]$, which shows that (JK1) holds.

Identities (JK2-JK8) can be readily verified using the triple product $\{x, y, z\} = [[x, y], z]$, the map $\kappa(x, z) = [x, z]$, and the Jacobi identity for L. For example, if $x, z \in L_\sigma$ and $y \in L_{-\sigma}$ then $\kappa(x, z) \circ y = [[x, z], y] = [[x, y], z] - [[z, y], x] = \{x, y, z\} - \{z, y, x\}$ justifying identity (JK3). Thus, the pair (J, M) is a JKP.

Assume now $\ker(\mathrm{ad}_\#) = 0$. We argue that L_0 is isomorphic to some subalgebra $\mathfrak{D}$ of $\mathfrak{str}(J, M)$ which contains $\mathfrak{instr}(J, M)$ and prove that L is isomorphic to $\mathfrak{L}(J, M, \mathfrak{D})$. The adjoint map $\mathrm{ad}_\# : L_0 \to \mathcal{E}(J, M) = \mathrm{End}(L_{-2}) \oplus \mathrm{End}(L_{-1}) \oplus \mathrm{End}(L_1) \oplus \mathrm{End}(L_2)$ is a Lie algebra monomorphism. Denote the image of L_0 in $\mathcal{E}(J, M)$ by $\mathfrak{D}$. The presence of the Jacobi identity for L implies that identities (STR1) and (STR2) hold. It is easy to check that $\mathrm{ad}\,[a, b] = \delta(a, b)$ and $\mathrm{ad}\,[x, y] = \upsilon(x, y)$ for any $a \in L_{2\sigma}$, $b \in L_{-2\sigma}$, $x \in L_\sigma$, $y \in L_{-\sigma}$. Thus, $\mathfrak{D}$ contains $\mathfrak{instr}(J, M)$. Now the fact that L is isomorphic to the Lie algebra $\mathfrak{L}(J, M, \mathfrak{D})$ follows from the definitions of the products in (J, M) and $\mathfrak{L}(J, M, \mathfrak{D})$.

Suppose in addition that $L_0 = [L_{-1}, L_1] + [L_{-2}, L_2]$. Then $\mathfrak{D} = \mathrm{ad}_\# \, L_0 = [\mathrm{ad}\ L_{-1}, \mathrm{ad}\ L_1] + [\mathrm{ad}\ L_{-2}, \mathrm{ad}\ L_2] = \mathfrak{instr}(\mathcal{P}(L))$. ∎

Corollary 4.6. *A Lie algebra L is isomorphic to a Lie algebra $\mathfrak{L}(J, M, \mathfrak{D})$ for a JKP (J, M) and a subalgebra $\mathfrak{D}$ of $\mathfrak{str}(J, M)$ which contains $\mathfrak{instr}(J, M)$*

if and only if L *is a 5-graded Lie algebra,* $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, *with* $\ker(\mathrm{ad}_{\#}) = 0$.

Now for every 5-graded Lie algebra L we can construct the JKP $\mathcal{P}(L)$, and for every JKP P we can construct the 5-graded Lie algebra $\mathfrak{L}(P)$. To compare these constructions we require the notion of morphisms of JKPs.

Definition 4.7. Let $P = (J, M)$ and $P' = (J', M')$ be two Jordan-Kantor pairs. A linear map $\rho : J_- \oplus M_- \oplus M_+ \oplus J_+ \to J'_- \oplus M'_- \oplus M'_+ \oplus J'_+$, $\rho(x) = x'$, is said to be a *JKP homomorphism* if $\rho(J_\sigma) \subseteq J'_\sigma$, $\rho(M_\sigma) \subseteq M'_\sigma$, and

$$\{a,b,c\}' = \{a',b',c'\}, \quad \{x,y,z\}' = \{x',y',z'\},$$
$$(a \circ x)' = a' \circ x', \quad \kappa(x,z)' = \kappa(x',z')$$

for all $a, c \in J_\sigma$, $b \in J_{-\sigma}$, $x, z \in M_\sigma$, and $y \in M_{-\sigma}$.

We say that ρ is an *isomorphism* if it is bijective, and in this case, write $P \cong P'$.

Lemma 4.8.

(i) *Let* L *and* L' *be 5-graded Lie algebras. If* $L \cong L'$, *then* $\mathcal{P}(L) \cong \mathcal{P}(L')$.

(ii) *Let* P *and* P' *be JKPs. If* $P \cong P'$, *then* $\mathfrak{L}(P) \cong \mathfrak{L}(P')$.

(iii) *For every JKP* P *one has* $\mathcal{P}(\mathfrak{L}(P)) = P$.

(iv) *Let* L *be a 5-graded Lie algebra. Then* $\mathfrak{L}(\mathcal{P}(L)) \cong L$ *if and only if* $L \cong \mathfrak{L}(P)$ *for some JKP* P.

Proof. Statements (i) and (ii) are immediate. Assertion (iii) follows from (4.2) and (i)-(iii) of Theorem 4.5.

If $L \cong \mathfrak{L}(P)$, then from (i)-(iii) of this lemma we have that $\mathfrak{L}(\mathcal{P}(L)) \cong \mathfrak{L}(\mathcal{P}(\mathfrak{L}(P))) \cong \mathfrak{L}(P) \cong L$. The converse is obvious. ∎

Recall that an element h of a $\mathbf{Z}$-graded Lie algebra $L = \bigoplus_{i \in \mathbf{Z}} L_i$ is called a *grading element* if $\mathrm{ad}\, h|_{L_i} = i\,\mathrm{Id}|_{L_i}$ for every $i \in \mathbf{Z}$. It is unique up to a central element. In this terminology, the grading derivation of $\mathfrak{str}(J, M)$ in (3.10) is a grading element of the Lie algebra $\mathfrak{L}(J, M, \mathfrak{D})$.

Theorem 4.9. *Every* BC_1*-graded algebra* L *of type* D_1 *is a central extension of a Lie algebra* $\mathfrak{L}(P)$ *for a JKP* P *such that* $\mathfrak{instr}(P)$ *contains a grading derivation.*

Proof. A BC_1-graded algebra L of type D_1 is 5-graded, $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ relative to $\mathfrak{h} = \mathbf{F}h$, and $L_0 = [L_{-1}, L_1] + [L_{-2}, L_2]$ contains the grading element h. The latter fact implies that $Z(L) \subseteq L_0$. Thus by Theorem 4.5, the algebra $\overline{L} = L/Z(L)$ is isomorphic to $\mathfrak{L} = \mathfrak{L}(P, \mathfrak{D})$ where $P = \mathcal{P}(\overline{L})$ and $\mathfrak{D}$ is some subalgebra of $\mathfrak{str}(P)$ which contains $\mathfrak{instr}(P)$. Clearly the coset containing h is a grading element in $\overline{L}$. Moreover, $\mathfrak{D} \subseteq \mathfrak{L}_0 \subseteq [\mathfrak{L}_{-1}, \mathfrak{L}_1] + [\mathfrak{L}_{-2}, \mathfrak{L}_2] \subseteq \mathfrak{instr}(P)$, so $\mathfrak{L}(P, \mathfrak{D}) = \mathfrak{L}(P)$. ∎

Lemma 4.10. *Let* $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ *be a 5-graded Lie algebra with an invariant symmetric bilinear form* $\bar{f}$ *such that* $\bar{f}(L_i, L_j) = 0$ *if* $i + j \neq 0$.

Then the restriction of $\bar{f}$ to $P = \mathcal{P}(L)$ is an invariant form of the JKP P, and the restriction of $\bar{f}$ to L_0 is an invariant symmetric bilinear form of the Lie algebra L_0.

Conversely, if f is an invariant form of the JKP $P = \mathcal{P}(L)$ and f_0 is an invariant symmetric bilinear form of the Lie algebra L_0 such that

$$f_0([l_{-\sigma}, l_\sigma], l_0) = f(l_{-\sigma}, [l_\sigma, l_0]) \tag{4.11}$$

for all $l_0 \in L_0$ and $(l_{-\sigma}, l_\sigma) \in (L_{-\sigma}, L_\sigma) \bigcup (L_{-2\sigma}, L_{2\sigma})$, then the map $\bar{f}$ defined by

$$\bar{f}(p + l_0, p' + l_0') = f(p, p') + f_0(l_0, l_0')$$

for $p, p' \in P$ and $l_0, l_0' \in L_0$ is an invariant symmetric bilinear form of the Lie algebra L.

Proof. The first part of the lemma is an immediate consequence of the Jacobi identity and the invariance of f.

For the second part one has to check that the equation

$$\bar{f}([l_i, l_j], l_k) = \bar{f}([l_i, [l_j, l_k])$$

holds for $l_i \in L_i$, $l_j \in L_j$, $l_k \in L_k$. Both sides of this relation are equal to zero when one of i, j, k is zero, and the sum of other two is nonzero. If one of i, j, k is zero, and the sum of other two is zero, the equation follows from (4.11). If i, j, k are all nonzero, or all zero, it follows from properties of f and f_0. ∎

5. Central extensions of $\mathfrak{L}(J, M)$ and the classification of $\mathbf{BC}_1$-graded Lie algebras of type $\mathbf{D}_1$

5.1 The aim of this section is to describe the central extensions of the Lie algebra $\mathfrak{L}(J, M)$ for $P = (J, M)$ a JKP. Recall that $\mathfrak{L}(J, M)$ or $\mathfrak{L}(P)$ is our shorthand for the Lie algebra $\mathfrak{L}(J, M, \mathfrak{instr}(J, M))$ in (4.1). Our results here are adaptations of ideas from [10], [21], and [7] to our particular setting.

Full skew-dihedral homology

5.2 First we construct the Lie algebra $\{P, P\}$. The space $\{P, P\}$ is the direct sum $(J_+ \otimes J_-) \bigoplus (M_+ \otimes M_-)$ modulo the subspace $\mathcal{U}$ spanned by the elements

(S1) $\{a, b, c\} \otimes d - c \otimes \{b, a, d\} + \{c, d, a\} \otimes b - a \otimes \{d, c, b\}$,

(S2) $\{x, y, z\} \otimes w - z \otimes \{y, x, w\} + \{z, w, x\} \otimes y - x \otimes \{w, z, y\}$,

(S3) $\kappa(z, x) \otimes b - x \otimes (b \circ z) + z \otimes (b \circ x)$,

(S4) $a \otimes \kappa(y, w) - (a \circ y) \otimes w + (a \circ w) \otimes y$.

Here $x, z \in M_+$, $y, w \in M_-$, $a, c \in J_+$, $b, d \in J_-$. Let $\{a, b\}$ and $\{x, y\}$ denote the cosets $a \otimes b + \mathcal{U}$ and $x \otimes y + \mathcal{U}$ respectively in $\{P, P\}$. Then, $\{P, P\} = \{J_+, J_-\} + \{M_+, M_-\}$. If we add (S3) with $z = a \circ y$ to (S4) with $w = b \circ x$, we obtain another element of $\mathcal{U}$,

(S5) $\kappa(a\circ y,x)\otimes b - a\otimes\kappa(b\circ x,y) + a\circ(b\circ x)\otimes y - x\otimes b\circ(a\circ y)$.

5.3 Since J_+, J_-, M_+, and M_- are $\mathfrak{str}(J,M)$ – modules, the space $(J_+\otimes J_-)\oplus(M_+\otimes M_-)$ carries a natural $\mathfrak{str}(J,M)$-module structure given by:

$$T\big(\sum_i a_i\otimes b_i + \sum_j x_j\otimes y_j\big) = \sum_i Ta_i\otimes b_i + \sum_i a_i\otimes Tb_i + \sum_j Tx_j\otimes y_j + \sum_j x_j\otimes Ty_j$$

for any $T\in\mathfrak{str}(J,M)$. Thus, the elements in (S1) (S2) and (S5) can be rewritten as

(S1) $\delta(a,b)(c\otimes d) + \delta(c,d)(a\otimes b)$,

(S2) $\upsilon(x,y)(z\otimes w) + \upsilon(z,w)(x\otimes y)$,

(S5) $\upsilon(x,y)(a\otimes b) + \delta(a,b)(x\otimes y)$.

Lemma 5.4. *The subspace $\mathcal{U}$ is invariant under the action of $\mathfrak{str}(J,M)$.*

Proof. It suffices to note that the images of the elements in (S1-S4) under the action of $\mathfrak{str}(J,M)$ are in $\mathcal{U}$. If $T\in\mathfrak{str}(J,M)$, then

$$\begin{aligned}
&T\big(\delta(a,b)(c\otimes d)\big) + T\big(\delta(c,d)(a\otimes b)\big)\\
&= [T,\delta(a,b)](c\otimes d) + \delta(a,b)(T(c\otimes d)) + [T,\delta(c,d)](a\otimes b) + \delta(c,d)(T(a\otimes b))\\
&= \delta(Ta,b)(c\otimes d) + \delta(a,Tb)(c\otimes d) + \delta(a,b)(Tc\otimes d) + \delta(a,b)(c\otimes Td)\\
&\quad + \delta(Tc,d)(a\otimes b) + \delta(c,Td)(a\otimes b) + \delta(c,d)(Ta\otimes b) + \delta(c,d)(a\otimes Tb)\\
&= \delta(Ta,b)(c\otimes d) + \delta(c,d)(Ta\otimes b) + \delta(a,Tb)(c\otimes d) + \delta(c,d)(a\otimes Tb)\\
&\quad + \delta(a,b)(Tc\otimes d) + \delta(Tc,d)(a\otimes b) + \delta(a,b)(c\otimes Td) + \delta(c,Td)(a\otimes b) \in \mathcal{U};
\end{aligned}$$

$$\begin{aligned}
T(\kappa(z,x)\otimes b) - {}&T(x\otimes b\circ z) + T(z\otimes b\circ x)\\
&= T(\kappa(z,x))\otimes b + \kappa(z,x)\otimes Tb - Tx\otimes b\circ z - x\otimes T(b\circ z)\\
&\qquad + Tz\otimes b\circ x + z\otimes T(b\circ x)\\
&= \kappa(Tz,x)\otimes b + \kappa(z,Tx)\otimes b + \kappa(z,x)\otimes Tb\\
&\qquad - Tx\otimes b\circ z - x\otimes Tb\circ z - x\otimes b\circ Tz\\
&\qquad\qquad + Tz\otimes b\circ x + z\otimes Tb\circ x + z\otimes b\circ Tx \in \mathcal{U}.
\end{aligned}$$

Similarly one can check that the images of (S2) and (S4) under T are in $\mathcal{U}$. ■

5.5 Lemma 5.4 implies that the space $\{P,P\}$ is a $\mathfrak{str}(J,M)$-module satisfying the following identities:

(P1) $\delta(a,b)\{c,d\} = -\delta(c,d)\{a,b\}$,

(P2) $\delta(a,b)\{x,y\} = -\upsilon(x,y)\{a,b\}$,

(P3) $\upsilon(x,y)\{z,w\} = -\upsilon(z,w)\{x,y\}$,

(P4) $\{\kappa(z,x),b\} = \{x,b\circ z\} - \{z,b\circ x\}$,

(P5) $\{a,\kappa(y,w)\} = \{a\circ y,w\} - \{a\circ w,y\}$,

(P6) $T\{a,b\} = \{Ta,b\} + \{a,Tb\}$,

(P7) $T\{x,y\} = \{Tx,y\} + \{x,Ty\}$,

for all $x,z\in M_+$, $y,w\in M_-$, $a,c\in J_+$, $b,d\in J_-$, and $T\in\mathfrak{str}(J,M)$.

The next result will enable us to define a Lie product on $\{P,P\}$.

Lemma 5.6. *Let L be a Lie algebra and Q be an L-module. Assume that there is a linear map $\lambda : Q \to L$ such that for all $x, y \in Q$*

(i) $\lambda(x)y = -\lambda(y)x,$ *, and*

(ii) $[\lambda(x), \lambda(y)] = \lambda(\lambda(x)y)$. *Then, Q forms a Lie algebra under the product $[x, y] = \lambda(x)y$. Moreover, λ is a Lie algebra homomorphism.*

Proof. Anticommutativity of the product follows from (i) and the Jacobi identity from (ii). The assertion about λ being a homomorphism follows from (ii) also. ∎

Remark 5.7. If under the conditions of Lemma 5.6, the space Q possesses a symmetric bilinear form f, and L is represented by skew-symmetric transformations relative to that form, then f is an invariant form of the Lie algebra Q: $f([x, y], z) = -f(\lambda(y)x, z) = f(x, \lambda(y)z) = f(x, [y, z])$.

Proposition 5.8. *There is a unique linear map $\lambda : \{P, P\} \to \mathfrak{instr}(J, M)$ such that $\lambda\{a, b\} = \delta(a, b)$ and $\lambda\{x, y\} = \upsilon(x, y)$. The space $\{P, P\}$ with the product $[p, q] = \lambda(p)q$ is a Lie algebra which satisfies these relations:*

$$
\begin{aligned}
&[\{a, b\}, \{c, d\}] = \{\delta(a, b)c, d\} + \{c, \delta(a, b)d\},\\
&[\{a, b\}, \{x, y\}] = \{\delta(a, b)x, y\} + \{x, \delta(a, b)y\},\\
&[\{x, y\}, \{z, w\}] = \{\upsilon(x, y)z, w\} + \{z, \upsilon(x, y)w\},
\end{aligned}
\tag{5.9}
$$

for $a, c \in J_+$, $b, d \in J_-$ $x, z \in M_+$, $y, w \in M_-$. Moreover, λ is a Lie algebra homomorphism.

Proof. There is a unique map $\mu : (J_+ \otimes J_-) \bigoplus (M_+ \otimes M_-) \to \mathfrak{instr}(J, M)$ such that $\mu(a \otimes b) = \delta(a, b)$ and $\mu(x \otimes y) = \upsilon(x, y)$. We claim that $\mu(\mathcal{U}) = 0$. Indeed,

$$
\begin{aligned}
&\mu\big(\delta(a, b)c \otimes d + c \otimes \delta(a, b)d + \delta(c, d)a \otimes b + a \otimes \delta(c, d)b\big)\\
&\quad = \delta(\delta(a, b)c, d) + \delta(c, \delta(a, b)d) + \delta(\delta(c, d)a, b) + \delta(a, \delta(c, d)b)\\
&\quad = [\delta(a, b), \delta(c, d)] + [\delta(c, d), \delta(a, b)] = 0, \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
&\mu\big(\kappa(z, x) \otimes b - x \otimes b \circ z + z \otimes b \circ x\big)\\
&\qquad = \delta(\kappa(z, x), b) - \upsilon(x, b \circ z) + \upsilon(z, b \circ x) = 0 \ \text{ by (4.4)}.
\end{aligned}
$$

In the same manner one can show that μ sends (S2) and (S4) to zero. Thus, $\mu(\mathcal{U}) = 0$, and we have a well-defined map $\lambda : \{P, P\} \to \mathfrak{instr}(J, M)$, $\lambda(p + \mathcal{U}) = \mu(p)$ for which $\lambda\{a, b\} = \delta(a, b)$ and $\lambda\{x, y\} = \upsilon(x, y)$. Next, we verify that λ satisfies the conditions of Lemma 5.6. First,

$$\lambda(\{a, b\})\{c, d\} = \delta(a, b)\{c, d\} = -\delta(c, d)\{a, b\} = -\lambda(\{c, d\})\{a, b\}$$

by (P1). Analogously, (P2) and (P3) imply $\lambda(\{a, b\})\{x, y\} = -\lambda(\{x, y\})\{a, b\}$ and $\lambda(\{x, y\})\{z, w\} = -\lambda(\{z, w\})\{x, y\}$. Since, the elements $\{a, b\}$ and $\{x, y\}$ span $\{P, P\}$, (i) of Lemma 5.6 holds in $\{P, P\}$. Next we have

$$
\begin{aligned}
[\lambda(\{a, b\}), \lambda(\{c, d\})] &= [\delta(a, b), \delta(c, d)] = \delta(\delta(a, b)c, d) + \delta(c, \delta(a, b)d)\\
&= \lambda(\{\delta(a, b)c, d\} + \{c, \delta(a, b)d\})\\
&= \lambda(\delta(a, b)\{c, d\}) = \lambda(\lambda(\{a, b\})\{c, d\}),
\end{aligned}
$$

because of Lemma 3.9 and (P5). Similarly,

$$[\lambda(\{a,b\}),\lambda(\{x,y\})]=\lambda(\lambda(\{a,b\})\{x,y\})$$

and $[\lambda(\{x,y\}),\lambda(\{z,w\})]=\lambda(\lambda(\{x,y\})\{z,w\})$. Therefore, (ii) of Lemma 5.6 is true for $\{P,P\}$, and the result follows from that lemma. ∎

5.10 In concluding our investigations on $\{P,P\}$, we note that

$$\ker(\lambda)=\{\sum_i\{a_i,b_i\}+\sum_j\{x_j,y_j\}\mid\sum_i\delta(a_i,b_i)+\sum_j\upsilon(x_j,y_j)=0$$
$$\text{for } a_i\in J_+, b_i\in J_-, x_j\in M_+, y_j\in M_-\}$$

is a central ideal of $\{P,P\}$, which we term the *full skew-dihedral homology group* of P and denote by $\mathrm{HF}(P)$.

Proposition 5.11. *Assume that (,) is an invariant form on a JKP P. Then there is a unique invariant symmetric bilinear form f on the Lie algebra $\{P,P\}$ such that*

$$\begin{aligned} &f(\{a,b\},\{c,d\})=(\delta(a,b)c,d),\\ &f(\{a,b\},\{x,y\})=(\delta(a,b)x,y)=(x,\delta(a,b)y),\\ &f(\{x,y\},\{z,w\})=(\upsilon(x,y)z,w)\end{aligned}\tag{5.12}$$

for $a,c\in J_+,\ b,d\in J_-\ x,z\in M_+,\ y,w\in M_-$.

Proof. There is a unique form $\bar f$ on the vector space $(J_+\otimes J_-)\bigoplus(M_+\otimes M_-)$ such that

$$\begin{aligned}\bar f(a\otimes b,c\otimes d)&=(\delta(a,b)c,d),\\ \bar f(a\otimes b,x\otimes y)&=(\delta(a,b)x,y),\\ \bar f(x\otimes y,a\otimes b)&=(\upsilon(x,y)a,b),\\ \bar f(x\otimes y,z\otimes w)&=(\upsilon(x,y)z,w).\end{aligned}$$

First, we show that $\bar f$ is symmetric. Indeed for $a,c\in J_+,\ b,d\in J_-$,

$$\begin{aligned}\bar f(a\otimes b,c\otimes d)&=(\delta(a,b)c,d)=(\delta(c,b)a,d)=-(a,\delta(c,b)d)=-(a,\delta(c,d)b)\\ &=(\delta(c,d)a,b)=\bar f(c\otimes d,a\otimes b).\end{aligned}$$

Secondly, we note that for $a\in J_+,\ b\in J_-\ x\in M_+,\ y\in M_-$, (IF5) and (IF3) imply

$$\begin{aligned}\bar f(a\otimes b,x\otimes y)&=(\delta(a,b)x,y)=(a\circ(b\circ x),y)=(x,b\circ(a\circ y))=(\kappa(a\circ y,x),b)\\ &=(\upsilon(x,y)a,b)=\bar f(x\otimes y,a\otimes b).\end{aligned}$$

To complete the argument, we observe that for $x,z\in M_+,\ y,w\in M_-$ it follows from (IF4) that

$$(x,\kappa(y,w)\circ z)=(\kappa(z,x),\kappa(y,w))=(w,\kappa(z,x)\circ y).$$

This last identity, along with (JK3) and (IF2), implies that

$$\begin{aligned}\bar{f}(x\otimes y, z\otimes w) &= (\upsilon(x,y)z, w)\\ &= (\upsilon(z,y)x, w) + (\kappa(x,z)\circ y, w)\\ &= (x, \upsilon(y,z)w) - (x, \kappa(y,w)\circ z)\\ &= (x, \upsilon(w,z)y) + (x, \kappa(y,w)\circ z) - (x, \kappa(y,w)\circ z)\\ &= (\upsilon(z,w)x, y) = \bar{f}(z\otimes w, x\otimes y).\end{aligned}$$

Now an easy verification shows that the subspace $\mathcal{U}$ is contained in the radical of this form. Hence there is an induced form f on $\{P,P\}$ having the properties in (5.12). To complete the proof it suffices to argue that $\mathfrak{instr}(P)$ is represented by skew-symmetric transformations of $\{P,P\}$ (see Remark 5.7). Assume that $T \in \mathfrak{instr}(P)$, $a, c \in J_+$, $b, d \in J_-$. Then from (P6), (IF1-4), and Lemma 3.9 we have

$$\begin{aligned}f(T\{a,b\},\{c,d\}) &= f(\{Ta,b\}+\{a,Tb\},\{c,d\})\\ &= f(\delta(Ta,b)c,d) + f(\delta(a,Tb)c,d)\\ &= f(T\delta(a,b)c,d) - f(\delta(a,b)Tc,d)\\ &= -f(\delta(a,b)c,Td) - f(\delta(a,b)Tc,d)\\ &= -f(\{a,b\},\{c,Td\}) - f(\{a,b\},\{Tc,d\})\\ &= -f(\{a,b\},T\{c,d\})\end{aligned}$$

as desired. For the other choices of elements from $\{P,P\}$, the calculation is virtually identical and consequently is omitted here. ∎

The Lie algebra $\widehat{\mathfrak{L}}(J,M)$

5.13 Next we construct a Lie algebra $\widehat{\mathfrak{L}}(J,M)$ using the algebra $\{P,P\}$ instead of $\mathfrak{instr}(J,M)$. On the space

$$\widehat{\mathfrak{L}}(J,M) = J_- \oplus M_- \oplus \{P,P\} \oplus M_+ \oplus J_+ \tag{5.14}$$

there is a unique anticommutative product $[\,,\,]$ such that $\{P,P\}$ is a subalgebra of $\widehat{\mathfrak{L}}(J,M)$ and the following formulas hold:

$$\begin{aligned}&[p,(y,b)_-] = (\lambda(p)y, \lambda(p)b)_-, \qquad [p,(x,a)_+] = (\lambda(p)x, \lambda(p)a)_+,\\ &[(y,b)_-,(w,d)_-] = (0,\kappa(y,w))_-, \qquad [(x,a)_\sigma,(z,c)_\sigma] = (0,\kappa(x,z))_\sigma,\\ &[(x,a)_+,(y,b)_-] = (-b\circ x, 0)_- + \{a,b\} + \{x,y\} + (a\circ y, 0)_+\,,\end{aligned} \tag{5.15}$$

for all pairs $(x,a)_+$, $(z,c)_+ \in M_+ \oplus J_+$, $(y,b)_-, (w,d)_- \in M_- \oplus J_-$, and all $p \in \{P,P\}$. Here λ is the homomorphism in Proposition 5.8.

Theorem 5.16. *Let $P = (J, M)$ be a Jordan-Kantor pair.*

(i) *The space $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(J, M)$ with the product in (5.15) is a Lie algebra. It has a 5-grading $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}_{-2} \oplus \widehat{\mathfrak{L}}_{-1} \oplus \widehat{\mathfrak{L}}_0 \oplus \widehat{\mathfrak{L}}_1 \oplus \widehat{\mathfrak{L}}_2$, where $\widehat{\mathfrak{L}}_{-2} = J_-$, $\widehat{\mathfrak{L}}_{-1} = M_-$, $\widehat{\mathfrak{L}}_0 = \{P, P\}$, $\widehat{\mathfrak{L}}_1 = M_+$, $\widehat{\mathfrak{L}}_2 = J_+$.*

(ii) *$Z(\widehat{\mathfrak{L}}) \cap \widehat{\mathfrak{L}}_0 = \mathrm{HF}(P)$ for the center $Z(\widehat{\mathfrak{L}})$ of $\widehat{\mathfrak{L}}$. Moreover, if the algebra $\mathfrak{instr}(J, M)$ contains a grading derivation then $Z(\widehat{\mathfrak{L}}) = \mathrm{HF}(P)$.*

Proof. The fact that $\widehat{\mathfrak{L}}$ is a graded algebra follows directly from (5.15). We write $|X| = i$ if $X \in \widehat{\mathfrak{L}}_i$. Define the linear map $\rho : \widehat{\mathfrak{L}} \to \mathfrak{L}(J, M)$ by

$$\rho\big((y, b)_- + p + (x, a)_+\big) = (y, b)_- + \lambda(p) + (x, a)_+ \ . \tag{5.17}$$

Evidently, ρ is an extension of λ and $\ker(\rho) \subseteq \{P, P\}$. Moreover, it is easy to see using (4.2) and (5.15) that ρ is a homomorphism. It follows that $\mathfrak{j}(X, Y, Z) = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$ is zero whenever $|X| + |Y| + |Z| \neq 0$.

Assume that $|X| + |Y| + |Z| = 0$. Because $\mathfrak{j}(X, Y, Z)$ is an alternating function and because of the $+/-$ symmetry, it suffices to check that $\mathfrak{j}(X, Y, Z) = 0$ in the following cases: $(|X|, |Y|, |Z|) = (2, -1, -1)$ or $(0, 2, -2)$ or $(0, 1, -1)$. Here are the calculations in those particular cases:

$$\begin{aligned}
\mathfrak{j}\big((0, a)_+, (y, 0)_-, (w, 0)_-\big) &= \{a \circ y, w\} - \{a, \kappa(y, w)\} - \{a \circ w, y\} = 0 \quad \text{by (P5)},\\
\mathfrak{j}\big(p, (0, a)_+, (0, b)_-\big) &= \{\lambda(p)a, b\} - \lambda(p)\{a, b\} + \{a, \lambda(p)b\} = 0 \quad \text{by (P6)},\\
\mathfrak{j}\big(p, (x, 0)_+, (y, 0)_-\big) &= \{\lambda(p)x, y\} - \lambda(p)\{x, y\} + \{x, \lambda(p)y\} = 0 \quad \text{by (P7)}.
\end{aligned}$$

To prove (ii) we note that the inclusion $\mathrm{HF}(P) = \ker(\lambda) \subseteq Z(\widehat{\mathfrak{L}}) \cap \widehat{\mathfrak{L}}_0$ follows from (5.15). On the other hand, if $z \in Z(\widehat{\mathfrak{L}}) \cap \widehat{\mathfrak{L}}_0$, then for every $i \neq 0$ and $y_i \in \widehat{\mathfrak{L}}_i$ one has $\lambda(z)y_i = [z, y_i] = 0$. Thus, $z \in \mathrm{HF}(P)$.

Finally, if $h = \sum_i \delta(a_i, b_i) + \sum_j \upsilon(x_j, y_j) \in \mathfrak{instr}(J, M)$ is a grading derivation, then the element $\widehat{h} = \sum_i \{a_i, b_i\} + \sum_j \{x_j, y_j\}$ is a grading element of $\widehat{\mathfrak{L}}(J, M)$. In particular, for every $z \in Z(\widehat{\mathfrak{L}})$, one has $[z, \widehat{h}] = 0$, hence $z \in \widehat{\mathfrak{L}}_0$. Therefore, $Z(\widehat{\mathfrak{L}}) = \mathrm{HF}(P)$. ∎

The algebra $\widehat{\mathfrak{L}}(J, M)$ just constructed is universal in the following sense.

Theorem 5.18. *Let $L = \bigoplus_{i=-2}^{2} L_i$ be a 5-graded Lie algebra and (J, M) be a JKP. For any JKP homomorphism $\rho : (J, M) \to ((L_{-2}, L_2), (L_{-1}, L_1))$ there is a unique Lie algebra homomorphism from $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(J, M)$ to L which extends ρ.*

Proof. Let $\rho(x) = x'$ be such a JKP homomorphism. It follows from the assumptions that

$$\begin{gathered}
\{a, b, c\}' = [[a', b'], c'], \quad \{x, y, z\}' = [[x', y'], z'],\\
(a \circ x)' = [a', x'], \quad \kappa(x, z)' = [x', z'],
\end{gathered}$$

for all $a, c \in J_\sigma$ and $b \in J_{-\sigma}$ and $x, z \in M_\sigma$ and $y \in M_{-\sigma}$.

There is a unique linear map $\varphi : (J_+ \otimes J_-) \bigoplus (M_+ \otimes M_-) \to [L_2, L_{-2}] + [L_1, L_{-1}]$ such that $\varphi(a \otimes b) = [a', b']$ and $\varphi(x \otimes y) = [x', y']$. To show that it induces a map ψ from $\{P,P\}$ to L_0 we need to prove that $\varphi(\mathcal{U})=0$ where $\mathcal{U}$ is as in (5.2).

For elements of type (S1) we have

$$\begin{aligned}\varphi\big(&\delta(a,b)c\otimes d+c\otimes\delta(a,b)d+\delta(c,d)a\otimes b+a\otimes\delta(c,d)b\big)\\&=[(\delta(a,b)c)',d']+[c',(\delta(a,b)d)']+[(\delta(c,d)a)',b']+[a',(\delta(c,d)b)']\\&=[[[a',b'],c'],d']-[c',[[b',a'],d']]+[[[c',d'],a'],b']-[a',[[d',c'],b']]=0.\end{aligned}$$

The verification for elements of type (S2) is completely analogous.

For (S3), observe that

$$\begin{aligned}\varphi\big(&\kappa(z,x)\otimes b-x\otimes b\circ z+z\otimes b\circ x\big)\\&=[[z',x'],b']-[x',[b',z']]+[z',[b',x']]=0\end{aligned}$$

and similarly for (S4). Thus, $\varphi(\mathcal{U})=0$. Define a map $\psi:\widehat{\mathfrak{L}}\to L$ to be ρ on $\widehat{\mathfrak{L}}_i$, for $i\neq 0$, and $\psi|_{\{P,P\}}$ to be the map induced by φ. We claim that ψ is a homomorphism.

First, we note that the formulas $\psi([a,b])=[\psi(a),\psi(b)]$ and $\psi([x,y])=[\psi(x),\psi(y)]$ follow immediately from the definition of ψ. Next, observe that

$$\begin{aligned}\psi([\{a,b\},c])&=\psi(\delta(a,b)c)=[[a',b'],c']=[\psi(\{a,b\}),\psi(c)],\\ \psi([\{a,b\},d])&=\psi(\delta(a,b)d)=-[[b',a'],d']=[\psi(\{a,b\}),\psi(d)],\\ \psi([\{a,b\},x])&=\psi(\delta(a,b)x)=[a',[b',x']]=[[a',b'],x']=[\psi(\{a,b\}),\psi(x)],\\ \psi([\{a,b\},y])&=\psi(\delta(a,b)y)=-[b',[a',y']]=[[a',b'],y']=[\psi(\{a,b\}),\psi(y)],\end{aligned}$$

and similarly

$$\begin{aligned}\psi([\{x,y\},a])&=[\psi(\{x,y\}),\psi(a)],\\ \psi([\{x,y\},b])&=[\psi(\{x,y\}),\psi(b)],\\ \psi([\{x,y\},z])&=[\psi(\{x,y\}),\psi(z)],\\ \psi([\{a,b\},y])&=[\psi(\{x,y\}),\psi(w)].\end{aligned}$$

These equations imply in particular that $(\lambda(p)X)'=[\psi(p),X']$ for any $X\in\widehat{\mathfrak{L}}_i, i\neq 0$, and $p\in\{P,P\}$. Therefore,

$$\begin{aligned}\psi\big([p,\{a,b\}]\big)&=\psi(\{\lambda(p)a,b\}+\{a,\lambda(p)b\})\\&=[(\lambda(p)a)',b']+[a',(\lambda(p)b)']=[[\psi(p),a'],b']+[a',[\psi(p),b']]\\&=[\psi(p),[a',b']]=[\psi(p),\psi(\{a,b\})],\end{aligned}$$

and $\psi([p,a,b])=[\psi(p),\psi(\{x,y\})]$. Thus, ψ is indeed the desired homomorphism. The uniqueness of such a homomorphism is apparent. ∎

5.19 Before proceeding further we recall a few general facts about central extensions of Lie algebras (see [22] and [27]). A *central extension* of a Lie algebra L is a pair $(\widetilde{L},\pi)$, where $\widetilde{L}$ is a Lie algebra and $\pi:\widetilde{L}\to L$ is a surjective homomorphism whose kernel is contained in the center of $\widetilde{L}$. A *covering* $(\widehat{L},\pi)$

of L is a central extension for which $\widehat{L}$ is perfect, i.e. $\widehat{L} = [\widehat{L}, \widehat{L}]$. The covering $(\widehat{L}, \pi)$ is said to be *universal* if for every central extension $(\widetilde{L}, \tau)$ of L, there is a homomorphism $\psi : \widehat{L} \to \widetilde{L}$ so that $\tau \circ \psi = \pi$. A covering $(\widehat{L}, \pi)$ is universal if and only if $\widehat{L}$ is *centrally closed*, i.e., every central extension of $\widehat{L}$ splits [22, pp. 14-15]. We refer to a universal covering as the *universal central extension*, as it is unique up to isomorphism.

5.20 For any JKP (J, M), the Lie algebra $\widehat{\mathfrak{L}}(J, M)$ with the homomorphism ρ defined in (5.17) is a central extension of $\mathfrak{L}(J, M)$. Now, assume that $\mathfrak{instr}(J, M)$ contains a grading derivation h, and $h = \sum_i \delta(a_i, b_i) + \sum_j \upsilon(x_j, y_j)$. Then the element $\widehat{h} = \sum_i \{a_i, b_i\} + \sum_j \{x_j, y_j\}$ is a grading element of $\widehat{\mathfrak{L}}(J, M)$. In this case $\widehat{\mathfrak{L}}(J, M)$ is perfect and hence is a covering of $\mathfrak{L}(J, M)$.

Theorem 5.21. *Assume that $\mathfrak{instr}(J, M)$ contains a grading derivation h. Then the Lie algebra $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(J, M)$ is centrally closed.*

Proof. Suppose $(\widetilde{L}, \pi)$ is a central extension of $\widehat{\mathfrak{L}}$, and denote $\ker(\pi)$ by I. Set $D = \mathrm{ad}\, \widehat{h}$, and let $\widetilde{D} = \mathrm{ad}\, \widetilde{h}$ for a preimage $\widetilde{h}$ of $\widehat{h}$. Since $D|_{\widehat{\mathfrak{L}}_i} = i\mathrm{Id}$, and $D(D^2 - 1)(D^2 - 4) = 0$, we have $\widetilde{D}(\widetilde{D}^2 - 1)(\widetilde{D}^2 - 4)(\widetilde{L}) \subseteq I$. It follows that $\widetilde{D}^2(\widetilde{D}^2 - 1)(\widetilde{D}^2 - 4) = 0$ because I is central. As the polynomials $x^2, x - 1, x + 1, x - 2, x + 2$ are relatively prime, there is a decomposition

$$\widetilde{L} = \widetilde{L}_{x+2} \oplus \widetilde{L}_{x+1} \oplus \widetilde{L}_{x^2} \oplus \widetilde{L}_{x-1} \oplus \widetilde{L}_{x-2}, \tag{5.22}$$

where $\widetilde{L}_{p(x)}$ is the null space of $p(\widetilde{D})$.

Denote $\widetilde{L}_{x-i}$ by L_i for $i = -2, -1, 1, 2$. Then $\pi(L_i) \subseteq \widehat{\mathfrak{L}}_i$, and $\pi(\widetilde{L}_{x^2}) \subseteq \widehat{\mathfrak{L}}_0$. Since $I \subseteq \widetilde{L}_{x^2}$, we have $I \bigcap L_i = (0)$ and therefore we can define a linear map $\rho : \widehat{\mathfrak{L}}_{-2} \oplus \widehat{\mathfrak{L}}_{-1} \oplus \widehat{\mathfrak{L}}_1 \oplus \widehat{\mathfrak{L}}_2 \to \widetilde{L}$ such that $\pi \circ \rho = \mathrm{Id}$.

Next, we observe that $[L_i, L_j] \subseteq L_{i+j}$ when $i+j \neq 0$, and $[[L_i, L_j], L_k] \subseteq L_{i+j+k}$ when $i + j + k \neq 0$, since $\mathrm{ad}\, \widetilde{D}$ is a derivation and $\mathrm{ad}\, \widetilde{D}\,|_{L_i} = i\mathrm{Id}$. It follows that $L = \sum_{i=-2}^{2} L_i$, where $L_0 = [L_{-2}, L_2] + [L_{-1}, L_1]$, is a subalgebra of $\widetilde{L}$. Moreover, because $L_0 \subseteq \widetilde{L}_{x^2}$, the algebra $L = \bigoplus_{i=-2}^{2} L_i$ is 5-graded.

We claim that $\rho : (J, M) \to \big((L_{-2}, L_2), (L_{-1}, L_1)\big)$ is a JKP homomorphism. First, we note that $(J, M) = \big((\widehat{\mathfrak{L}}_{-2}, \widehat{\mathfrak{L}}_2), (\widehat{\mathfrak{L}}_{-1}, \widehat{\mathfrak{L}}_1)\big)$ as a JKP. Let $i, j, k \neq 0$ and $x \in \widehat{\mathfrak{L}}_i$, $y \in \widehat{\mathfrak{L}}_j$, and $z \in \widehat{\mathfrak{L}}_k$. Assume that $i + j \neq 0$. Since π is a Lie algebra homomorphism and $\pi \circ \rho = \mathrm{Id}$, $\rho([x, y]) - [\rho(x), \rho(y)] \in I$. On the other hand, $\rho([x, y]) - [\rho(x), \rho(y)] \in L_{i+j}$ and $L_{i+j} \bigcap I = (0)$. Thus, $\rho([x, y]) = [\rho(x), \rho(y)]$. Similarly, if $i + j + k \neq 0$, then $\rho([[x, y], z]) = [[\rho(x), \rho(y)], \rho(z)]$ proving that ρ is a JKP homomorphism.

Now according to Theorem 5.18, there is a Lie algebra homomorphism $\psi : \widehat{\mathfrak{L}} \to L \subseteq \widetilde{L}$ which extends ρ, so $\pi \circ \psi|_{\widehat{\mathfrak{L}}_i} = \mathrm{Id}$ for $i \neq 0$. It follows that $\pi \circ \psi = \mathrm{Id}$, because $\widehat{\mathfrak{L}}$ is generated by subspaces $\widehat{\mathfrak{L}}_i$, $i \neq 0$. Thus the extension $\pi : \widetilde{L} \to \widehat{\mathfrak{L}}$ splits. ∎

Corollary 5.23. *The Lie algebra $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(J, M)$ with the homomorphism ρ from (5.17) is the universal central extension of $\mathfrak{L}(J, M)$.*

Description of the BC_1-graded Lie algebras of type D_1

Theorem 5.24. *Let $P = (J, M)$ be a JKP and assume $\mathfrak{instr}(P)$ contains a grading element. Then for every subspace $\mathcal{H} \subseteq \mathrm{HF}(P)$, the Lie algebra $\mathfrak{L}(P, \mathcal{H}) = J_- \oplus M_- \oplus (\{P, P\}/\mathcal{H}) \oplus M_+ \oplus J_+$ is BC_1-graded of type D_1. Every BC_1-graded Lie algebra of type D_1 can be obtained in this way.*

Proof. For a JKP $P = (J, M)$ and a subspace $\mathcal{H} \subseteq \mathrm{HF}(P)$, the Lie algebra $\mathfrak{L}(P, \mathcal{H})$ has a grading element and a 5-grading with the required property inherited from $\widehat{\mathfrak{L}}(P)$. As a consequence, $\mathfrak{L}(P, \mathcal{H})$ is BC_1-graded of type D_1.

Conversely, consider an arbitrary BC_1-graded Lie algebra $L = \bigoplus_{i=-2}^{2} L_i$ of type D_1. According to the Theorem 4.9, L is a central extension of $\mathfrak{L}(P)$ for a JKP P such that $\mathfrak{instr}(P)$ contains a grading derivation. Therefore, there exists a homomorphism ψ from the universal central extension $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(P)$ into L. Since $\ker(\rho) \subseteq \widehat{\mathfrak{L}}_0$ and L is generated by the subspaces L_i with $i \neq 0$, the map ψ is surjective. Furthermore, $\ker(\psi) \subseteq \ker(\rho) \subseteq Z(\widehat{\mathfrak{L}}) = \mathrm{HF}(P)$. Thus, $L \cong \widehat{\mathfrak{L}}(P)/\mathcal{H} = \mathfrak{L}(P, \mathcal{H})$, where $\mathcal{H} = \ker(\psi) \subseteq \mathrm{HF}(P)$. ∎

6. J-ternary algebras and the classification of BC_1-graded Lie algebras of type C_1

6.1 In this section we introduce a nonassociative object needed for the coordinatization of BC_1-graded Lie algebras of type C_1. Jordan-Kantor pairs with an invertible element in the Jordan pair can be used for this purpose. Alternatively, since JKPs of this kind can be described by another set of axioms, these Lie algebras may be coordinatized by J-ternary algebras. Following [8], we elect to use J-ternary algebras for coordinates. We establish connections between Jordan-Kantor pairs and J-ternary algebras, which enable us to apply the results of previous sections to classify the BC_1-graded Lie algebras of type C_1.

Jordan-Kantor pairs and J-ternary algebras

6.2 First, we recall background information about J-ternary algebras and related Lie algebra constructions. Our exposition here follows [2] and [8].

Let $\mathfrak{J}$ be a Jordan algebra with unit element $\mathfrak{e}$, and $\mathfrak{M}$ be a special unital $\mathfrak{J}$-module; that is, $\mathfrak{e} \circ x = x$ for $x \in \mathfrak{M}$ and

$$\textbf{(JT0)} \qquad (ab) \circ x = \frac{1}{2}\Big(a \circ (b \circ x) + b \circ (a \circ x)\Big).$$

for all $a, b \in \mathfrak{J}$, $x \in \mathfrak{M}$. The space $\mathfrak{M}$ is said to be a *J-ternary algebra* if there is a skew bilinear map $\langle\, ,\, \rangle : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{J}$, and a trilinear map $\langle\, ,\, ,\, \rangle : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ satisfying:

(JT1) $\quad a \circ \langle v, w\rangle = \frac{1}{2}\langle av, w\rangle + \frac{1}{2}\langle v, aw\rangle,$

(JT2) $\quad a\langle u, v, w\rangle = \langle au, v, w\rangle - \langle u, av, w\rangle + \langle u, v, aw\rangle,$

(JT3) $\langle u,w\rangle v = \langle w,v,u\rangle - \langle u,v,w\rangle\,,$

(JT4) $\langle u,v\rangle w = \langle u,v,w\rangle - \langle v,u,w\rangle\,,$

(JT5) $\langle\langle u,v,w\rangle,x\rangle + \langle w,\langle u,v,x\rangle\rangle = \langle u,\langle w,x\rangle v\rangle\,,$

(JT6) $\langle u,v,\langle w,x,y\rangle\rangle = \langle\langle u,v,w\rangle,x,y\rangle + \langle w,\langle v,u,x\rangle,y\rangle + \langle w,x,\langle u,v,y,\rangle\rangle\,.$

We denote a J-ternary algebra by $(\mathfrak{J},\mathfrak{M})$ or by ${}_{\mathfrak{J}}\mathfrak{M}$. The first is useful in drawing parallels with JKPs, while the second is handy as a shorthand in more complicated expressions.

Remark. In the definition above and in what follows we adopt the left operator version of J-ternary algebras not the right operator version used in [2]. The correspondence between the two different notions of J-ternary algebras is explained in [8, Remark 3.17]. This switch of side is necessary to make the results of this section consistent with those in previous sections. Also, the left operator notation was used in [8] for the other BC_r-graded algebras.

One may note also that the use of $\frac{1}{2}$ in (JT0) is not consistent with (JK1). In previous sections on JKPs we have avoided introducing $\frac{1}{2}$ because it makes the identities for JKP and Lie algebra constructions simpler, but omitting this factor departs from traditional Jordan theory. Since we use the definition of J-ternary algebras given in [8], which follows usual Jordan customs, we have to bear this inconsistency.

6.3 A 5-graded Lie algebra $\mathcal{K}(\mathfrak{J},\mathfrak{M})$, the so-called *Tits-Kantor-Koecher Lie algebra,* can be associated with any J-ternary algebra $(\mathfrak{J},\mathfrak{M})$. Here we recall the construction of this algebra from [2] and [8].

Let $\mathcal{E} = \mathrm{End}_{\mathbf{F}}(\mathfrak{J}) \oplus \mathrm{End}_{\mathbf{F}}(\mathfrak{M})$ with its canonical embedding into the algebra $\mathrm{End}_{\mathbf{F}}(\mathfrak{J}\oplus\mathfrak{M})$. For $a\in\mathfrak{J}$, define the multiplication operator $\mathrm{L}_a\in\mathcal{E}$ by $\mathrm{L}_a(x,b) = (\frac{1}{2}a\cdot x, a.b)$. Setting $\mathrm{L}_{a,b} = 2(\mathrm{L}_{ab} + [\mathrm{L}_a,\mathrm{L}_b])$ we note that $\mathrm{L}_{a,b}(x,c) = \big(a\circ(b\circ x)+b\circ(a\circ x), 2((ab)c+(cb)a)-(ca)b)\big)$. Also, for $x,y\in\mathfrak{M}$, suppose $\mathrm{L}_{x,y}\in\mathcal{E}$ is given by $\mathrm{L}_{x,y}(z,a) = (\langle x,y,z\rangle, \langle x, a\cdot y\rangle)$.

For $A\in\mathcal{E}$, assume $A^\varepsilon = A - 2\mathrm{L}_{A\mathfrak{e}}$, where $\mathfrak{e}$ is the unit of the Jordan algebra $\mathfrak{J}$. Let $\mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ denote the set of elements A of $\mathcal{E}$ which satisfy

$$[A+A^\varepsilon,\mathrm{L}_a] = \mathrm{L}_{Aa+A^\varepsilon a} \text{ and} \tag{6.4}$$

$$[A,\mathrm{L}_{x,y}] = \mathrm{L}_{Ax,y} + \mathrm{L}_{x,A^\varepsilon y} \tag{6.5}$$

for all $a\in\mathfrak{J}$ and $x,y\in M$.

As was observed in [2], $\mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ is a Lie subalgebra of $\mathcal{E}$, and $A\in\mathcal{E}$ lies in $\mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ if and only if A satisfies the relations

$$A(ab) = (Aa)b + a(Ab) - \{a, A\mathfrak{e}, b\}_{\mathfrak{J}}, \tag{6.6}$$

$$A(a\circ u) = (Aa)\circ u + a\circ(A^\varepsilon u), \tag{6.7}$$

$$A\langle u,v,w\rangle = \langle Au,v,w\rangle + \langle u,A^\varepsilon v,w\rangle + \langle u,v,Aw\rangle, \tag{6.8}$$

$$A\langle u,v\rangle = \langle Au,v\rangle + \langle u,Av\rangle, \tag{6.9}$$

for all $a,b\in\mathfrak{J}$ and $u,v,w\in M$, where $\{a,c,b\}_{\mathfrak{J}} = (ac)b+(bc)a-(ab)c$. We use $\{\ ,\ ,\ \}_{\mathfrak{J}}$ for this trilinear operation because we want to reserve $\{\ ,\ ,\ \}$ for operations in JKPs.

Note that (6.6) is equivalent to the identity

$$A\{a,c,b\}_{\mathfrak{J}} = \{Aa,c,b\}_{\mathfrak{J}} + \{a,A^{\varepsilon}c,b\}_{\mathfrak{J}} + \{a,c,Ab\}_{\mathfrak{J}}. \tag{6.10}$$

6.11 Let $\mathfrak{D}$ be a Lie subalgebra of $\mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ which contains all the operators L_a and $\mathrm{L}_{x,y}$. Now suppose

$$\mathcal{K} = \mathcal{K}(\mathfrak{J},\mathfrak{M},\mathfrak{D}) = \widetilde{\mathfrak{J}} \oplus \widetilde{\mathfrak{M}} \oplus \mathfrak{D} \oplus \mathfrak{M} \oplus \mathfrak{J},$$

where $\widetilde{\mathfrak{J}} = \{\tilde{a} \mid a \in \mathfrak{J}\}$ and $\widetilde{\mathfrak{M}} = \{\tilde{x} \mid x \in \mathfrak{M}\}$ are second copies of $\mathfrak{J}$ and $\mathfrak{M}$. Then $\mathcal{K}$ can be equipped with an anticommutative multiplication $[\,,\,]$ for which $\mathfrak{D}$ is a subalgebra by defining

$$[T,(x,a)] = (Tx,Ta), \quad [T,(x,a)^{\sim}] = (T^{\varepsilon}x,T^{\varepsilon}a)^{\sim},$$
$$[(x,a),(y,b)] = (0,-\langle x,y\rangle)), \quad [(x,a)^{\sim},(y,b)^{\sim}] = (0,\langle x,y\rangle)^{\sim},$$
$$[(x,a),(y,b)^{\sim}] = (-b\cdot x,0)^{\sim} + \mathrm{L}_{x,y} + \mathrm{L}_{a,b} + (a\cdot y,0),$$

for $T \in \mathfrak{D}$, $x,y \in \mathfrak{M}$, $a,b \in \mathfrak{J}$. Relative to this product $\mathcal{K}(\mathfrak{J},\mathfrak{M},\mathfrak{D})$ is a 5-graded Lie algebra: $\mathcal{K} = \mathcal{K}_{-2} \oplus \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ with $\mathcal{K}_{-2} = \widetilde{\mathfrak{J}}$, $\mathcal{K}_{-1} = \widetilde{\mathfrak{M}}$, $\mathcal{K}_0 = \mathfrak{D}$, $\mathcal{K}_1 = \mathfrak{M}$, $\mathcal{K}_2 = \mathfrak{J}$. Furthermore, $Z(\mathcal{K}) \bigcap \mathcal{K}_0 = (0)$. Thus by Theorem 4.5, the algebra $\mathcal{K} = \mathcal{K}(\mathfrak{J},\mathfrak{M},\mathfrak{D})$ can be described in terms of a JKP (J,M) whose structure is determined by the J-ternary algebra in the following way:

Theorem 6.12. *Assume $(\mathfrak{J},\mathfrak{M})$ is a J-ternary algebra. Then*

(i) *$J = (J_-,J_+)$ with $J_+ = \mathfrak{J} = J_-$ and with the product $\{a_\sigma,b_{-\sigma},c_\sigma\} = 2[(ab)c + (bc)a - (ac)b]_\sigma$ is a Jordan pair;*
(ii) *$M = (M_-,M_+)$ with $M_+ = \mathfrak{M} = M_-$ is a special J-bimodule relative to the action $a \circ x = ax \in M_\sigma$ for $a \in J_\sigma$, $x \in M_{-\sigma}$;*
(iii) *the pair $P = (J,M)$ with the skew maps $\kappa : M_\sigma \times M_\sigma \to J_\sigma$, $\kappa(x,y) = -\sigma\langle x,y\rangle$, and the triple product $\{x,y,z\} = \sigma\langle x,y,z\rangle$ for $x,z \in M_\sigma$ and $y \in M_{-\sigma}$ forms a JKP.*

For a J-ternary algebra $\mathfrak{M}$ we denote the JKP described in this theorem by $\mathcal{P}(\mathfrak{J},\mathfrak{M})$. Note that $\mathfrak{instr}(\mathcal{P}(\mathfrak{J},\mathfrak{M}))$ contains the grading derivation $2\mathrm{L}_{\mathfrak{e}}$.

6.13 To describe Jordan-Kantor pairs of the form $\mathcal{P}(\mathfrak{J},\mathfrak{M})$ we introduce

Definition 6.14. Let (J,M) be a JKP. An element $a_\sigma \in J_\sigma$ is said to be *J-invertible* if the linear maps $Q_{a_\sigma} : J_{-\sigma} \to J_\sigma$, $Q_{a_\sigma}(b_{-\sigma}) = \{a_\sigma,b_{-\sigma},a_\sigma\}$, and $\mathrm{L}_{a_\sigma} : M_{-\sigma} \to M_\sigma$, $\mathrm{L}_{a_\sigma}(y_{-\sigma}) = a_\sigma \circ y_{-\sigma}$, are invertible.

Note that in this case, $J_- \cong J_+$ and $M_- \cong M_+$ as vector spaces. Our term for such an element comes from Jordan theory. The first condition of the definition is merely the condition for a_σ to be an invertible element of the Jordan pair J (see [28] for the definition and further details).

Proposition 6.15. *A JKP $P = (J, M)$ is isomorphic to $\mathcal{P}(\mathfrak{J}, \mathfrak{M})$ if and only if P possesses a J-invertible element.*

Proof. First we note that the identity element $\mathfrak{e} \in J_+ = \mathfrak{J}$ is a J-invertible element of the JKP $\mathcal{P}(\mathfrak{J}, \mathfrak{M})$.

To establish the converse we consider a JKP $P = (J, M)$ with J-invertible element a_σ. Since Q_{a_σ} is invertible, there is a unique element $a_{-\sigma} \in J_{-\sigma}$ such that $\{a_\sigma, a_{-\sigma}, a_\sigma\} = 2a_\sigma$.

For every element $c_\sigma \in J_\sigma$ one has $c_\sigma = Q_{a_\sigma}(b_{-\sigma})$ for some $b_{-\sigma} \in J_{-\sigma}$, and therefore $\{a_\sigma, a_{-\sigma}, c_\sigma\} = \{a_\sigma, a_{-\sigma}, Q_{a_\sigma}(b_{-\sigma})\} = \{a_\sigma, b_{-\sigma}, Q_{a_\sigma}(a_{-\sigma})\} = 2Q_{a_\sigma}(b_{-\sigma}) = 2c_\sigma$. Here, the second equality follows from a Jordan pair identity $\{a, b, Q_a(c)\} = \{a, c, Q_a(b)\}$ ([28, Eqn. (1) of Sec. 2.1]).

In addition, since L_{a_σ} is invertible, corresponding to any $x_\sigma \in M_\sigma$, there is a $y_{-\sigma} \in J_{-\sigma}$ so that $x_\sigma = a_\sigma \circ y_{-\sigma}$ and $a_\sigma \circ (a_{-\sigma} \circ x_\sigma) = a_\sigma \circ (a_{-\sigma} \circ (a_\sigma \circ y_{-\sigma})) = (1/2)\{a_\sigma, a_{-\sigma}, a_\sigma\} \circ y_{-\sigma} = a_\sigma \circ y_{-\sigma} = x_\sigma$ (which follows from (JK1)).

In a similar way one can prove that $\{a_{-\sigma}, a_\sigma, b_{-\sigma}\} = 2b_{-\sigma}$ and $a_{-\sigma} \circ (a_\sigma \circ y_{-\sigma}) = y_{-\sigma}$ for every $b_{-\sigma} \in J_{-\sigma}$ and $y_{-\sigma} \in J_{-\sigma}$. These equalities imply that the element $\delta(a_\sigma, a_{-\sigma})$ is a grading derivation of P.

It follows then that elements a_σ, $a_{-\sigma}$ and $\delta(a_\sigma, a_{-\sigma})$ form an $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{L}(P) := \mathfrak{L}(J, M, \mathfrak{instr}(J, M))$. Moreover, this $\mathfrak{sl}_2$-triple satisfies the requirements of Proposition 2.4, as can be seen from the representation theory of $\mathfrak{sl}_2$.

Thus, the algebra $\mathfrak{L}(P)$ is isomorphic to one constructed from a J-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$.

Finally, one can check that the operations on P and ${}_{\mathfrak{J}}\mathfrak{M}$ are related in the way described in Theorem 6.11. ∎

6.16 Next we want to show that the Lie algebra constructions for $(\mathfrak{J}, \mathfrak{M})$ and $\mathcal{P}(\mathfrak{J}, \mathfrak{M})$ are essentially the same.

Proposition 6.17. *The map $\rho : \mathfrak{str}(\mathcal{P}(\mathfrak{J}, \mathfrak{M})) \to \mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ defined by $\rho(T) = T|_{\mathfrak{J} \oplus \mathfrak{M}}$ is a Lie algebra isomorphism. Moreover, the Lie algebra $\mathfrak{L}(J, M, \mathfrak{D})$ is isomorphic to the Lie algebra $\mathcal{K}(\mathfrak{J}, \mathfrak{M}, \rho(\mathfrak{D}))$.*

Proof. Assume $T \in \mathfrak{str}(\mathcal{P}(\mathfrak{J}, \mathfrak{M}))$ and $T|_{\mathfrak{J} \oplus \mathfrak{M}} = 0$. Then (STR2) implies that $T(b_-) = (1/2)\{\mathfrak{e}_+, T(b_-), \mathfrak{e}_+\} = 0$ for $\mathfrak{e}_+ = \mathfrak{e} \in J_+$ and every $b_- \in J_-$, and (STR2) implies that $T(y_-) = \mathfrak{e}_- \circ (\mathfrak{e}_+ \circ T(y_-)) = \mathfrak{e}_- \circ T(\mathfrak{e}_+ \circ y_-) = 0$ for $\mathfrak{e}_- = \mathfrak{e} \in J_-$, $\mathfrak{e}_+ = \mathfrak{e} \in J_+$ and every $y_- \in M_-$. Injectivity of ρ follows.

Comparing (6.7-6.10) with (STR1-STR2), one can see that for $A \in \mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ the map (A^ε, A) from $\mathrm{End}(J_- \oplus M_- \oplus M_+ \oplus J_+)$ belongs to the algebra $\mathfrak{str}(\mathcal{P}(\mathfrak{J}, \mathfrak{M}))$. Thus ρ is surjective. Then one can readily check that the map $\bar{\rho} : (x, a)^{\sim} + D + (y, b) \mapsto (x, a)^{\sim} + \rho(D) + (y, b)$ is an isomorphism of $\mathfrak{L}(J, M, \mathfrak{D})$ and $\mathcal{K}(\mathfrak{J}, \mathfrak{M}, \rho(\mathfrak{D}))$. ∎

Full skew-dihedral homology of J-ternary algebras

6.18 We proceed to construct the universal central extension of the Lie algebra $\mathcal{K}(\mathfrak{J}, \mathfrak{M})$. This can be accomplished using the fact that $(\mathfrak{J}, \mathfrak{M})$ can be represented as a JKP $\mathcal{P} = \mathcal{P}(\mathfrak{J}, \mathfrak{M})$ and then applying the construction of $\{\mathcal{P}, \mathcal{P}\}$ from 5.2. However, the Jordan pair in $\mathcal{P}$ actually comes from a unital Jordan algebra. This enables us to simplify one of the defining identities for $\{\mathcal{P}, \mathcal{P}\}$ responsible for "the Jordan pair" part of the picture.

Theorem 6.19. *Let ${}_{\mathfrak{J}}\mathfrak{M}$ be a J-ternary algebra and $\mathcal{P} = \mathcal{P}(\mathfrak{J}, \mathfrak{M})$ be the corresponding JKP. Then the subspace $\mathcal{U}$ of $(\mathfrak{J}\otimes\mathfrak{J})\bigoplus(\mathfrak{M}\otimes\mathfrak{M})$ described in 5.2 is spanned by the elements*

(**JTS1**) $\quad a^2 \otimes a - \mathfrak{e} \otimes a^3\,,$

(**JTS2**) $\quad \langle x, y, z\rangle \otimes w + z \otimes \langle y, x, w\rangle + \langle z, w, x\rangle \otimes y + x \otimes \langle w, z, y\rangle\,,$

(**JTS3**) $\quad \langle x, z\rangle \otimes b - x \otimes (b \circ z) + z \otimes (b \circ x),$

(**JTS4**) $\quad a \otimes \langle y, w\rangle - (a \circ y) \otimes w + (a \circ w) \otimes y.$

Here $a, b \in J$ and $x, y, z, w \in M$.

Proof. The expressions in (JTS2-JTS4) are merely those in (S2-S4) written in terms of the J-ternary operations.

It is left to show that elements of the form (JTS1) and (S1) span the same subspace in $(\mathfrak{J}\otimes\mathfrak{J})\bigoplus(\mathfrak{M}\otimes\mathfrak{M})$. Note that $(S1)$ is simply a linearization of $\{a, b, a\} \otimes b - a \otimes \{b, a, b\}$. Using the bilinear product in $\mathfrak{J}$ instead of the trilinear one, we obtain the following form of this element,

(**S1$'$**) $\quad 2(ab)a \otimes b - a^2b \otimes b - 2a \otimes (ba)b + a \otimes b^2a$

and hence (S1) and (S1$'$) span the same subspace, which we denote by $\mathcal{U}_1$.

First, we prove that elements in (JTS1) belong to the span of those in (S1$'$). Setting $b = c = d = \mathfrak{e}$ in (S1$'$), we see that the elements

$$a \otimes \mathfrak{e} - \mathfrak{e} \otimes a \tag{6.20}$$

belong to $\mathcal{U}_1$. Next, (S1$'$) with $c = \mathfrak{e}$ and $b = d = a$ implies that

$$2a^2 \otimes a - a \otimes a^2 - \mathfrak{e} \otimes a^3 \tag{6.21}$$

is in $\mathcal{U}_1$; and (S1$'$) with $b = \mathfrak{e}$ and $c = d = a$ gives that

$$a^2 \otimes a - 2a \otimes a^2 + a^3 \otimes \mathfrak{e} \tag{6.22}$$

belongs to $\mathcal{U}_1$. Subtracting (6.22) from (6.21) multiplied by 2 and using (6.20), we have that elements of the form (JTS1) are in $\mathcal{U}_1$.

Now we prove that the elements in (S1$'$) belong to the subspace $\mathcal{U}_2$ spanned by elements (JTS1). For this we write $X \equiv Y$ if $X - Y$ is in $\mathcal{U}_2$. To begin note that linearization of (JTS1) yields

$$ab \otimes c + bc \otimes a + ca \otimes b \equiv \mathfrak{e} \otimes (ab)c + \mathfrak{e} \otimes (bc)a + \mathfrak{e} \otimes (ca)b. \tag{6.23}$$

This identity with $b = c = \mathfrak{e}$ gives us

$$a \otimes \mathfrak{e} \equiv \mathfrak{e} \otimes a, \tag{6.24}$$

while with $b = a, c = \mathfrak{e}$ it says

$$a \otimes a \equiv \mathfrak{e} \otimes a^2. \tag{6.25}$$

Linearization of (6.25) provides us with the relation

$$a \otimes b + b \otimes a \equiv 2\mathfrak{e} \otimes ab. \tag{6.26}$$

Now (6.25) and (6.24) imply that $a \otimes a^2 - a^3 \otimes \mathfrak{e} \equiv -a^2 \otimes a + 2\mathfrak{e} \otimes a^3 - a^3 \otimes \mathfrak{e} \equiv -a^2 \otimes a + \mathfrak{e} \otimes a^3 \equiv 0$. It follows then that the subspace $\mathfrak{U}_2$ is invariant under the linear transformation τ of $(\mathfrak{J} \otimes \mathfrak{J}) \bigoplus (\mathfrak{M} \otimes \mathfrak{M})$ which sends $a \otimes b$ to $b \otimes a$ and $x \otimes y$ to $y \otimes x$.

We are now ready to prove that $(\mathrm{S1}')$ is in $\mathfrak{U}_2$. Letting $c = ab$ in (6.23) and applying (6.25) and (6.26), we have

$$(ab)a \otimes b + a \otimes (ba)b \equiv \mathfrak{e} \otimes ((ab)a)b - \mathfrak{e} \otimes ((ba)b)a. \tag{6.27}$$

On the other hand, (6.23) with $a = a^2$ and $c = b$ gives

$$2a^2b \otimes b + b^2 \otimes a^2 \equiv 2\mathfrak{e} \otimes (a^2b)b + \mathfrak{e} \otimes b^2a^2. \tag{6.28}$$

Interchanging a and b in (6.28) and applying τ shows that

$$2a \otimes b^2a + b^2 \otimes a^2 \equiv 2\mathfrak{e} \otimes (b^2a)a + \mathfrak{e} \otimes b^2a^2. \tag{6.29}$$

Then (6.29) subtracted from (6.28) gives

$$a^2b \otimes b - a \otimes b^2a \equiv \mathfrak{e} \otimes (a^2b)b - \mathfrak{e} \otimes (b^2a)a, \tag{6.30}$$

and subtracting (6.30) from (6.27) multiplied by 2 yields $2(ab)a \otimes b - a^2b \otimes b - 2a \otimes (ba)b + a \otimes b^2a \equiv \mathfrak{e} \otimes \Big(2((ab)a)b - (a^2b)b - 2((ba)b)a + (b^2a)a\Big)$. However, the identity $2((ab)a)b - (a^2b)b - 2((ba)b)a + (b^2a)a = 0$ being a partial linearization of $a(ba^2) = (ab)a^2$ holds in any Jordan algebra $\mathfrak{J}$. ∎

6.31. Let us denote $(\mathfrak{J} \otimes \mathfrak{J} \oplus \mathfrak{M} \otimes \mathfrak{M})/\mathfrak{U}$ by $\{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\}$ and set $\{a, b\} = a \otimes b + \mathfrak{U}$ and $\{x, y\} = x \otimes y + \mathfrak{U}$.

We have a Lie algebra homomorphism $\rho : \{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\} \to \mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M})$ defined by $\rho(\{a, b\}) = \mathrm{Ł}_{a,b}$ and $\rho(\{x, y\}) = \mathrm{Ł}_{x,y}$. Naturally, we call the kernel of this homomorphism the *full skew-dihedral homology group* $\mathrm{HF}(\mathfrak{J}, \mathfrak{M})$ of the $\mathfrak{J}$-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$.

The following result is a consequence of Proposition 6.17, Theorem 5.16, and Corollary 5.23.

Theorem 6.32. *For a J-ternary algebra* ${}_{\mathfrak{J}}\mathfrak{M}$*, let* $N = \{(x,a) \mid x \in \mathfrak{M}, a \in \mathfrak{J}\}$ *and suppose* $N^{\sim}$ *is a second copy of* N*.*

(i) *The space* $\widehat{\mathcal{K}}(\mathfrak{J},\mathfrak{M}) = N^{\sim} \oplus \{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\} \oplus N$ *with the anticommutative product* $[\,,\,]$ *defined by the formulas*

$$
\begin{aligned}
&[p,q] \text{ is the product in } \{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\},\\
&[q,(x,a)] = (\rho(q)(x), \rho(q)(a)),\\
(6.33)\quad &[q,(x,a)^{\sim}] = (\rho(q)^{\varepsilon}(x), \rho(q)^{\varepsilon}(a))^{\sim},\\
&[(x,a),(y,b)] = (0, -\langle x,y\rangle)\ , \qquad [(x,a)^{\sim},(y,b)^{\sim}] = (0,\langle x,y\rangle)^{\sim},\\
&[(x,a),(y,b)^{\sim}] = -(b\cdot x, 0)^{\sim} + \{x,y\} + \{a,b\} + (a\cdot y, 0),
\end{aligned}
$$

for $p,q \in \{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\}$*,* $x,y \in \mathfrak{M}$*,* $a,b \in \mathfrak{J}$*, is a 5-graded Lie algebra.*

(ii) $\mathrm{HF}(\mathfrak{J},\mathfrak{M})$ *is the center of* $\widehat{\mathcal{K}}(\mathfrak{J},\mathfrak{M})$*.*

(iii) $\widehat{\mathcal{K}}(\mathfrak{J},\mathfrak{M})$ *is the universal central extension of* $\mathcal{K}(\mathfrak{J},\mathfrak{M})$*.*

A description of the BC_1-graded algebras of type C_1

Theorem 6.34. *Let* ${}_{\mathfrak{J}}\mathfrak{M}$ *be a J-ternary algebra and* $\mathcal{H}$ *be a subspace of* $\mathrm{HF}(\mathfrak{J},\mathfrak{M})$*. Then the Lie algebra* $\mathcal{K}(\mathfrak{J},\mathfrak{M},\mathcal{H}) = N^{\sim} \oplus (\{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\}/\mathcal{H}) \oplus N$ *is* BC_1*-graded Lie algebra of type* C_1*, and every* BC_1*-graded algebra of type* C_1 *can be obtained in this way.*

Proof. The Lie algebra $\mathcal{K} = \mathcal{K}(\mathfrak{J},\mathfrak{M},\mathcal{H})$ is 5-graded with $\mathcal{K}_0 = [\mathcal{K}_{-1},\mathcal{K}_1] + [\mathcal{K}_{-2}\mathcal{K}_2]$. Consider the elements $e = (0,\mathfrak{e})$, $f = (0,2\mathfrak{e})^{\sim}$, and $h = 2\{\mathfrak{e},\mathfrak{e}\} + \mathcal{H}$. It follows from (6.33) that $\langle e,f,h\rangle$ is an $\mathfrak{sl}_2$-triple and h is a grading element. According to Theorem 2.9, $\mathcal{K}$ is a BC_1-graded algebra of type C_1.

Conversely, assume L is a BC_1-graded algebra of type C_1 with a grading subalgebra $\mathfrak{g}$ spanned by an $\mathfrak{sl}_2$-triple $\langle e,f,h\rangle$. Since L is also BC_1-graded of type D_1 by Theorem 2.9, $L \cong \mathfrak{L}(P,\mathcal{H})$ for some JKP $P = (J,M)$ and a subspace $\mathcal{H} \subseteq \mathrm{HF}(P)$. The image of e is an invertible element of J. Hence $P \cong \mathcal{P}(\mathfrak{J},\mathfrak{M})$ and $L \cong \mathfrak{L}(P,\mathcal{H}) \cong \mathcal{K}(\mathfrak{J},\mathfrak{M},\mathcal{H})$ for some J-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$. ∎

7. Kantor pairs, structurable algebras, and the classification of BC_1-graded Lie algebras of type B_1

7.1 In this section we establish connections between Jordan-Kantor pairs and structurable algebras. This enables us to apply the results of previous sections to classify the BC_1-graded Lie algebras of type B_1.

Kantor pairs and Jordan-Kantor pairs

Definition 7.2. (See [9].) A pair of spaces $M = (M_-, M_+)$ is said to be a *Kantor pair* if it possesses a triple product $\{x,y,z\} = V_{x,y}z \in M_\sigma$ for $x,z \in M_\sigma$ and $y \in M_{-\sigma}$ such that

(KP1) $\qquad [V_{x,y}, V_{z,w}] = V_{V_{x,y}z,w} - V_{z,V_{y,x}w}\,,$

(KP2) $\qquad K_{x,z}V_{w,u} + V_{u,w}K_{x,z} = K_{K_{x,z}w,u}$

hold for $K_{x,z}y := \{x,y,z\} - \{z,y,x\}$ and $u \in M_\sigma, w \in M_{-\sigma}$.

7.3 We briefly describe the various relationships among the notions of Jordan, Kantor, and Jordan-Kantor pairs.

The triple product $V_{a,b}c = \{a,b,c\}$ in a Jordan pair $J = (J_-, J_+)$ satisfies (KP1), so a Jordan pair is just a special case of a Kantor pair with $K_{x,z} = 0$ for all x, z.

Assume now that (J,M) is a JKP and let $K_{x,z}y = \kappa(x,z) \circ y \overset{\text{(JK3)}}{=} \{x,y,z\} - \{z,y,x\}$. Identity (JK2) is the same as (KP1), and identity (JK5) implies (KP2). Therefore, the pair $M = (M_-, M_+)$ with its product $V_{x,y}z = \{x,y,z\}$ and with $K_{x,z}y = \kappa(x,z) \circ y$ is a Kantor pair. Thus, any JKP (J,M) consists of a Jordan pair J and a Kantor pair M whose structures are interrelated via $\circ$ and κ and the identities listed in (3.1).

7.4 Any Kantor pair $M = (M_-, M_+)$ can be embedded into a JKP $\mathcal{P}(M)$ in the following way. The exposition below is sketchy because we will not need it later on.

The 5-graded Lie algebra $\mathfrak{L} = \mathfrak{L}_{-2} \oplus \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2$ constructed from M in [9, Sec. 4] satisfies all the conditions of Theorem 4.5 (iv). Thus, it determines a JKP $\mathcal{P}(\mathfrak{L}) = ((\mathfrak{L}_{-2}, \mathfrak{L}_2), (\mathfrak{L}_{-1}, \mathfrak{L}_1))$. It follows from the construction that the Kantor pair $(\mathfrak{L}_{-1}, \mathfrak{L}_1)$ is isomorphic to M and that $(\mathfrak{L}_{-2}, \mathfrak{L}_2)$ is the Jordan pair $K = (K_-, K_+)$, where K_σ is the subspace of $\mathrm{Hom}(M_{-\sigma}, M_\sigma)$ spanned by the operators $K_{x,z}$ with the product $\{A,B,C\} = ABC + CBA$. The action of K on M is given $K_{x,z} \circ y = K_{x,z}(y)$. The skew maps $\kappa : M_\sigma \times M_\sigma \to K_\sigma$ are defined by $\kappa(x,y) = K_{x,y}$. We denote the JKP obtained in this manner by $\mathcal{P}(M)$. These JKPs admit the following description.

Proposition 7.5. *A JKP (J,M) is isomorphic to $\mathcal{P}(M)$ for some Kantor pair M if and only if*

(i) *J acts faithfully on the bimodule M, and*

(ii) *$J_\sigma = \kappa(M_\sigma, M_\sigma)$ for $\sigma = \pm$.*

Proof. The "only if" part follows from the description of $\mathcal{P}(M)$. For the other direction note that if (i) and (ii) hold, then a straightforward verification shows that the map $\rho : (J,M) \to \mathcal{P}(M)$ defined by

$$\rho((j_-, m_-, m_+, j_+)) = (\mathrm{L}_{j_-}, m_-, m_+, \mathrm{L}_{j_+}),$$

where $\mathrm{L}_{j_\sigma}(m_{-\sigma}) = j_\sigma \circ m_{-\sigma}$, is an isomorphism. ∎

Structurable algebras

7.6. Structurable algebras can be considered as a subclass of Kantor pairs, and as such, as a subclass of JKPs. They play an important role in our classification of BC_1-graded algebras of type B_1 as we have already seen in Theorem 2.6.

Recall that a unital algebra $(X, ^-)$ with an involution "$^-$" is said to be *structurable* provided the operators $V_{x,y}z = \{x,y,z\} = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$ satisfy identity (KP1).

If $(X,^-)$ is a structurable algebra, then according to [9], the pair of spaces (X, X) with the triple product $\{\,,\,,\,\}$ forms a Kantor pair. As in (7.4), we can associate a JKP to $(X,^-)$. This time we provide more details.

7.7. We start with a description of the corresponding Lie algebra construction (see [3] for details). For a structurable algebra $(X,^-)$, the set $S = \{s \in X \mid \bar{s} = -s\}$ of skew-hermitian elements is a Jordan triple system with respect to the product $\{r, s, t\} = (rs)t + (ts)r$. That is to say, the pair of spaces (S, S) with this product is a Jordan pair.

It follows from (KP1) that the subspace $V_{X,X}$ of $\mathrm{End}(X)$ spanned by operators $V_{x,y}$ is a Lie subalgebra of $\mathrm{End}(X)$. Let $N = \{(x, s) \mid x \in X, s \in S\}$, and denote a second copy of N by $N^{\sim}$. On the vector space

$$\mathcal{K} = N^{\sim} \oplus V_{X,X} \oplus N$$

define the anti-commutative bilinear product $[\,,\,]$ which extends the product on $V_{X,X}$ and satisfies

$$\begin{aligned}
&[V_{x,y}, (z, s)] = (V_{x,y}(z), x(\bar{y}s) + (sy)\bar{x}),\\
&[V_{x,y}, (z, s)^{\sim}] = (-V_{y,x}(z), -y(\bar{x}s) - (sx)\bar{y})^{\sim},\\
(7.8)\qquad &[(x, r), (y, s)] = (0, x\bar{y} - y\bar{x})\ , \qquad [(x, r)^{\sim}, (y, s)^{\sim}] = (0, x\bar{y} - y\bar{x})^{\sim},\\
&[(x, r), (y, s)^{\sim}] = -(sx, 0)^{\sim} + V_{x,y} + \mathrm{L}_r\mathrm{L}_s + (ry, 0),
\end{aligned}$$

for $x, y, z \in X$, $r, s \in S$, where L_s denotes left multiplication by s and $\mathrm{L}_r\mathrm{L}_s = (1/2)(V_{rs,1} - V_{r,s})$.

Note: The construction above is from [3]. It differs from [8] by some multiples of 2.

The Lie algebra defined above is referred to as the *Tits-Kantor-Koecher Lie algebra* of $(X,^-)$, and usually it is denoted $\mathcal{K}(X,^-)$ or $\mathcal{K}$ for short. It is not difficult to see that it possesses a 5-grading

$$\mathcal{K} = \mathcal{K}_{-2} \oplus \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2,$$

where $\mathcal{K}_{-2} = (0, S)^{\sim}$, $\mathcal{K}_{-1} = (X, 0)^{\sim}$, $\mathcal{K}_0 = V_{X,X}$, $\mathcal{K}_1 = (X, 0)$, $\mathcal{K}_2 = (0, S)$.

7.9. Let us calculate the JKP $\mathcal{P}(\mathcal{K})$. From the formulas in (7.8) we can read off the operations in $\mathcal{P}(\mathcal{K})$. The Jordan pair $(\mathcal{K}_{-2}, \mathcal{K}_2)$ is simply the Jordan triple system S with the product $\{r, s, t\} = (rs)t + (ts)r$, because

$$\begin{aligned}
[[(0, r), (0, s)^{\sim}], (0, t)] &= \frac{1}{2}[V_{rs,1} - V_{r,s}, (0, t)]\\
&= \frac{1}{2}(0, t(sr) + (rs)t + (ts)r + r(st))\\
&= (0, (rs)t + (ts)r).
\end{aligned}$$

The last equality follows from the identity $t(sr) + r(st) = (rs)t + (ts)r$ which is valid in any structurable algebra. Next, the Kantor pair $(\mathcal{K}_{-1}, \mathcal{K}_1)$ is just the Kantor triple system X under the product $\{x, y, z\} = V_{x,y}z$. The action of S on X is given by $s \circ x = sx$ and the map κ is defined by $\kappa(x, z) = x\bar{z} - z\bar{x}$. Let us denote this JKP by $\mathcal{P}(X,^-)$. Thus, $\mathcal{P}(X,^-) = \mathcal{P}(\mathcal{K}(X,^-))$. In addition $V_{1,1}$ is a grading derivation in $\mathfrak{instr}(\mathcal{P}(X,^-))$. To describe all JKPs obtained from structurable algebras we require the following concept.

Definition 7.10. Let (J,M) be a JKP. An element $x_\sigma \in M_\sigma$ is said to be *conjugate invertible* provided $\sigma v(x_\sigma, x_{-\sigma})$ is a grading derivation of (J,M) for some $x_{-\sigma} \in M_{-\sigma}$.

Note that the element $1 \in X$ is conjugate invertible in the JKP $\mathcal{P}(X,^-)$ because $\operatorname{ad} v(1,1)|_{\mathcal{K}_i} = i \operatorname{Id}|_{\mathcal{K}_i}$.

Proposition 7.11. *A JKP $P=(J,M)$ is isomorphic to $\mathcal{P}(X,^-)$ for a structurable algebra $(X,^-)$ if and only if there is a conjugate invertible element $x_\sigma \in M_\sigma$. In this case $\mathfrak{L}(P) \cong \mathcal{K}(X,^-)$.*

Proof. We need to prove only the "if" direction. Let $\mathfrak{L}=\mathfrak{L}(P)$ be the Lie algebra constructed from a JKP $P=(J,M)$, and assume P contains a conjugate invertible element $x_\sigma \in M_\sigma$. It follows from (4.2) that

$$e=(2x_\sigma,0), \quad f=(x_{-\sigma},0), \quad h=2\sigma v(x_\sigma,x_{-\sigma})$$

is an $\mathfrak{sl}_2$-triple in $\mathfrak{L}$.

One can verify that this $\mathfrak{sl}_2$-triple satisfies the conditions of Proposition 2.5 and therefore $P \cong \mathcal{P}(\mathfrak{L}(P)) \cong \mathcal{P}(\mathcal{K}(X,^-)) = \mathcal{P}(X,^-)$. ∎

As we have just shown, structurable algebras provide particular examples of JKPs. In the rest of this section we see how the general considerations of Sections 4-7 greatly simplify in this special case. We begin with an easier construction of the structure algebra. Assume that $(X,^-)$ is a structurable algebra. It follows from the isomorphism in Proposition 7.11 that $\mathfrak{instr}(\mathcal{P}(X,^-)) \cong V_{X,X}$. Now we want to realize this isomorphism explicitly. In fact we will establish that $\mathfrak{str}(\mathcal{P}(X,^-))$ is isomorphic to the structure Lie algebra of $(X,^-)$ which is described in [3, Cor. 5] as follows:

$$\mathfrak{str}(X,^-) = \{E \in \operatorname{End}(X) \mid [E,V_{x,y}] = V_{Ex,y} + V_{x,E'y} \text{ for some } E' \in \operatorname{End}(X)\}.$$

Proposition 7.12. *Let $(X,^-)$ be a structurable algebra. The map $\phi: T \mapsto T|_X$ is an isomorphism from $\mathfrak{str}(\mathcal{P}(X,^-))$ onto $\mathfrak{str}(X,^-)$. Moreover, $\phi(\mathfrak{instr}(\mathcal{P}(X,^-))) = V_{X,X}$.*

Proof. We use the auxiliary fact that if $y \in X$ has the property that $\{x,y,z\}=0$ for all $x,z \in X$ then $y=0$. Indeed letting $x=z=1$, we have $\{1,y,1\} = 2\bar{y} - y = 0$. Applying the involution to $2\bar{y}=y$ we see that $2y=\bar{y}$ and hence $2y=\bar{y}=(1/2)y$. Thus, $y=0$.

Let us show that the map $\phi: T \mapsto T|_X$ from $\mathfrak{str}(\mathcal{P}(X,^-))$ to $\operatorname{End}(X)$ is injective. In this calculation we identify $(X,0)$, $(X,0)^\sim$, $(0,S)$, and $(0,S)^\sim$ with X, $X^\sim$, S, and $S^\sim$. It follows from (STR1) that $\{x,T(y^\sim),z\} = T\{x,y^\sim,z\} - \{Tx,y^\sim,z\} - \{x,y^\sim,Tz\}$. If $T|_X=0$, then $\{x,T(y^\sim),z\}=0$ and therefore $T(y^\sim)=0$ for every $y^\sim$. For this T, the equations $T|_S=0$ and $T|_{S^\sim}=0$ follow from (STR2). Hence $T=0$. Surjectivity of ϕ is clear. Thus ϕ is an isomorphism.

To prove the second assertion we note that $V_{X,X} \subseteq \phi(\mathfrak{instr}(\mathcal{P}(X,^-)))$. On the other hand, for every $r,s \in S$ we claim that

$$2\delta(r,s) = v(rs,1) - v(r,s). \tag{7.13}$$

According to the discussion above, it is necessary to check (7.13) just on X. However, on X it amounts to $2\mathrm{L}_r\mathrm{L}_s = V_{rs,1} - V_{r,s}$ which is established in [3, Eq. 14]. ∎

Full skew-dihedral homology of structurable algebras

The universal central extension of the Lie algebra $\mathcal{K}(X,^-)$ is described in [10, Cor. 5.19]. Our aim here is to give an alternative description which is a consequence of the general theory developed for JKPs in Section 5.

We observe how the definition of the Lie algebra $\{\mathcal{P},\mathcal{P}\}$ simplifies for a structurable algebra viewed as a JKP.

Proposition 7.14. *If $(X,^-)$ is a structurable algebra and $\mathcal{P} = \mathcal{P}(X,^-)$, then $\{\mathcal{P},\mathcal{P}\} \cong (X \otimes X)/\mathcal{R}$ where $\mathcal{R}$ is the subspace of $X \otimes X$ spanned by the elements of the form*

(**R1**) $\qquad V_{x,y}z \otimes w - z \otimes V_{y,x}w + V_{z,w}x \otimes y - x \otimes V_{w,z}y$

for $x, y, z, w \in X$

Proof. Recall that $\{\mathcal{P},\mathcal{P}\} = ((S \otimes S) \bigoplus (X \otimes X))/\mathcal{U}$ where $\mathcal{U}$ is the subspace spanned by the elements of the form (S1-S4) in (5.2). In this case S is a subspace of X, and therefore to avoid confusion we need to distinguish elements from $X \otimes X$ and $S \otimes S$. We use the subscripts X and S for this purpose; thus $(r \otimes s)_X \in X \otimes X$ but $(r \otimes s)_S \in S \otimes S$.

First we simplify the set of generators for $\mathcal{U}$ by showing that $\mathcal{U}$ is spanned by the elements of the form (R1) and

(**R2**) $\qquad 2(r \otimes s)_S - (rs \otimes 1)_X + (r \otimes s)_X$

for $r, s \in S$.

It is easy to see that the subspace $\mathcal{U}_1$ spanned by the elements in (R1-R2) is contained in $\mathcal{U}$. Indeed, (R1) and (S2) are the same, and if we set $w = 1$, $y = s$, $a = r$ in (S4) we obtain (R2).

To establish the reverse containment, it suffices to verify that the generators (S1-S4) of $\mathcal{U}$ belong to $\mathcal{U}_1$. In this proof we write $\xi \equiv \zeta$ to mean $\xi - \zeta \in \mathcal{U}_1$.

In particular, from (5.3) we see that (R1) and (R2) are equivalent to the following in this notation:

$$\upsilon(x,y)(z \otimes w)_X \equiv -\upsilon(z,w)(x \otimes y)_X \; ; \tag{7.15}$$

$$2(r \otimes s)_S \equiv (rs \otimes 1)_X - (r \otimes s)_X. \tag{7.16}$$

Using these relations and (7.13) we obtain that

$$\begin{aligned}
\upsilon(x,y)(r \otimes s)_S + &\delta(r,s)(x \otimes y)_X \\
&\equiv \frac{1}{2}\upsilon(x,y)(rs \otimes 1)_X - \frac{1}{2}\upsilon(x,y)(r \otimes s)_X + \delta(r,s)(x \otimes y)_X \\
&\equiv -\frac{1}{2}\upsilon(rs,1)(x \otimes y)_X + \frac{1}{2}\upsilon(r,s)(x \otimes y)_X + \delta(r,s)(x \otimes y)_X \\
&\equiv 0.
\end{aligned}$$

In turn, this identity (along with (7.13) and (7.16)) implies that

$$\begin{aligned}\delta(r,s)(t\otimes u)_S &+ \delta(t,u)(r\otimes s)_S\\ &\equiv \frac{1}{2}\delta(r,s)(tu\otimes 1)_X - \frac{1}{2}\delta(r,s)(t\otimes u)_X + \delta(t,u)(r\otimes s)_S\\ &\equiv -\frac{1}{2}\upsilon(tu,1)(r\otimes s)_S + \frac{1}{2}\upsilon(t,u)(r\otimes s)_S + \delta(t,u)(r\otimes s)_S\\ &\equiv 0.\end{aligned}$$

Thus, elements of the form (S1) belong to $\mathcal{U}_1$. As (S2) is the same as (R1), those elements belong to $\mathcal{U}_1$ also.

Next we note that (7.15) with $z = w = 1$ gives us

$$\upsilon(x,y)(1\otimes 1)_X \equiv -2(x\otimes y)_X \ .$$

We use this identity and relations (7.16) and (7.13) to modify (S3) as follows:

$$\begin{aligned}(\kappa(z,x)\otimes s)_S &- (x\otimes sz)_X + (z\otimes sx)_X\\ &\equiv \frac{1}{2}(\kappa(z,x)s\otimes 1)_X - \frac{1}{2}(\kappa(z,x)\otimes s)_X - (x\otimes sz)_X + (z\otimes sx)_X\\ &\equiv -\frac{1}{2}\Big(\frac{1}{2}\upsilon(\kappa(z,x)s,1) - \frac{1}{2}\upsilon(\kappa(z,x),s) - \upsilon(x,sz) + \upsilon(z,sx)\Big)(1\otimes 1)_X\\ &\equiv -\frac{1}{2}\Big(\delta(\kappa(z,x),s) - \upsilon(x,sz) + \upsilon(z,sx)\Big)(1\otimes 1)_X\end{aligned}$$

It is left only to note that $\delta(\kappa(z,x),s) - \upsilon(x,sz) + \upsilon(z,sx) = 0$, because $(\delta(\kappa(z,x),s) - \upsilon(x,sz) + \upsilon(z,sx))|_X = \L_{\kappa(z,x)}\L_s - V_{x,sz} + V_{z,sx} = 0$ in any structurable algebra (see [4, Eq. (14)]). Thus, (S3) is in $\mathcal{U}_1$.

The proof for (S4) is similar with the sole exception being that one applies the identity $\L_s\L_{\kappa(z,x)} - V_{sx,z} + V_{sz,x} = 0$, which follows from [4, Eqs. (4), (8), and (14)].

Now we claim that $\mathcal{R} = (X\otimes X)\cap\mathcal{U}$. The inclusion $\mathcal{R}\subseteq(X\otimes X)\cap\mathcal{U}$ is clear. To show the reverse inclusion it suffices to prove that $\sum_i(r_i\otimes s_i)_S = 0$ implies that $\sum_i(r_is_i\otimes 1)_X - \sum_i(r_i\otimes s_i)_X = 0$. Here we can assume that $\{s_i\}$ is linearly independent, and therefore every $r_i = 0$, which proves our claim.

It follows then that the kernel of the map $\psi : X\otimes X \to \{\mathcal{P},\mathcal{P}\} = ((S\otimes S)\oplus(X\otimes X))/\mathcal{U}$ defined by $x\otimes y\mapsto x\otimes y+\mathcal{U}$ is precisely $\mathcal{R}$. Surjectivity of ψ follows from (7.16). ■

7.17. Let us denote $(X\otimes X)/\mathcal{R}$ by $\{X,X\}$ and $\{x,y\} = x\otimes y+\mathcal{R}$. We have a chain of homomorphisms: $\psi : \{X,X\}\to\{\mathcal{P},\mathcal{P}\}$ defined in 7.14, $\lambda : \{\mathcal{P},\mathcal{P}\}\to\mathfrak{str}(\mathcal{P}(X,^-))$ defined in 5.8, and $\phi : \mathfrak{str}(\mathcal{P}(X,^-))\to\mathfrak{str}(X,^-)$ defined in 7.12. We denote their composition by ϑ. Thus $\vartheta(\{x,y\}) = V_{x,y}$. Define the *full skew-dihedral homology group* $\mathrm{HF}(X,^-)$ of a structurable algebra $(X,^-)$ to be the kernel of the map ϑ; that is,

$$\mathrm{HF}(X,^-) = \{\sum_i\{x_i,y_i\}\in\{X,X\}\mid\sum_i V_{x_i,y_i} = 0\}.$$

This definition agrees with the original definition from [10] because $\mathrm{HF}(X,{}^-)$ is the center of the universal central extension of $\mathcal{K}(X,{}^-)$ (see Theorem 7.18 below).

The following result is a consequence of Theorem 5.16, Propositions 7.11 and 7.14, and Corollary 5.23.

Theorem 7.18. *For a structurable algebra $(X,{}^-)$ with skew-hermitian elements S, let $N = \{(x,s) \mid x \in X, s \in S\}$ and suppose $N^{\sim}$ is a second copy of N.*

(i) *The space $\widehat{\mathcal{K}}(X,{}^-) = N^{\sim} \oplus \{X,X\} \oplus N$ with the anticommutative product $[\,,\,]$ defined by the formulas,*

$$\begin{aligned}
&[\{x,y\},\{z,w\}] = \{V_{x,y}z,w\} - \{z,V_{y,x}w\},\\
&[\{x,y\},(z,s)] = (V_{x,y}(z),(sy)\bar{x} + x(\bar{y}s),\\
&[\{x,y\},(z,s)^{\sim}] = (-V_{y,x}(z),-(sx)\bar{y} - y(\bar{x}s))^{\sim},\\
&[(x,r),(y,s)] = (0,x\bar{y} - y\bar{x})\ ,\\
&[(x,r)^{\sim},(y,s)^{\sim}] = (0,x\bar{y} - y\bar{x})^{\sim},\\
&[(x,r),(y,s)^{\sim}] = -(sx,0)^{\sim} + \{x,y\} + (1/2)\{rs,1\}\\
&\qquad\qquad\qquad - (1/2)\{r,s\} + (ry,0),
\end{aligned} \tag{7.19}$$

is a 5-graded Lie algebra.

(ii) $\mathrm{HF}(X,{}^-)$ *is the center of* $\widehat{\mathcal{K}}(X,{}^-)$*.*

(iii) $\widehat{\mathcal{K}}(X,{}^-)$ *is the universal central extension of* $\mathcal{K}(X,{}^-)$*.*

A description of the BC_1-graded algebras of type B_1

Theorem 7.20. *Let $(X,{}^-)$ be a structurable algebra and $\mathcal{H}$ be a subspace of $\mathrm{HF}(X,{}^-)$. The Lie algebra $\mathcal{K}(X,{}^-,\mathcal{H}) = N^{\sim} \oplus (\{X,X\}/\mathcal{H}) \oplus N$ is BC_1-graded Lie algebra of type B_1, and every BC_1-graded algebra of type B_1 can be obtained in this way.*

Proof. The Lie algebra $\mathcal{K} = \mathcal{K}(X,{}^-,\mathcal{H})$ is 5-graded with $\mathcal{K}_0 = [\mathcal{K}_{-1},\mathcal{K}_1] + [\mathcal{K}_{-2}\mathcal{K}_2]$. It follows from (7.19) that the elements $e = (2,0)$, $f = (1,0)^{\sim}$, and $h = 2\{1,1\} + \mathcal{H}$ determine an $\mathfrak{sl}_2$-triple, and $(1/2)h$ is a grading element. According to Theorem 2.9, $\mathcal{K}$ is a BC_1-graded algebra of type B_1.

Conversely, assume L is a BC_1-graded algebra of type B_1 with a grading subalgebra $\mathfrak{g}$ spanned by an $\mathfrak{sl}_2$-triple $\langle e,f,h\rangle$. Since L is also of type D_1, $L \cong \mathfrak{L}(P,\mathcal{H})$ for some JKP $P = (J,M)$ and a subspace $\mathcal{H} \subseteq \mathrm{HF}(P)$. The image of e is a conjugate invertible element of M_+. Hence $P \cong \mathcal{P}(X,{}^-)$ and $L \cong \mathfrak{L}(P,\mathcal{H}) \cong \mathcal{K}(X,{}^-,\vartheta(\mathcal{H}))$ for some structurable algebra $(X,{}^-)$. ∎

8. Derivations of BC_1-graded Lie algebras

8.1 In this section we describe the derivations of BC_1-graded Lie algebras. First, we consider a 5-graded Lie algebra $L = \bigoplus_{i=-2}^{2} L_i$ and assume that h

is a grading element of L so that $\operatorname{ad} h\,|_{L_i} = i\,\mathrm{Id}_{L_i}$. The grading of L gives rise to an associated grading $\mathrm{Der}(L) = \bigoplus_{i\in\mathbf{Z}} \mathrm{Der}(L)_i$ of the derivation algebra $\mathrm{Der}(L)$ of L such that $\mathrm{Der}(L)_i = \{D \in \mathrm{Der}(L) \mid D(L_j) \subseteq L_{i+j} \text{ for all } j\}$. This follows from the fact that $\mathrm{Der}(L)_i$ is the eigenspace of $\operatorname{ad} h$ corresponding to the eigenvalue i. Also, it is clear that $\mathrm{Der}(L)_i = (0)$ whenever $|i| > 4$, and that $\operatorname{ad} L_i \subseteq \mathrm{Der}(L)_i$.

8.2 For such a 5-graded Lie algebra L we consider the JKP $\mathcal{P}(L)$ associated with it (see Theorem 4.5). As a vector space $L = \mathcal{P}(L) \oplus L_0$. It follows from the definition of $\mathcal{P}(L)$ that the restriction map $\rho : D \mapsto D|_{\mathcal{P}(L)}$, for $D \in \mathrm{Der}(L)_0$, determines a Lie homomorphism $\rho : \mathrm{Der}(L)_0 \to \mathfrak{str}(\mathcal{P}(L))$. Moreover, ρ is an embedding if $\mathcal{P}(L)$ generates L.

8.3 We assume now that the 5-graded Lie algebra $L = \bigoplus_{i=-2}^{2} L_i$ has a realization as in Theorem 5.24,

$$L = \mathfrak{L}(P, \mathcal{H}) = P_- \oplus \Big(\{P, P\}/\mathcal{H}\Big) \oplus P_+,$$

where $P = \mathcal{P}(L)$ a JKP and $\mathcal{H}$ is a subspace of $\mathrm{HF}(P)$. Since $\mathcal{P}(L)$ generates L, the map ρ is an embedding. To describe the image of $\mathrm{Der}(L)_0$, recall from (P6) and (P7) that there is a representation of $\mathfrak{str}(P)$ on $\{P, P\}$, where the action of $T \in \mathfrak{str}(P)$ is given by

$$T\{a, b\} = \{Ta, b\} + \{a, Tb\},$$
$$T\{x, y\} = \{Tx, y\} + \{x, Ty\}$$

for $\{a, b\}, \{x, y\} \in \{P, P\}$. We define

$$\mathfrak{str}(P, \mathcal{H}) = \{T \in \mathfrak{str}(P) \mid T(\mathcal{H}) \subseteq \mathcal{H}\}. \tag{8.4}$$

Lemma 8.5. $\rho(\mathrm{Der}(L)_0) = \mathfrak{str}(P, \mathcal{H})$.

Proof. Consider an arbitrary element $D \in \mathrm{Der}(L)_0$ as an endomorphism of $L_0 = \{P, P\}/\mathcal{H}$. Then for all $a \in J_+$ and $b \in J_-$,

$$\begin{aligned} D(\{a, b\} + \mathcal{H}) = D([a, b]) = [Da, b] + [a, Db] &= \{Da, b\} + \{a, Db\} + \mathcal{H} \\ &= \rho(D)\{a, b\} + \mathcal{H}, \end{aligned}$$

and similarly $D(\{x, y\} + \mathcal{H}) = \rho(D)\{x, y\} + \mathcal{H}$ for all $x \in M_+$ and $y \in M_-$. This implies that D is the map induced by $\rho(D)$ on the quotient space $\{P, P\}/\mathcal{H}$ and therefore $\rho(D) \in \mathfrak{str}(P, \mathcal{H})$.

The reverse inclusion follows from the fact that every element $T \in \mathfrak{str}(P, \mathcal{H})$ induces a derivation of $L = P \oplus \big(\{P, P\}/\mathcal{H}\big)$. ∎

Theorem 8.6. *Let L be a BC_1-graded Lie algebra. Then the Lie algebra $\mathrm{Der}(L)$ of derivations of L has a $\mathbf{Z}$-grading $\mathrm{Der}(L) = \bigoplus_{i\in\mathbf{Z}} \mathrm{Der}(L)_i$ induced from the 5-grading on L, with $\mathrm{Der}(L)_i = (0)$ if $|i| > 2$, and with $\mathrm{Der}(L)_i = \operatorname{ad} L_i$ for $i = \pm 1, \pm 2$. Moreover,*

(i) *when* L *has type* D_1, *then* $L = \mathfrak{L}(P, \mathcal{H}) = J_- \oplus M_- \oplus (\{P, P\}/\mathcal{H}) \oplus M_+ \oplus J_+$ *for some JKP* $P = \mathcal{P}(L) = (J, M)$ *and some subspace* $\mathcal{H}$ *of* $\mathrm{HF}(P)$, *and* $\mathrm{Der}(L)_0 \cong \mathfrak{str}(P, \mathcal{H})$;

(ii) *when* L *has type* C_1, *then* $L = \mathcal{K}(\mathfrak{J}, \mathfrak{M}, \mathcal{H})$ *for some* $\mathfrak{J}$*-ternary algebra* ${}_{\mathfrak{J}}\mathfrak{M}$ *and some subspace* $\mathcal{H}$ *of* $\mathrm{HF}(\mathfrak{J}, \mathfrak{M})$, *and* $\mathrm{Der}(L)_0 \cong \mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M}, \mathcal{H}) \stackrel{\text{def}}{=} \{T \in \mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M}) \mid T(\mathcal{H}) \subseteq \mathcal{H}\}$;

(iii) *when* L *has type* B_1, *then* $L = \mathcal{K}(X, {}^-, \mathcal{H})$ *for some structurable algebra* $(X, {}^-)$ *and some subspace* $\mathcal{H}$ *of* $\mathrm{HF}(X, {}^-)$, *and* $\mathrm{Der}(L)_0 \cong \mathfrak{str}(X, {}^-, \mathcal{H}) \stackrel{\text{def}}{=} \{T \in \mathfrak{str}(X, {}^-) \mid T(\mathcal{H}) \subseteq \mathcal{H}\}$.

Proof. It is enough to argue that $\mathrm{Der}(L) \subseteq \mathrm{Der}(L)_0 + \sum_{i \neq 0} \mathrm{ad}\, L_i$. Suppose $D \in \mathrm{Der}(L)$, and let h be a grading element of L. Then $D(h) = \sum_{i=-2}^{2} x_i$ for some $x_i \in L_i$. For every $y_j \in L_j$,

$$jD(y_j) = D([h, y_j]) = [\sum_{i=-2}^{2} x_i, y_j] + [h, D(y_j)].$$

The $(i+j)$th-component of this equality is $iD(y_j) = [x_i, y_j]$. Therefore the ith-component of D acts as $i^{-1}\, \mathrm{ad}\, x_i \in \mathrm{ad}\, L_i$ for $i \neq 0$, and $D - \sum_{i \neq 0} i^{-1}\, \mathrm{ad}\, x_i \in \mathrm{Der}(L)_0$. This, together with Lemma 8.5, gives (i).

Now Proposition 6.17 and Lemma 5.4 imply that for a J-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$, the space $\{{}_{\mathfrak{J}}\mathfrak{M}, {}_{\mathfrak{J}}\mathfrak{M}\}$ is a module over the Lie algebra $\mathfrak{str}({}_{\mathfrak{J}}\mathfrak{M}) \cong \mathfrak{str}(\mathcal{P}(\mathfrak{J}, \mathfrak{M}))$. The results in (i) applied to the JKP $\mathcal{P}(\mathfrak{J}, \mathfrak{M})$ now produce the desired conclusions.

Analogously, for a structurable algebra $(X, {}^-)$ the space $\{X, X\}$ is a module over the Lie algebra $\mathfrak{str}(X, {}^-) \cong \mathfrak{str}(\mathcal{P}(X, {}^-))$ by Propositions 7.12 and 7.14, so that the result follows from (i) with the JKP $\mathcal{P}(X, {}^-)$. ■

9. Invariant bilinear forms on BC_1-graded Lie algebras

9.1 Our aim in this section is to describe the invariant bilinear forms on BC_1-graded Lie algebras in terms of invariant forms on their coordinate JKP. Recall from (3.11-3.12) that an invariant form on a JKP $P = (J, M)$ can be viewed as a symmetric bilinear form g on the vector space $J_- \oplus M_- \oplus M_+ \oplus J_+$ and that the transformations $\delta(a, b)$ and $v(x, y)$ on that space are skew-symmetric relative to this form.

Invariant forms on J-ternary algebras

9.2 As a J-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$ is just a special case of JKP, the notion of an invariant form on ${}_{\mathfrak{J}}\mathfrak{M}$ should come from the notion of an invariant form on a JKP, but we would hope to have a simpler description in this particular case. Suppose $P = (\mathfrak{J}, \mathfrak{M})$ is the JKP associated to ${}_{\mathfrak{J}}\mathfrak{M}$ as described in Theorem 6.12, and let $(\ ,\ ) : J_+ \times J_- \to \mathbf{F}$, $(\ ,\ ) : M_+ \times M_- \to \mathbf{F}$ be an invariant form on P. One obvious simplification comes from the fact that $J_+ = \mathfrak{J} = J_-$ and $M_+ = \mathfrak{M} = M_-$, so we can set $\prec a, b \succ = (a_+, b_-)$ and $\prec x, y \succ = (x_+, y_-)$ to obtain bilinear forms on $\mathfrak{J}$ and $\mathfrak{M}$ respectively. Here we use the notation x_σ for an element $x \in \mathfrak{J} \cup \mathfrak{M}$ to mean that x is considered as an element of $J_\sigma \cup M_\sigma$.

Since $b_- = \frac{1}{2}D_{\mathfrak{e}_-,b_+}\mathfrak{e}_-$, identity (IF1) (in 3.11) implies that $\prec a,b \succ = (a_+,b_-) = (a_+,\frac{1}{2}D_{\mathfrak{e}_-,b_+}\mathfrak{e}_-) = (\frac{1}{2}D_{b_+,\mathfrak{e}_-}a_+,\mathfrak{e}_-) = ((ab)_+,\mathfrak{e}_-)$ and therefore $\prec a,b \succ$ is symmetric.

On the other hand it follows from (IF5) that $\prec x,y \succ = (x_+,y_-) = (x_+,\mathfrak{e}_- \circ y_+) = -(y_+,\mathfrak{e}_- \circ x_+) = - \prec y,x \succ$, that is $\prec x,y \succ$ is skew-symmetric.

Identity (IF1) can be rewritten as

$$\prec ab,c \succ = \prec a,bc \succ, \tag{9.3}$$

while (IF2) and (IF3) can be expressed as

$$\prec \langle x,y,z\rangle, w \succ = - \prec z, \langle y,x,w\rangle \succ \qquad \text{and} \tag{9.4}$$

$$\prec \langle x,z\rangle, b \succ = \prec x, b \circ z \succ . \tag{9.5}$$

Then (IF4) follows from the fact that $\prec x,y \succ$ is skew-symmetric. These considerations justify the following

Definition 9.6. An *invariant form* on a J-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$ consists of a symmetric bilinear form $\prec\ ,\ \succ : \mathfrak{J} \times \mathfrak{J} \to \mathbf{F}$ and a skew-symmetric bilinear form $\prec\ ,\ \succ : \mathfrak{M} \times \mathfrak{M} \to \mathbf{F}$ for which identities (9.3)-(9.5) hold.

Remark. Note that (9.3) says that $\prec\ ,\ \succ : \mathfrak{J}\times\mathfrak{J} \to \mathbf{F}$ is an invariant symmetric bilinear form relative to the Jordan product on $\mathfrak{J}$.

Every invariant form $\prec\ ,\ \succ$ on a $\mathfrak{J}$-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$ defines a unique invariant form $(\ ,\ )$ on the JKP $P = \mathcal{P}(\mathfrak{J},\mathfrak{M})$ by $(a_+,b_-) = \prec a,b \succ$ and $(x_+,y_-) = \prec x,y \succ$.

Invariant forms on structurable algebras

Definition 9.7. A bilinear form f on a structurable algebra $(X,{}^-)$ is *symmetric* provided $f(x,y) = f(y,x)$ for all $x,y \in X$, and it is *invariant* if

$$f(\bar{x},\bar{y}) = f(x,y) \qquad \text{and} \qquad f(xz,w) = f(z,\bar{x}w), \tag{9.8}$$

for all $x,z,w \in X$.

Lemma 9.9. *If f is a symmetric invariant form on the structurable algebra $(X,{}^-)$, then there is an invariant form $\hat{f}$ defined on the JKP $\mathcal{P}(X,{}^-)$ by $\hat{f}|_{X\times X} = f|_{X\times X}$ and $\hat{f}|_{S\times S} = f|_{S\times S}$.*

Proof. The fact that $\hat{f}$ is invariant can be verified directly, or it can be found in [31, Eqs. (8-9)] and [3, Eq. (42)]. ∎

This brings us to the main result on forms,

Theorem 9.10.

(i) *Let L be a BC_1-graded Lie algebra of type D_1:* $L = \mathfrak{L}(P, \mathcal{H})$ *where* $P = (J, M)$ *is a JKP and* $\mathcal{H} \subseteq \mathrm{HF}(P)$. *If f is an invariant form on P, then there is a unique symmetric invariant form g on L such that*

$$g(a,b) = f(a,b) \quad \text{and} \quad g(x,y) = f(x,y) \tag{9.11}$$

for all $a \in J_+$, $b \in J_-$, $x \in M_+$, and $y \in M_-$. Conversely, for every symmetric invariant form g on L, there is a unique invariant form f on P satisfying (9.11).

(ii) *Let L be a BC_1-graded Lie algebra of type C_1:* $L = \mathcal{K}(\mathfrak{J}, \mathfrak{M}, \mathcal{H})$ *for some $\mathfrak{J}$-ternary algebra ${}_{\mathfrak{J}}\mathfrak{M}$ and some subspace $\mathcal{H} \subseteq \mathrm{HF}(\mathfrak{J}, \mathfrak{M})$. If $\prec\ ,\ \succ$ is an invariant form on ${}_{\mathfrak{J}}\mathfrak{M}$, then there is a unique symmetric invariant form g on L such that*

$$g(a,b) = \prec a, b \succ \quad \text{and} \quad g(x,y) = \prec x, y \succ \tag{9.12}$$

for all $a, b \in \mathfrak{J}$, $x, y \in \mathfrak{M}$. Conversely, for every symmetric invariant form g on L, there is a unique invariant form $\prec\ ,\ \succ$ on ${}_{\mathfrak{J}}\mathfrak{M}$ satisfying (9.12).

(iii) *Let L be a BC_1-graded Lie algebra of type B_1:* $L = \mathcal{K}(X, {}^-, \mathcal{H})$ *for a structurable algebra $(X, {}^-)$ and a subspace $\mathcal{H} \subseteq \mathrm{HF}(X, {}^-)$. If f is a symmetric invariant form on $(X, {}^-)$, then there is a unique symmetric invariant form g on L such that*

$$g((x,0), (y,0)^{\sim}) = f(x,y) \tag{9.13}$$

for all $x, y \in X$. Conversely, for every symmetric invariant form g on L, there is a unique symmetric invariant form f on $(X, {}^-)$ satisfying (9.13).

Proof. (i) We can assume that $P = \mathcal{P}(L)$. Let f be an invariant form of P. Then there is an invariant symmetric bilinear form f' on the Lie algebra $\{P, P\}$ described in (5.12). It is easy to see that $\mathrm{HF}(P)$ is contained in the radical of this form. Hence there is an induced form f_0 on $L_0 = \{P, P\}/\mathcal{H}$. The identities in (5.12) guarantee that condition (4.11) holds. As a consequence, Lemma 4.10 implies that $g(p + l_0, p' + l_0') = f(p, p') + f_0(l_0, l_0')$ is an invariant symmetric bilinear form on L. The uniqueness of g follows from the fact that L is generated by the subspaces L_i, $i \neq 0$.

For the other direction, assume g is an invariant symmetric bilinear form on L and h is a grading element of L. Then for every $l_i \in L_i$ and $l_j \in L_j$, one has $ig(l_i, l_j) = g([h, l_i], l_j) = -g(l_i, [h, l_j]) = -jg(l_i, l_j)$, which implies that

$$g(L_i, L_j) = 0 \ \text{ unless } \ i + j = 0.$$

Now by Lemma 4.10, the restriction f of g to P is an invariant form on the JKP P. Clearly it is unique and satisfies (9.11).

(ii) The proof in this case follows from (i) and 9.2.

(iii) If f is a symmetric invariant form on $(X,{}^-)$, then by Lemma 9.9 there is an invariant form $\hat{f}$ on $\mathcal{P}(X,{}^-)$. Part (i) implies that $\hat{f}$ can be extended in a unique way on $\mathfrak{L}(\mathcal{P}(X,{}^-),\vartheta^{-1}(\mathcal{H}))\cong\mathcal{K}(X,{}^-,\mathcal{H})$ so that property (9.13) holds.

Now assume that g is a symmetric invariant form on $L=\mathcal{K}(X,{}^-,\mathcal{H})$. Define a bilinear form f on X by $f(x,y)=g((x,0),(y,0)^{\sim})$. In verifying f is symmetric and satisfies the identities in (9.8), we will use the $\mathfrak{sl}_2$-triple $\langle e,f,h\rangle=\langle(2,0),\,2\{1,1\}+\mathcal{H},\,(1,0)^{\sim}\rangle$ in L and the following equations,

$$(\bar{x},0)=(\mathrm{Id}-\frac{1}{2}FE)(x,0),\qquad(\bar{x},0)^{\sim}=(\mathrm{Id}-\frac{1}{2}EF)(x,0),$$

$$(x,0)^{\sim}=(\frac{1}{3}F^3E-F^2)(x,0)=(\frac{1}{3}EF^3-F^2)(x,0),$$

where $E=\operatorname{ad}e$, $F=\operatorname{ad}f$, and $H=\operatorname{ad}h$. All these relations except the last one are proved in [4] for the Lie algebra $\mathcal{K}(X,{}^-)$. In our situation they can be readily checked. The last identity follows from the equation $[E,F^3]=3F^2(H-2\mathrm{Id})$, which is a corollary of the $\mathfrak{sl}_2$-relations. From these equations and properties of g we have

$$\begin{aligned}f(\bar{x},\bar{y})&=g((\bar{x},0),(\bar{y},0)^{\sim})=g(((\mathrm{Id}-\frac{1}{2}FE)(x,0),0),(\bar{y},0)^{\sim})\\&=g((x,0),(\mathrm{Id}-\frac{1}{2}EF)(\bar{y},0)^{\sim})=g((x,0),(\bar{\bar{y}},0)^{\sim})\\&=f(x,y),\qquad\text{and}\\f(x,y)&=g((x,0),(y,0)^{\sim})=g((x,0),(\frac{1}{3}F^3E-F^2)(y,0))\\&=g((\frac{1}{3}EF^3-F^2)(x,0),(y,0))=g((x,0)^{\sim},(y,0))\\&=f(y,x).\end{aligned}$$

Finally, the invariance of g implies $f(V_{x,y}z,w)=f(z,V_{y,x}w)$. When $y=1$, this amounts to

$$f(xz+zx-z\bar{x},w)=f(z,\bar{x}w+w\bar{x}-wx).$$

If $\bar{x}=x$, it is immediate that $f(xz,w)=f(z,xw)$ as desired.

If instead $\bar{x}=-x$, then $f(xz+2zx,w)=f(z,-xw-2wx)$. Applying (9.8) to this equation and replacing $\bar{z},\bar{w}$ with z,w, we see that $f(zx+2xz,w)=f(z,-wx-2xw)$. Combining the last two equations, we obtain the desired conclusion $f(xz,w)=-f(z,xw)$. ∎

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